Cosmology

TUM WS 2016/2017

Lecture 2

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http://www.eso.org/~bleibund/Cosmology/

Recap Einstein Equations

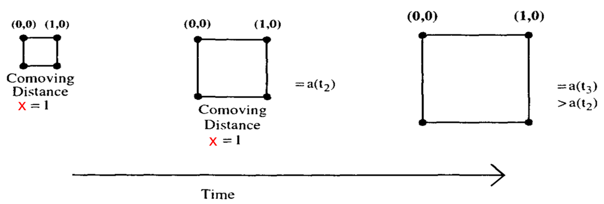
- Gravity is the dominant force in the universe → General Relativity
- Need the most general form of the metric → transformations between coordinate systems
  - find 'invariant' parameters
- Equation of motion for a force-free particle ($\ddot{x} = 0$) in GR leads to affine connections → Christoffel symbols
- Putting this together with the geometry and the energy content → Einstein Equations

3

Distances

- We separate the observed distances *r(t)* into the expansion factor *a(t)* and the fixed part *x* (called *comoving* distance)

$$r(t) = a(t)x$$



TUM WS19/20 Cosmology 2 Wolfgang Hillebrandt and Bruno Leibundgut 4

4

Friedmann equation

- The equation governing the expansion of the (flat) universe is

$$\left(\frac{\dot{a}}{a}\right)^2 \equiv H^2(t) = \frac{8\pi G}{3}\rho(t)$$

- and dividing by the Hubble constant H_0

$$\frac{H^2}{H_0^2} = \frac{\rho}{\rho_{crit}} \equiv \Omega$$

  – with $\rho_{crit} = \frac{3H_0^2}{8\pi G} \approx 10^{-26}\, kg/m^3$

- ρ*(t)* includes all energy forms in the universe

Curved geometry

- Consider the spatial part

$$dl^2 = d\mathbf{x}^2$$

- This is invariant under translations and rotations of the coordinate system

$$dl^2 = d\mathbf{x}^2 + dz^2; \mathbf{x}^2 + z^2 = a^2$$

- This is also true for the hyperbolic case

$$dl^2 = d\mathbf{x}^2 - dz^2; \mathbf{x}^2 - z^2 = a^2$$

- rescale with ***x'=ax*** and *z'=az*

Curved space

- gives

$$dl^2 = a^2[d\mathbf{x}^2 \pm dz^2]; z^2 \pm \mathbf{x}^2 = 1$$

- Differentiating $z^2 = 1 \mp \mathbf{x}^2$ gives $zdz = \mp \mathbf{x} d\mathbf{x}$
- $dl^2 = a^2\left[d\mathbf{x}^2 \pm \frac{(\mathbf{x}d\mathbf{x})^2}{1 \mp \mathbf{x}^2}\right]$
- and in general

$$dl^2 = a^2\left[d\mathbf{x}^2 + k\frac{(\mathbf{x}d\mathbf{x})^2}{1 - k\mathbf{x}^2}\right]$$

- with

$$k = \begin{cases} +1 & \text{spherical} \\ -1 & \text{hyperspherical} \\ 0 & \text{flat (Euclidian)} \end{cases}$$

7

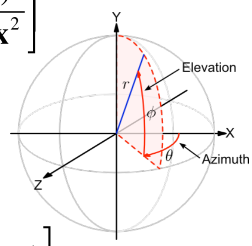
Curved space

- The line element becomes

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -c^2dt^2 + a^2(t)\left[d\mathbf{x}^2 + k\frac{(\mathbf{x}d\mathbf{x})^2}{1 - k\mathbf{x}^2}\right]$$

- Consider polar coordinates

$$d\mathbf{x}^2 = dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi\right)$$



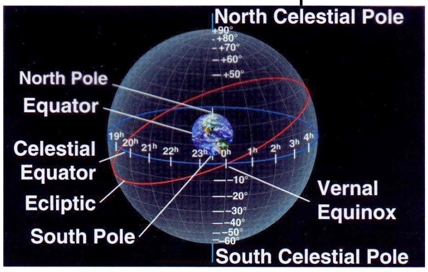
- leads to

$$ds^2 = -c^2dt^2 + a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

8

Robertson Walker metric

- As an observer at the origin of the coordinate system it is best to use polar coordinates
  - think of 'celestial sphere'
  - reason for right ascension and declination as coordinates on the sky
  - also longitude and latitude on Earth



