Modelling the line-of-sight contributions in substructure lensing

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ABSTRACT

We investigate the gravitational effect on Einstein rings and magnified arcs of small-mass dark-matter haloes placed along the line-of-sight of gravitational lens systems. By comparing the gravitational signature of line-of-sight haloes of different mass density profiles with that of substructures within the lensing galaxy, we derive a mass-redshift relation that allows to rescale the detection threshold (i.e. lowest detectable mass) for substructures to a detection threshold for line-of-sight haloes at any redshift. For a given substructure mass detection threshold we then use this relation to quantify the line-of-sight contribution to the total number of small mass haloes (including substructures) that can be detected through strong gravitational lensing. We then quantify the degeneracy between substructures and line-of-sight haloes of different mass and redshift to provide a statistical interpretation of current and future detections and shed more light into the dark matter components of our Universe. We find that for data with high enough signal-to-noise ratio and angular resolution the non-linear effects arising from a double lens plane configuration are such that one is able to observationally distinguish between the gravitational effect of a line-of-sight halo and a substructure and to recover the line-of-sight halo redshift with an absolute error precision of 0.15 at the 68 per cent confidence level. Our results have important implication for the constraints that can be placed on dark matter models using strong gravitational lensing.

Key words: galaxies: halos - cosmology: theory - dark matter - methods: numerical

1 INTRODUCTION

Strong gravitational lensing is a powerful tool to measure the total projected mass distribution of structures from galaxy clusters (Limousin et al. 2016; Meneghetti et al. 2016) to small sub-galactic scales (Keeton 2003; Keeton et al. 2003). Gravitational lensing effectively depends not only on the properties of the system acting as a main lens, but also on the mass distribution integrated along the line-of-sight between the observer and the source (Bartelmann & Schneider 2001; Bartelmann 2010). Understanding the contribution from those is, therefore, of primary importance for better discerning the matter density distribution within the Universe down to small scales.

Given the increasing resolution of the observational data, probing the line-of-sight contaminations is becoming more and more relevant and a number of recent works have addressed this problem, mainly on galaxy cluster-size systems (e.g. Birrer et al. 2016; McCully et al. 2017).

On galaxy-scale lenses a significant effort has been made over the years to understand the line-of-sight contribution to the flux ratio anomalies observed in gravitationally lensed quasars (e.g. Metcalf & Amara 2012; Xu et al. 2012, 2015).

The aim of this work is to investigate the gravitational lensing effect of small-scale line-of-sight haloes on the surface brightness distribution of lensed arcs and Einstein rings, and quantify their contribution to the total number of detectable objects. Our goal is also to provide a statistical interpretation for current (Vegetti & Koopmans 2009; Vegetti et al. 2010, 2012, 2014; Hezaveh et al. 2016) and possible future detections. In particular, we use simulated mock data to explore the relative lensing signal of line-of-sight haloes and substructures within a host lens galaxy as a function of redshift mass and density profile. Unlike, Li et al. (2016) who

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have carried out a similar analysis for a specific case, here we adopt a more general approach with the aim of obtaining results that are valid for a wide range of realistic strong lensing targeted observations.

In this work, we show that the line-of-sight contribution is of particular relevance when trying to distinguish between different dark matter models for four main reasons: (i) while low-mass substructures may be the surviving cores of progenitor haloes (van den Bosch et al. 2007; Gao et al. 2004; Giocoli et al. 2008), the number and the mass of line-of-sight haloes is less affected by baryonic or accretion processes; hence the detection of low-mass line-of-sight haloes provides a more direct constraint on the dark-matter mass function and in particular on models that predict a strong suppression of low-mass structures (e.g. warm dark matter, WDM); (ii) the number of detectable line-of-sight haloes can be larger or comparable to the number of detectable substructures (see Section 3), hence failing to detect a significant number of small-mass structures even with small samples of lens galaxies could potentially rule out all dark matter models that predicts a steeply rising halo mass function (e.g. cold dark matter, CDM); (iii) the lensing effect of a foreground line-of-sight halo is larger than the lensing effect of a substructure of the same mass, therefore for a given signal-to-noise ratio and angular resolution of the lensed images, line-of-sight structures allow one to probe the dark matter mass function at a lower mass regime, where different dark matter models differ the most (Viel et al. 2005; Lovell et al. 2014); (iv) finally, the combination of points (ii) and (iii) implies that smaller samples of lenses are required to set constraints on the nature of dark matter that are as tight as those that are derived from the substructure contribution only.

