Energy transport by convection

Convection in stars is a highly turbulent, 3-dimensional and non-local motion in compressible medium on dynamical timescales.

\[ \text{Reynolds number } \text{Re} := \frac{v l_m}{\eta} \approx 10^{10}; \ \eta \text{ viscosity; } v \text{ speed of blobs } \approx 10^3 \text{ cm/s} = 10^{-5} v_{\text{sound}}; \ l_m = 10^9 \text{ cm; in laboratory: turbulence sets in when } \text{Re} > 100; \]

Presently, no full hydro-solution possible, only \( \nabla \) from local approaches. Most widely used is the **Mixing Length Theory** (by Prandtl, Böhm-Vitense, . . .).

The Mixing Length Theory equation (formulation of Kippenhahn & Weigert):

Fluxes and gradients, definitions:

\[
F = \frac{L_r}{4\pi r^2} = F_{\text{conv}} + F_{\text{rad}} \quad (1)
\]
\[
F = \frac{4acG T^4 m}{3\kappa P r^2} \nabla_{\text{rad}} \quad (2)
\]
\[
F_{\text{rad}} = \frac{4acG T^4 m}{3\kappa P r^2} \nabla \quad (3)
\]
\[
F_{\text{conv}} = \rho v c_P (DT) \quad (4)
\]

\( \nabla_{\text{rad}} \) is the gradient needed if energy would be transported by radiation only. Of course \( \nabla_{\text{rad}} > \nabla_{\text{ad}} \).

\( F_{\text{rad}} \) is the flux actually carried by radiation, even if there is convection. If convection is adiabatic, this vanishes.
The Mixing Length picture is the following: We assume (as for the discussion of convective instability), a blob starts somewhere with $DT > 0$ and loses identity after a typical mixing length distance $l_m$. It dissolves into its surroundings and deposits its energy there.

We consider some sphere at radius $r$ and an average blob coming from below. On average that blob will have traveled $l_m/2$, so its excess temperature at $r$ will be

$$\frac{DT}{T} = \frac{1}{T} \frac{\partial (DT)}{\partial r} \frac{l_m}{2} = (\nabla - \nabla_e) \frac{l_m}{2} \frac{1}{H_P}$$

where the gradients are those of the surroundings and of the element ($e$), and we used the definition of $H_P = -\frac{dr}{d\ln P} = \frac{P}{g\rho}$

$D\rho$ can be calculated from the equation of state, remembering that $DP = 0$ because of hydrostatic equilibrium between the element and the surrounding pressure;

$$\frac{D\rho}{\rho} = -\delta \frac{DT}{T}$$

the resulting buoyancy force is therefore $f_b = -gD\rho/\rho = g\delta DT/T$. The work done by it is (assume 50% have acted on the element over $l_m/2$) then

$$\frac{1}{2} f_b \frac{l_m}{2} = g\delta (\nabla - \nabla_e) \frac{l_m^2}{8H_P}$$
Again, assume half of this goes into kinetic energy of the element, the rest be used up for “pushing away” the surrounding. Then,

\[ v^2 = g\delta (\nabla - \nabla_e) \frac{l_m^2}{8H_P} \]  \hspace{1cm} (5)

and we can compute the convective flux

\[ F_{\text{conv}} = \rho c_P T \sqrt{g\delta} \frac{l_m^2}{4\sqrt{2}} H_p^{-3/2} (\nabla - \nabla_e)^{3/2} \]  \hspace{1cm} (6)

At this point we have to take into account that convection may not be adiabatic, so there will be some extra heat exchange with the surrounding. \( T_e \) changes due to adiabatic cooling and radiation losses,

\[ \left( \frac{dT}{dr} \right)_e = \left( \frac{dT}{dr} \right)_{\text{ad}} - \frac{\lambda}{\rho V c_P v}, \]

or

\[ \nabla_e - \nabla_{\text{ad}} = \frac{\lambda H_P}{8VvT c_P}. \]  \hspace{1cm} (7)

\( \lambda \) describes the radiation loss relative to the blob’s energy. It can be formulated as

\[ \lambda = S f = \frac{8a c T^3}{3\kappa \rho} DT \frac{S}{d} \]
with \( S \) and \( d \) being surface and diameter of the blob, and \( f \) the non-radial flux from the blob into the surrounding, for which the radiative transport equation equally holds. The temperature gradient in normal direction is approximated by \( \frac{\partial T}{\partial n} \approx 2DT/d \).