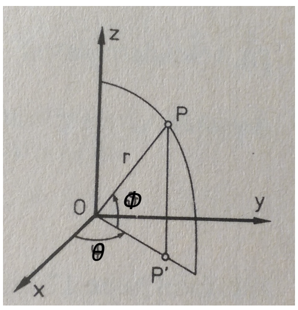
Robertson Walker metric

The line element has angular and radial components

$$ds^2 = -c^2dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right]$$

$$g_{00} = -1; g_{rr} = \frac{a^2(t)}{1-kr^2};$$

$$g_{\theta\theta} = a^2(t)r^2; g_{\phi\phi} = a^2(t)r^2\sin^2\theta$$



Curved space

- Going through the same steps again to calculate the contributions in the Einstein equations and then determine the Friedmann equation for curved space

$$\frac{\dot{a}^2}{a^2} + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\rho(t)$$

11

Cosmological Constant

- Einstein Equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

- Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} - \frac{8\pi G}{3}\Lambda c^2 = \frac{8\pi G}{3}\rho$$

12

Gravity in Einstein's Equations

- Consider an enclosed mass in a sphere

$$M(x) = \frac{4\pi}{3}\rho_0 x^3 = \frac{4\pi}{3}\rho(t) r^3(t) = \frac{4\pi}{3}\rho(t) a^3(t) x^3$$

  - here we converted the fixed density in comoving coordinates first into the density in the observed coordinate and then replaced it with the expansion factor
  - in principle this resulted in $\rho_0 = \rho(t) a^3(t)$

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13

Gravity in Einstein's Equations

- Acceleration of a particle on the surface of the sphere is

$$\ddot{r}(t) = \frac{d^2 r}{dt^2} = -\frac{GM(x)}{r^2} = -\frac{4\pi G}{3}\frac{\rho_0 x^3}{r^2}$$

  - now use $r(t) = a(t)x$ to change to the expansion factor

$$\ddot{a}(t) = \frac{\ddot{r}(t)}{x} = -\frac{4\pi G}{3}\frac{\rho_0}{a^2(t)} = -\frac{4\pi G}{3}\rho(t) a(t)$$

- This is the gravitational part of the field equations – GR modifies this part

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14

The Energy-Momentum Tensor

Use the form for the 'perfect fluid'

$$T^{\mu\nu} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

The energy conservation requires that the covariant derivative

$$0 = T^{\mu\nu}_{;\mu} = \frac{\partial T^{0\mu}}{\partial x^{\mu}} + \Gamma^{0}_{\mu\nu} T^{\nu\mu} + \Gamma^{\mu}_{\mu\nu} T^{0\nu} = \frac{\partial T^{00}}{\partial t} + \Gamma^{0}_{ij} + \Gamma^{i}_{i0} T^{00} = \frac{c^2 d\rho}{dt} + 3\frac{\dot{a}}{a}(p + \rho c^2)$$

$$c^2 \dot{\rho} + 3\frac{\dot{a}}{a}(p + \rho c^2) = 0$$

Energy-Momentum Tensor

- A general form is $p = \omega \rho c^2$. ω is the equation of state parameter.
- Inserting this into the conservation equation gives $\frac{\dot{\rho}}{\rho} = -3(1+\omega)\frac{\dot{a}}{a}$ which integrates to $\log(\rho) = -3(1+\omega)\log a + const.$
- Exponentiating yields

$$\rho \propto a^{-3(1+\omega)}$$

Expansion and contents

2[nd] derivative of the scale factor gives the dynamics of the expansion, i.e. differentiation of the Friedmann equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\rho c^2 + 3p)$$

– expect only deceleration ($\ddot{a} < 0$), since density (ρ) and pressure (p) are positive
– acceleration requires $(\rho c^2 + 3p) < 0$

Light ray coming towards us

- No angular dependence, hence

$$cdt = \pm a(t) \frac{dx}{\sqrt{1-kx^2}}$$

- and integrated

$$s = a \int_0^x \frac{dx}{\sqrt{1-kx^2}} = aS(x)$$

- with

$$S(x) = \begin{cases} \arcsin(x) & k = 1 \\ x & k = 0 \\ \operatorname{arcsinh}(x) & k = -1 \end{cases}$$