We structure this paper as follows: in Section 2 we describe the analytical models that we employ and our method for generating mock data; in Section 3 we derive the mass-redshift relation that allows us to compare the effect of substructures with that of line-of-sight haloes at different redshifts - and with different density profiles. We use these analytical relations for two purposes: (i) rescale the sensitivity function (Vegetti et al. 2014) in order to convert the lowest detectable substructure mass in a lowest detectable mass as a function of redshift; (ii) use this lowest detectable mass to correctly integrate the line-of-sight mass function, i.e. by considering only those haloes that would have a detectable lensing effect. In Section 4, we model our mock data with the lensing code by Vegetti & Koopmans (2009); Vegetti et al. (2012): to quantify the degeneracies in the mass-redshift space and test the limits of the analytical approach derived in the previous section. This allows us to statistically interpret individual detections from observations and quantify the probability that these arise from a line-of-sight halo. Finally, in Section 5 we conclude by summarizing our results.

2 MOCK DATA

In order to test the general validity of our results, we consider several mock data sets. These are characterized by different angular resolutions, signal-to-noise ratios, background source morphologies and different lens-source alignments as well as perturbers located at different redshifts. The main lens galaxies are assumed to have a singular isothermal ellipsoidal (SIE) mass density profile (Kormann et al. 1994) and a total mass of \( M \simeq 10^{15} M_\odot \). This corresponds roughly to the typical total mass of early-type lens galaxies at redshift \( z = 0.2 - 1 \) (e.g. Auger et al. 2009). The effect of different external shear contributions are also explored. More details on the considered lens systems are reported in Table 1. In our simplest model the source has a Gaussian light profile and, to avoid any influence from asymmetry, the main lens has a singular isothermal sphere (SIS) mass profile and no external shear is included. We use this simplified model as a red reference sample and to compare our results with those by Li et al. (2016). We then modify this model by adding ellipticity and external shear, in order to systematically test the effect of these components. The other lens systems are based on real observations instead: this means that the lens models include both ellipticity and external shear and the sources are not regular. In particular, we choose: (i) two systems from the SLACS survey (Bolton et al. 2006), which have already been used for substructure analysis by Vegetti et al. (2014) and Despali & Vegetti (2017); (ii) three systems from the BLAEs sample (Shu et al. 2016,?, Ritondale & Vegetti 2010); (iii) one system from the SHARP survey (Lagattuta et al. 2012; Fassnacht & Vegetti 2010) based on which Vegetti et al. (2012) have reported the detection of a \( \sim 2 \times 10^6 M_\odot \) substructure; (iv) one system with significantly-high angular resolution, similarly to what one could obtain with VLBI observations (Spingola et al. 2016).

For each of the lens system, we consider a so called smooth model (i.e. without substructures and line-of-sight haloes) and several perturbed models where substructures and line-of-sight haloes with different masses, redshifts and density profiles are included (see Section 2.1 for details).

Mock images created with the smooth models alone are shown in Fig. 1. In the next sections we provide the reader with more details on the properties of the perturbers.

2.1 Inclusion of haloes along the line-of-sight

In order to include only those line-of-sight haloes that can effectively perturb the lensed images, we consider as line-of-sight volume a double cone with a base of 1.5 times the Einstein radius of the main lens (see Figure 2). Within this cone, we sample the whole redshift range between the observer and the source - thus considering both foreground and background objects. Line-of-sight haloes are modeled as Navarro et al. (1997) profiles (hereafter NFW), while for the substructures we also consider the Pseudo-Jaffe (PJ) profile, as often used to model real observational datasets (e.g. Dalal & Kochanek (2002), Vegetti et al. (2014), Hezaveh et al. (2016)). Here, we assume the highest possible dark perturber mass to be \( \sim 10^{15} M_\odot h^{-1} \) since subhaloes with masses \( M < 10^6 M_\odot h^{-1} \) are mostly completely dark and at \( 10^{10} M_\odot h^{-1} \) the star fraction is still relatively small (\( f_s \lesssim 0.2 \)) (Schaller et al. 2015; Fiacconi et al. 2016; Despali & Vegetti 2017). Instead, the minimum mass is chosen in such a way to include line-of-sight haloes which are relevant for substructure detections down to \( 10^6 M_\odot \) and so we set it at \( 10^3 M_\odot \) (see Section 3).