Using this expression for \( \lambda \) we obtain an expression in which the geometry of the blob appears in the form \( l_mS/Vd \), which would be \( 6/l_m \) if the blob would be spherical with diameter \( l_m \). The usual approximation is

\[
\frac{l_mS}{Vd} \approx \frac{9/2}{l_m}
\]

With this and the expression for \( \lambda \) eq. (8) can be turned into

\[
\frac{\nabla_e - \nabla_{ad}}{\nabla - \nabla_e} = \frac{6acT^3}{\kappa \rho^2 c_P l_m v}
\]  

(8)

We now have the 5 equations (1), (3), (5), (6), (8) for five variables: \( v, F_{\text{conv}}, F_{\text{rad}}, \nabla \) and \( \nabla_e \) in terms of \( P, T, \rho, l_m, \nabla_{ad}, \nabla_{rad} \), which are all local variables plus the unknown mixing length \( l_m \).

An analytical solution is possible. We define

\[
U = \frac{3acT^3}{c_P \rho^2 \kappa l_m^2} \sqrt{\frac{8H_P}{g\delta}}
\]

\[
W = \nabla_{rad} - \nabla_{ad}
\]

\[
\xi^2 = \nabla - \nabla_{ad} + U^2
\]
We first eliminate $v$ with (5) from (8) and get
\[ \nabla e - \nabla_{ad} = 2U \sqrt{\nabla - \nabla_e} \]

Another equation is obtained by eliminating the energy fluxes in (1), (2), and (3), using the definition of $H_P$ and the hydrostatic equation:
\[ (\nabla - \nabla_3)^{3/2} = \frac{8}{9} U (\nabla_{rad} - \nabla) \]

If we rewrite the l.h.s. of the first one as $(\nabla - \nabla_{ad}) - (\nabla - \nabla_e)$, one realizes that that is a quadratic equation for $(\nabla - \nabla_e)^{1/2}$ with the solution
\[ \sqrt{\nabla - \nabla_e} = -U + \xi \]

We insert this in the second equation on the left hand side, and rewrite $\nabla$ on the right hand side with the definition of $\xi$ and now get the cubic equation of mixing length theory:
\[ (\xi - U)^3 + \frac{8U}{9} (\xi^2 - U^2 - W) = 0 \] \hspace{1cm} (9)

$W$ and $U$ can be calculated at any point (local!) and the cubic equation can be solved for $\xi$. From this we get $\nabla$, which is the final gradient established in the layer due to convection.
A useful quantity for diagnostic is
\[ \Gamma := \frac{(\nabla - \nabla_e)^{1/2}}{2U} = \frac{(\nabla - \nabla_e)}{(\nabla_e - \nabla_{ad})} \]
is the ratio of energy transported by the blob over that lost from it, or the “efficiency” of the convection.

If \( U \) is small, \( \Gamma \) is large and almost all flux is transported by convection and the resulting gradient is \( \approx \nabla_{ad} \). If \( U \) is large, \( \Gamma \) is small, and – although transport is by convection – the resulting gradient is almost \( \nabla_{rad} \). In general, convection is superadiabatic:

\[ \nabla = \nabla_{ad} + \delta \nabla \]

The equations still contain the mixing length, which is usually expressed as a multiple of the pressure scale height
\[ l_m = \alpha_{MLT} H_p. \]
\( \alpha_{MLT} \) is the mixing-length parameter. It is of order 1, and is determined usually by solar models; numerical values are \( \alpha_{MLT} \approx 1.2 \cdots 2.2 \), most recently they cluster around 1.8. However, there is no physical model that can determine \( \alpha_{MLT} \).

Example for \( \nabla \): Sun, \( r = R_\odot/2, m = M_\odot/2, T = 10^7 \), \( \rho = 1, \delta = \mu = 1 \)
\[ \rightarrow U = 10^{-8} \rightarrow \nabla = \nabla_{ad} + 10^{-5} = 0.4 \]
(as long as \( \nabla_{rad} < 100 \cdot \nabla_{ad} \))
at center, \( \nabla = \nabla_{ad} + 10^{-7} \).
Using MLT the various gradients for the Sun look like this:

...and for a $10M_\odot$ star they are shown on the left; the right panel displays the convective regions along the zero-age main-sequence.
Comments on Mixing Length Theory:

1. convection is turbulent with a spectrum of element sizes (and shapes)

2. MLT is purely local: overshooting from convective boundary or any other non-local effect is not included

3. MLT describes stationary situation: time-dependence of convection ignored

4. $\alpha_{\text{MLT}}$ is only average value; depth-, mass-, and composition-dependencies are all ignored, and not even empirically determined

5. there are several “flavours” of MLT (formulation, assumptions, hidden parameters, etc.), so $\alpha_{\text{MLT}}$ cannot be compared straightforwardly between different computations

6. empirical determination of $\alpha_{\text{MLT}}$ in fact also compensates for all other errors in physics that determine radius or temperature of star

7. MLT has worked surprisingly well!
Real convection:

1. Solar granulation (Big Bear Observatory):

2. Motion below photosphere from helioseismology (SOHO):
3. Downdrafts in simulations (Stein):