Strange consequences

- k=1
  - closed universe
  - distances increase and then decrease again with increasing x
- k=0
  - 'critical' universe
  - expanding forever
- k=-1
  - open universe
  - expands forever

Horizon

- Consider the causally connected region in the universe
  - distance travelled by light since t=0
  - (remember $dx = cdt/a$)

$$\eta = \int_0^t \frac{cdt}{a(t)}$$

  - this is the comoving distance for the horizon around every point in the universe
  - this is also called the conformal time

Redshift

- For two different times we get

$$\frac{dt_1}{a(t_1)} = \frac{dt_2}{a(t_2)}$$

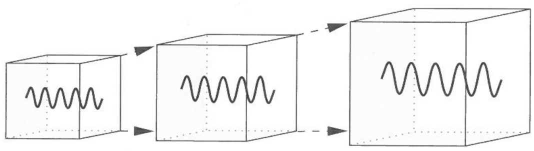
  – i.e. the time scales with the scale parameter
- If the time intervals *dt* are interpreted as oscillation periods, e.g. of a photon, then

$$\frac{dt_1}{dt_2} = \frac{\nu_2}{\nu_1} = \frac{a(t_1)}{a(t_2)} = \frac{1}{1+z}$$

- with z as the redshift between the two times

Redshift

- Redshift is directly related to the ratio of the scales between emission and absorption of a photon



- This is remarkably simple as a measurement in a spectrum tells the scale changes

Distances

- Different methods to measure distances
  - Luminosity distance

$l = \frac{L}{4\pi D^2}$; l *observed brightness*; L *emitted luminosity*; D *distance*

  - The distance is the comoving distance x_1 times the scale factor at the time of observation (for us 'today') $a(t_0)$

$$D = x_1 a(t_0)$$

Luminosity Distances

- The rate of the photons arrivals is reduced by a factor $\frac{a(t_1)}{a(t_0)} = \frac{1}{1+z}$ and the energy of the photons ($E=h\nu$) is also reduced by a factor $1+z$ (remember luminosity L is energy per time)

$$l = \frac{L}{4\pi x_1^2 a^2(t_0)(1+z)^2}$$

- Set $D_L = x_1 a(t_0)(1+z)$ and we recover the equation for the luminosity distance $l = \frac{L}{4\pi D_L^2}$

Angular size distance

– A different method is to measure the angle of a distant object of known size $D_A = \frac{l}{\theta}$

(here l is the size of the object; θ the observed angle)

– Inspection of the metric (here we only need the $g_{\theta\theta}$ part), which gives $l = x_1 a(t_1)\theta$ and inserting this in the equation above yields $D_A = x_1 a(t_1)$ and with $\frac{a(t_1)}{a(t_0)} = \frac{1}{1+z}$ we find $\frac{D_L}{D_A} = (1+z)^2$.

Distances

- This is quite remarkable for high redshifts
  - the physical distances differ for the same redshift!
  - an object for which we could measure the angular size distance and the luminosity distance would give a different number of kilometres!
  - a direct consequence of general relativity

Friedmann equation (last time)

- We can put the various densities into the Friedmann equation $\frac{\dot{a}^2}{a^2} = H^2 = \frac{8\pi G}{3}\rho(t)$
$$= \frac{8\pi G}{3}(\rho_{matter} + \rho_{rad} + \rho_{vac}) - \frac{k^2c^2}{a^2}$$
- We can define the critical density for a flat universe ($k = 0$) $\rho_{crit} = \frac{3H^2}{4\pi G}$ and we can define the ratio to the critical density $\Omega = \frac{\rho}{\rho_{crit}}$
- Most compact form of Friedmann equation $1 = \Omega_{matter} + \Omega_{rad} + \Omega_{vac} + \Omega_k$ with $\Omega_k = -\frac{kc^2}{a^2H^2}$