In the perturbed models substructures and the line-of-sight haloes have different projected positions on the image.
### Table 1. Properties of the lens models from Figure 1. For the first lens we used a Gaussian source, while in the other cases both the source and the SIE lens model come from real observations. In all cases the mass of the main lens is \( \approx 10^{13} \mu M_\odot \).
plane which are marked by empty circles in Fig. 1. As we want to perform a one-to-one comparison between the local lensing effect of the two different populations, the 3D position of the line-of-sight haloes is corrected with redshift so that its projected position on the plane of the main lens is conserved for each set of mocks. This means that the line-of-sight halo should always lie on the same line-of-sight, as sketched in Figure 2. In particular, we use the factor $\beta$ (see Section 3 - equation 13 for a definition) to rescale the position of the perturber in the background. For each perturbed model we only consider the presence of one perturber at the time: this is justified by the fact that we are interested in quantifying the relative lensing effect of substructure and line-of-sight haloes rather than their global effect on the data.

Below we remind the reader the main features of the considered mass profiles and the basic equations used to calculate their deflection angles. The density and deflection angle profiles for the these models are shown in Figure 3.

### 2.2 NFW profile

The NFW profile

$$\rho(r) = \frac{\rho_s}{\left(\frac{r}{r_s}\right)^2 \left(1 + \frac{r}{r_s}\right)}$$

is well defined in terms of the halo virial mass $M_{vir}$ and a concentration-mass relation, which is needed to compute the scale radius as $r_s = r_{vir}/\sqrt{2} c$. Here, we choose the concentration-mass relation by Meneghetti et al. (2014) which is characteristic of strong lensing systems, extrapolating it to galaxy scales. As we will show in Section 3, the choice of a particular concentration-mass relation is of second order importance for our purposes. The deflection angle can be written as

$$\alpha(x) = \frac{4k_s}{x} h(x),$$

where

$$h(x) = \ln \frac{x}{2} + \begin{cases} \frac{2}{\sqrt{x^2 - 1}} \arctan \sqrt{\frac{x^2 - 1}{x^2 + 1}} & \text{if } x > 1 \\ \frac{2}{\sqrt{1 - x^2}} \arctan \sqrt{\frac{1}{x^2 + 1}} & \text{if } x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

and

$$\sigma_v = \frac{c^2 + D_s}{4\pi G D_t D_s}, \quad k_s = \frac{r_s c^2}{\sigma_v^2}.$$  \hspace{1cm} (4)

For the main results of the paper we do not consider the scatter in the concentration-mass relation, meaning that we assign a deterministic value of the concentration always to each combination of mass and redshift. In Section 3, we demonstrate that for the main purposes of this paper, a different choice for the mass-concentration relation or allowing some scatter around the mean value, would introduce only second order effects.

### 2.3 PJ profile

The Pseudo-Jaffe profile is defined as

$$\rho(r) = \frac{\rho_s r_s^3}{r^2 (r^2 + r_s^2)};$$

where $r_s$ is the truncation radius and is often assumed to be well approximated by the substructure tidal radius

$$r_{tidal} = r \left(\frac{M_{sub}}{\Gamma M(< r)}\right)^{1/3},$$

where $M(< r)$ is the mass of the host lens galaxy-halo at the 3D position $r$ of the subhalo, $M_{sub}$ is the mass of the subhalo and $\Gamma$ depends on the assumptions made on the satellite orbit (it is typically set equal to 3 for the assumption of circular orbits). For an isothermal host lens and a substructure located exactly on the plane of the lens galaxy the truncation radius can be written as

$$r_{t \approx} \sqrt{\frac{E_{t \approx}}{r_{ko}^2}} \approx \sqrt{\frac{\pi k_{0, sub} k_0}{24\pi}} \approx \left(\frac{m}{\pi \sigma_c^2 E_{0,0}}\right)^{2/3},$$

where $k_0$ and $k_{0, sub}$ are the surface mass density normalizations of the main lens and substructure, respectively. Thus, the truncation of the profile depends on the redshift (via $\sigma_c$) and the mass (via $r_{ko}$) of the host lens galaxy. Finally, the profile deflection angle can be expressed as a function of the substructure projected position as

$$\alpha(R) = \frac{\kappa_0}{\sigma_c} \left(\frac{r_t + R - \sqrt{r_t^2 + R^2}}{R}\right).$$

Since the profile is truncated, it is possible to define a total subhalo mass as follows

$$M_{sub} = \pi \Sigma c_\kappa r_{ko} k_{0, sub}. $$

As the normalization of the PJ profile for a subhalo depends on the mass of the main halo it is embedded in, it would not be meaningful to define a virial mass or virial radius for this profile, as for the NFW. In Section 3.5 we investigate how to compare subhaloes with PJ and NFW profiles and how the NFW virial mass and PJ total mass can be related.

### 3 LINE-OF-SIGHT CONTRIBUTION

The aim of this section is to quantify the line-of-sight contribution to the total number of detectable objects (i.e. substructures plus line-of-sight haloes) for different lens-source redshift configurations. Both contributions can be quantified by integrating the relative mass function from the lowest detectable mass to the highest possible dark (sub)halo
Figure 3. Density (left) and deflection angle (right) profiles for a NFW and a PJ profile. We show the results for all the profiles at $z = 0.2$ and different line-styles stand for different masses ($10^7 - 10^9 M_\odot$ in the top panels and $10^6 - 10^8 M_\odot$ in the bottom panels); note that the mass definition differ between the NFW profiles - where $m_{\text{sub}}$ is the virial mass - and the PJ profile - for which it is possible to define a total mass (equation 9). These profiles help us notice that the lensing effect of the different models cannot be the same for the same mass and also that the relative strength varies with mass.

mass. Vegetti et al. (2014) have defined the lowest detectable substructure mass as the mass that can be detected with a statistical significance of 10-$\sigma$. We refer to their paper for a detailed discussion on how this mass is determined. Here, we assume that the lowest detectable substructure mass is given and we derive the corresponding lowest detectable line-of-sight halo mass as a function of redshift by comparing the relative lensing effect of substructures and line-of-sight haloes. As detailed in the previous section, both substructures and line-of-sight haloes have masses between $10^5$ and $10^{10} M_\odot$.

3.1 Lensing effect

Most of previous studies on the effect of line-of-sight haloes on gravitationally lensed images have mainly been focusing on multiply-imaged quasars (e.g. Metcalf 2005; Xu et al. 2012). Therefore, the relative gravitational effect of substructures and line-of-sight haloes was quantified in terms of local changes to the lensing magnification. In this paper, we focus instead on Einstein rings and magnified arcs. As demonstrated by Koopmans (2005) perturbations to the lensing potential locally affect the observed surface brightness distribution with a strength that can be expressed as the inner product of the gradient of the background source surface brightness distribution ($\nabla s$; evaluated in the source plane) dotted with the gradient of the potential perturbation due to (sub)structures ($\nabla \delta \psi$; evaluated in the image plane): $\nabla I = -\nabla s \cdot \nabla \delta \psi$. Since the (sub)structure deflection angle is related to its potential as $\delta \alpha = \nabla \delta \psi$, for a given background source brightness distribution, we quantify the relative gravitational effect of substructures and line-of-sight haloes in terms of their deflection angles. In particular, for a substructure of a given mass and projected position relative to the main lensing galaxy, at each redshift $0 \leq z \leq z_s$ we look for the line-of-sight halo mass that for the same projected position minimizes the following deflection angle residuals

$$d\alpha = \frac{1}{N_{\text{pix}}} \left( \sum_{i=1}^{N_{\text{pix}}} \left( \Delta \alpha_{\text{line-of-sight}} - \Delta \alpha_{\text{sub}} \right)^2 \right)^{1/2}$$

where $\Delta \alpha_i$ is the difference in the deflection angle between the perturbed and the smooth model. In the simple case of two lenses at the same redshift both the lensing potential and deflection angle can be written as the linear sum of the individual contributions of the two lenses and the lens equation is written as

$$\mathbf{u} = \mathbf{x} - [\alpha_1(\mathbf{x}) + \alpha_2(\mathbf{x})],$$

instead, when two lenses are separated along the line-of-sight enough that their caustics are distinct, a recursive lens equation is derived (Schneider 1992)

$$\mathbf{u} = \mathbf{x} - \alpha_1(\mathbf{x}) - \alpha_2 [\mathbf{x} - \beta \alpha_1(\mathbf{x})],$$
where the factor
\[ \beta = \frac{D_{12} D_{os}}{D_{os} D_{12}} \] (13)
encodes the redshift difference in terms of distance ratio; \( \beta \) vanishes if the two lenses have the same redshift and approaches unity for redshifts close to the observer or the source. This means that the combined effect of the two lenses cannot be calculated by simply summing the deflection angles, but the effect of the foreground object on the background one must be taken into account. When comparing the lensing effect of a given substructure with line-of-sight haloes via equation (10), we first order the lenses in redshift and then apply equation (12). As shown by McCully et al. (2017), since the deflection angle of the foreground lens enters the lens equation inside the argument of the deflection angle of the background lens, this may induce complicated effects because it creates a difference between the coordinates we see on the sky and those on the plane of the second lens. Thus, if the mass of the foreground perturber is high enough, it introduces non-linear lensing effects. However, the masses of our perturbers are much lower than the mass of the main lens (between 7 and 3 order of magnitude) and thus when the former is in the foreground the effect on the main lens is small, while the opposite holds when the perturber is in the background and its deflection angle is influenced by the presence of the main lens.