Matter

- The pressure in matter is negligible compared to the mass content (think *mc*²) and hence $\omega = 0$
- Thus $\rho_{matter} \propto a^{-3}$
- Inserting this in the Friedmann equation for a flat universe (k=0) provides the time dependence of the scale factor

$$a(t) \propto t^{\frac{2}{3}}$$

Radiation

- Radiation decreases with the volume (i.e. number of photons), but has one additional factor due to the redshift $\omega = \frac{1}{3}$ and hence $\rho_{rad} \propto a^{-4}$
- The time dependence here is now

$$a(t) \propto \sqrt{t}$$

Vacuum energy

- A special case is $\rho_{vacuum} = const.$
- In this case the density is associated to the vacuum
- Now the scale factor grows exponentially

$$a(t) \propto e^{Ht}$$

Dependence on scale parameter

- For the different contents there were different dependencies for the scale parameter
$$\rho_{matter} \propto a^{-3} \qquad \rho_{rad} \propto a^{-4} \qquad \rho_{\Lambda} = const.$$
- Combining this with the critical densities we can write the density as
$$\rho = \frac{3H_0^2}{8\pi G}\left[\Omega_{matter}\left(\frac{a_0}{a}\right)^3 + \Omega_{rad}\left(\frac{a_0}{a}\right)^4 + \Omega_{\Lambda}\right]$$
and the Friedmann equation
$$H^2 = H_0^2\left[\Omega_{matter}(1+z)^3 + \Omega_{rad}(1+z)^4 + \Omega_{\Lambda} + \Omega_k(1+z)^2\right]$$

Lookback Time

- Consider
$$H = \frac{\dot{a}}{a} = \frac{da}{dt}\frac{1}{a} = dt \ln\left(\frac{a(t)}{a_0}\right) = \frac{1}{dt}\ln\left(\frac{1}{1+z}\right) = \frac{-1}{1+z}\frac{dz}{dt}$$
- Inserting into the Friedmann equation we find the equation for the time interval
$$dt = \frac{-dz}{H_0(1+z)\sqrt{\Omega_{matter}(1+z)^3 + \Omega_{rad}(1+z)^4 + \Omega_{\Lambda} + \Omega_k(1+z)^2}}$$
and integrating
$$t_0 - t_1 = \frac{1}{H_0}\int_0^{z_1} \frac{dz}{(1+z)\sqrt{\Omega_{matter}(1+z)^3 + \Omega_{rad}(1+z)^4 + \Omega_{\Lambda} + \Omega_k(1+z)^2}}$$
- Age in a matter dominated universe (t_1=0, z=∞) $t_{0,matter} = \frac{1}{H_0}\int_0^{\infty}\frac{dz}{(1+z)^{5/2}} = \frac{2}{3H_0}$ and $t_{0,rad} = \frac{1}{2H_0}$

Distances (last time)

- We can now also express the luminosity distance $D_L = a_0 x_1 (1+z)$ in these terms
  - from the metric for a light ray coming towards us we have $\frac{dr}{cdt} = \frac{\sqrt{1-kx^2}}{a(t)}$ which turns into $\frac{\mathrm{a}_0}{\mathrm{c}} \frac{dx}{\sqrt{1-kx^2}} = (1+z)dt$
  - after integration we have (using dt from above) $\frac{a_0}{c}\int_0^{x_1} \frac{dx}{\sqrt{1-kx^2}} = \int_0^{z_1} \frac{dz}{H_0\sqrt{\Omega_{matter}(1+z)^3+\Omega_{rad}(1+z)^4+\Omega_\Lambda+\Omega_k(1+z)^2}}$
  - solutions of the left side are $\frac{a_0}{c} \times \begin{cases} \frac{\arcsin(x_1\sqrt{k})}{\sqrt{k}} & k > 0 \\ x_1 & k = 0 \\ \frac{\operatorname{arcsinh}(x_1\sqrt{-k})}{\sqrt{-k}} & k < 0 \end{cases}$

Luminosity Distance

- Putting this together with the appropriate trigonometric functions gives

$$\mathrm{D}_L = a_0 x_1 (1+z) = \frac{c(1+z)}{H_0\sqrt{|\Omega_k|}} S\left(\sqrt{|\Omega_k|}\int_0^z \frac{dz'}{\sqrt{\Omega_{matter}(1+z')^3+\Omega_{rad}(1+z')^4+\Omega_\Lambda+\Omega_k(1+z')^2}}\right)$$

with $S(y) = \begin{cases} \sin(y) & k > 0 \\ y & k = 0 \\ \sinh(y) & k < 0 \end{cases}$

- We now have the luminosity distance as a function of today's measurements (H_0, Ω's) and the redshift z