We have tested how much our results based on equations (10) and (12) depend on the region within which the deflection angle residuals are calculated. We have found that for configurations in which the perturber is located in the foreground, small level residuals around the main galaxy position arise because of the aforementioned effect. Thus, in the following sections when referring to equation (12) we always mean as calculated inside the entire image plane, minus a small region around the centre of the main lens.

Finally, it should be stressed that equation (10) does not contain any information about the data signal-to-noise ratio, the effect of the point-spread-function (it does however reflect the angular resolution of the data) nor the degeneracy among the main lens parameters, the source surface brightness distribution and the mass of the perturber. For this reason, it does not provide a strict mass-redshift degeneracy (see Section 4), but, as shown in the following sections, it provides a functional approach with which rescale the substructure detection threshold as a function of redshift and determine the relative contribution of substructures and line-of-sight haloes. We refer to Section 4 for the quantification of a proper degeneracy between substructures and line-of-sight haloes.

### 3.2 Deflection angles at different redshifts

Before investigating the effects due to the double-lens-plane coupling, we want to study how the optical properties of NFW lenses evolve as a function of redshift. To this end we choose a NFW lens with a mass of \( M_{ref} = 10^7 M_\odot \) and a redshift of \( z_{ref} = 0.2 \) as a reference point. Then, at each redshift in the considered range we find the mass of the NFW lens that minimizes the difference in the deflection angle relative to the reference case.

Results are presented in Figure 4. The black contours mark the regions within which the absolute relative difference is less than 10, 50 and 100 percent; as expected from geometrical arguments, for a given mass, the deflection angle decreases with increasing redshift. Therefore given a certain \((z_{ref}, M_{ref})\), a similar deflection angle may result from lower masses at lower redshift or higher masses at higher redshifts.

The curve that best fits the minimum of the distribution at each redshift (black solid line) marks a clear distinction between combinations that generate a stronger (red) or weaker (blue) deflection, allowing us to separate the lenses in two categories that will become important for the rescaling of the sensitivity function as we will discuss in more details in the next sections.

We find that these results do not depend on the specific choice of \( M_{ref} \) and \( z_{ref} \), with the solid black curve simply rescaling vertically with \( M_{ref} \) and horizontally with \( z_{ref} \).

We also find that our results are not significantly affected by our choice of mass-concentration relation. In particular, we find that significant differences would arise only close to the observer or the source redshift - thus far from the lens and in regimes where the probed volume and hence the number of possible line-of-sight contaminations is very small. In the lower panel of Figure 4 we plot the relative difference in the best fit curve derived using the mass-concentration relations by Meneghetti et al. (2014) and Zhao et al. (2009), computing the latter both for the Cold Dark Matter (CDM) and different Warm Dark Matter (WDM) cases (dashed and dot-dashed lines). In particular, to model the effect of WDM within the Zhao et al. (2009) mass-concentration relation we proceed as follows: (i) we modify the CDM initial power spectrum of our reference cosmology generated by CAMB (Lewis et al. 2000) to the corresponding WDM mass as presented in Bode et al. (2001), (ii) we compute the corresponding mass variances \( \sigma(M) \) (Lacey & Cole 1993; Sheth & Tormen 1999); Despali et al. (2016), (iii) we adopt the Giocoli et al. (2012b) mass accretion history model to recover the time \( t_{0.04} \) at which the main halo progenitor assemble 4% of its mass needed by the Zhang et al. (2009) concentration-mass model. The WDM trend is opposite to the CDM one, because in WDM models the concentration peaks at intermediate masses and decreases both at the high and low mass end, behaving similarly to the WDM power spectrum (Ludlow et al. 2016). The contours in the same figure show the effect of choosing a concentration \( 1\sigma \) or \( 2\sigma \) away from the average concentration given by the Meneghetti et al. (2014) relation. We find differences which are generally within the 10% level, and they become larger only towards \( z = 0 \) where the number of line-of-sight haloes is very small. Hence, it can be concluded that the specific choice of mass-concentration relation is of secondary importance.

### 3.3 Double lens-plane coupling with a simple lens

We now want to quantify the effect of the coupling between two lens planes and how much the results of the previous section and Figure 4 are affected by the main lens properties, such as position relative to the source, ellipticity and the presence of an external shear. In order to do so we quantify the difference in the deflection angle (i.e. equation 10) by taking into account the contribution of the main lens and by considering the recursive lens equation (12). Since this is the most common and better theoretically motivated
Figure 4. **Upper panel**: relative difference in the deflection angles in the \(z-\log(M)\) plane, for NFW haloes at different redshifts. The black dot marks the reference case used for the comparison: the color scale shows the relative difference of all the other combinations with respect to this particular case. The dotted line shows the minimum of the residuals at each redshift and the solid contours enclose the points within 50% difference. **Lower panel**: influence of the concentration-mass relation both in cold and warm dark matter cases. We estimate the scatter that would be induced by using a different concentration-mass relation, comparing the models of Meneghetti et al. (2014) and Zhao et al. (2009). The choice for isolated dark matter haloes, we assume line-of-sight haloes to be described by a NFW profile; at this stage we also model substructures with NFW profiles, while we refer to Section 3.5 for cases in which the two populations are allowed to have a different mass profile. The main lens is located at \(z_l = 0.2\), has a mass of \(10^{13} M_\odot\) and is perfectly aligned with the background source - the first model in Table 1. We model it at first as a singular isothermal sphere and then as a singular isothermal ellipsoid, to explore the effect of ellipticity and external shear with this toy model. For a substructure of given projected position, we look for the line-of-sight halo mass that minimizes equation (10) at each possible redshift and at the same projected position on the main lens plane. Figure 5 shows the mass-redshift relation for different positions of the perturber and different choices of ellipticity and external shear. The black curve shows the best fit curve derived in the previous section, i.e. without the double-lens plane contribution. We find that for a SIS lens with no external shear, the results are consistent with those derived in the previous section at the 5% level and do not significantly depend on the position of the perturber, these results are in agreement with those derived by Li et al. (2016). For a perfectly symmetric case, the non-linear effects arising from a double-lens-plane configuration are therefore not significant. Instead, as we increase the main lens ellipticity and the strength of the external shear we find increasing deviations from the symmetric and the single-lens-plane cases. In particular, the effects are stronger for background line-of-sight objects, as expected from equation (12): high ellipticity and strong shear in the main lens model break the perfect symmetry of the system and thus the main lens deflection angle \(\alpha_1(x)\) - entering the calculation of the background perturber deflection angle - can vary significantly with the position.

### 3.4 Realistic lenses

We now generalize the results of Section 3.3 by considering more realistic lens configurations. Since each lens system among current and future observations has a different combinations of lens and source redshift, we use the following rescaled quantities

\[
y = \log\left(\frac{M}{M_{\text{ref}}}\right) \quad \text{(where } M_{\text{ref}} \text{ is the mass of the substructure in the lens)}
\]

\[
x = \begin{cases} 
\frac{z}{z_l} - 1, & \text{if } (z < z_l) \\
\frac{x - z_l}{x - z_s}, & \text{if } (z > z_l) \\
0, & \text{if } (z = z_l)
\end{cases} \quad (14)
\]

so that \(x = -1\) corresponds to the observer and \(x = 1\) to the source redshift, respectively. As can be seen from Figure 6, this rescaling allows to plot all the redshift combinations in the same parameter space and thus obtain a general mass-redshift relation which reads as

\[
y = 1 + 0.0571 x + 0.0983 x^2 + 0.1582 x^3. \quad (15)
\]

The best fit parameters are obtained by performing a least-squares fit to the data points coming from the whole sample of lenses and positions. The main panel of Figure 6 shows the points corresponding to all the considered positions and

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**Figure 5.** Mass-redshift relation for all the considered variation of our first considered lens system and for a NFW perturber. All the blue points stand for different line-of-sight projected positions in the SIE model, falling exactly on the Einstein radius of the main lens or with a certain offset. Instead, the red and green points show the degeneracy-relation for the case of the NFW profile for different choices of ellipticity and/or external shear in the main lens.
3.5 Comparison between different profiles

Contrary to the previous sections and Li et al. (2016), we now allow the substructures and the line-of-sight haloes to have different mass density profiles. It is well known that isolated dark matter haloes and subhaloes do not have the same profiles, since the latter have been subjected to tidal interactions with the main halo after infall and have been stripped of significant amounts of mass (Hayashi et al. 2003; Giocoli et al. 2008). In particular, PJ profiles are employed to model substructure lensing observations since the truncation radius ensures convergence and thus the possibility to define a total mass. In this section, we model line-of-sight haloes as NFWs and the substructures as PJs and derive a relation that allows one to map the NFW virial mass into the PJ total mass in terms of their relative lensing effect. In particular, by comparing the two profiles via equations (10) and (11), we find that for each PJ subhalo of given truncation radius $r_t$, the most similar lensing effect is given by the NFW - at the same redshift - for which the total enclosed mass within a radius of $r_p \approx 5r_t$ is the same as that of the reference PJ at this radius. As shown in Figure 7, the PJ enclosed mass profile plateau at larger radii and $M(<r)$ approaches $M_{tot}$ at $r_p$, which is marked by the dotted vertical lines for two different cases. This confirms that the total PJ mass is a robust estimator of the subhalo mass. As we will detail in Section 4, this mass mapping is confirmed also by a systematic analysis based on the lens modeling of the mock images. More generally we find that, for each PJ total mass the corresponding NFW virial mass can be calculated from

$$\log(M_{vir}) = 1.07 \times \log(M_{tot}) + 0.38,$$

implying that the NFW virial mass must be roughly one order of magnitude larger. The dashed curves in Figure 7 show the NFW profiles for these corresponding masses, showing how the enclosed mass at $r_p$ is indeed the same for the two profiles. We can now combine equations (15) and (16), using the latter to rescale the zero-point of the former and obtain a more general mass-redshift relation:

$$\log M_{vir}(z) = (1 + 0.0571x + 0.0983x^2 + 0.1582x^3) \times (1.07 \times \log(M_{tot} + 0.38))$$

Taking into account the differences in the mass and density profile definitions is crucial to interpret correctly the line-of-sight contribution, since it may result in a very different number of effective line-of-sight perturbers, as we will show in the next sections. Since Pseudo-Jaffe and NFW profiles are not the only possible choices, especially for subhaloes, we plan to explore this effect in more details in the future.

3.6 Rescaling of the sensitivity function

Our analysis so far does not take into account for observational related quantities (e.g. the signal-to-noise ratio and the effect of the point-spread-function), nor for the degeneracy among the parameters of the main lens and of the perturber, neither for the fact that in a real situation the background source is unknown and its structure, which has to be inferred from the data, can readjust to partly absorb the effects of the perturbers (see Section 4). As such, our findings do not strictly quantify the mass-redshift degeneracy between line-of-sight haloes and substructures. However, given the lowest detectable substructure mass (derived in a way that takes into account all the above issues, see Vegetti et al. 2014), they provide an heuristic method to rescale the sensitivity function (i.e. the smallest detectable mass as...
3.7 Integrating the line-of-sight mass function

It is generally assumed that the line-of-sight-halo contribution dominates with respect to the one given by substructures, more evidently in the CDM than in the WDM models. This is obviously true if one could detect the whole halo population between $z = 0$ and $z_s$. However, given a certain substructure mass that we aim to detect or a series of observations with a certain $M_{\text{low}}(z = z_l)$ sensitivity, we need to take into account only those line-of-sight structures that would produce a detectable effect, that is, an effect equal or larger to that of $M_{\text{low}}(z = z_l)$. In practice, the lower integration limits $M_{\text{low}}(z)$ of the halo mass function depends on the redshift, as pointed out in (Li et al. 2016) and is set by equation (15) or (17) for any chosen subhalo mass.

Corresponds to taking into account only the line-of-sight structures in the red region of Figure 4 and it has a different quantitative effect on the CDM and WDM mass function, as we show in the following paragraphs. To calculate the effective number of line-of-sight haloes, we integrate the CDM halo mass function as

$$N_{\text{halos}} = \int_0^{z_s} \int_{M_{\text{low}}(z)}^{M_{\text{max}}} n(m,z)dm dz,$$

where we use the Sheth & Tormen (1999) halo mass function. As discussed above the upper integration limit $M_{\text{max}}$ is set equal to $10^{13} M_\odot$. However, we find that increasing $M_{\text{max}}$ does not significantly change our results, due to the
exponential cutoff at the high mass regime of the halo mass function. Since in WDM models the initial power spectrum is suppressed below a certain scale, the WDM halo mass function can be derived from the CDM one using this relation

\[ n(M)_{WDM} = \left(1 + \frac{M_h}{M}\right)^\beta n(M)_{CDM}, \quad (19) \]

where \(M_h\) is the mass corresponding to the cut-off in the power spectrum. For a 3.5 keV sterile neutrino model, we have \(M_h = 1.3 \times 10^8 M_\odot\) and \(\beta = -1.3\). The same relation holds for the subhalo mass function.

In Figure 9 we show the result of the integration, i.e. the total projected number of line-of-sight halos per arcsec^2, for two different choices of lens and source redshifts - which correspond to the lowest and highest \(z_l\) in our sample - together with the corresponding subhalo mass functions, in order to allow for a direct comparison. First, the top panels show the projected number of line-of-sight halos as a function of redshift for different values of \(M_{low}(z = z_l)\) (lower integration limit) and under the assumption that both subhalos and line-of-sight halos have the same profile (in this case an NFW profile). We find that, increasing the lowest detectable substructure mass \(M_{low}\) produces a drastic cut in the number of observable line-of-sight halos, especially for the CDM case. Moreover, the number of line-of-sight halos decreases both towards the observer and the source, due to the double cone geometry of the considered volume. Second, the bottom panels show the mass function integrated in redshift. We see that the cut imposed by different \(M_{low}\) has a larger impact on the number of background than foreground line-of-sight halos, since the corresponding mass cut for line-of-sight halos increases rapidly in the background. It also causes more dramatic changes in the CDM mass function than in the WDM one - since for the latter the number of low mass structures is already reduced. The lower integration limit \(M_{low}(z)\) is calculated from equation (15); if, instead of using this median relation for each lens, one would choose the relation derived for each specific case (as the points in different colors in Figure 6, the difference in the results would be within the 4% level.

Finally, in the last two panels we show what happens when the subhaloes have a PJ profile. In this case, we use equation 17 to derive the corresponding mass to for the NFW profile in any \(z\). Since the equivalent NFW masses are on average one order of magnitude larger, this has strong effects on the line-of-sight mass function and proves to play a fundamental role in the comparison. The importance of the mass definition is clear also from the comparison between the Figures 10 and 11, where we show the ratios between the projected number of line-of-sight halos and subhaloes (per arcsec^2) in the whole \(z_1\)-\(z_2\) range, generalizing the results of Table 2.

Taking the top right panel of Figure 11 as an example, here we integrate the subhalo mass function from \(10^8 M_\odot\) but for the line-of-sight halos we have \(M_{low}(z = z_l) = 8 \times 10^8 M_\odot\). Given that the halo mass function rapidly decreases with mass, the difference in the mass limit can have an important effect especially in the CDM models.

4 MASS-REDSHIFT DEGENERACY

In this section, we focus on the mass-redshift degeneracy between line-of-sight haloes and substructures. We are interested in deriving a statistical interpretation for the detections in the context of these two populations. In particular we want to quantify the probability that a detection, defined in terms of a substructure of measured mass, arises instead from of a line-of-sight halo with different mass and redshift. Our aim is also to determine under which observational configurations the non-linear effects arising from the double lens plane are such that this degeneracy can be broken or alleviated. To this end, we use the lens modeling code by Veggetti & Koopmans (2009) to model the realistic lens systems presented in Section 3. For each modelled system the free parameters of the model are the main lens geometrical parameters (Einsteins radius, position, mass density flattening, position angle and slope, and the external shear strength and position angle), the background source surface brightness distribution and regularization, and the perturber mass, projected position and redshift. As in Section 3, line-of-sight haloes have a NFW profile, while substructures can either be PJ or NFW.

In order to test the validity of the results from the previous sections, we create mock images with a main lens and a PJ subhalo and then we model the perturber as a NFW line-of-sight-halo, or the other way around. We then check that the derived parameters produce a consistent lens and source model and that the residuals with respect to the original mock image are small.

In Figure 12, we show an example of the parameter posterior probability distributions. We take the HST lens 2, where the mock image is created adding a PJ \(10^8 M_\odot\) subhalo at the coordinates \((x, y) = (0, 1.15)\), and we model it imposing that the perturber is: (i) a PJ subhalo (blue contours), (ii) a NFW subhalo (gray contours), (iii) a NFW line-of-sight halo - thus optimizing also for its redshift (red contours). The last three rows of the Figure show the results relative to the mass and projected position of the perturber. We see that all the models recover the true perturber position quite well; the true PJ mass is recovered for case (i), while cases (ii) and (iii) correspond to a higher mass, in agreement with what expected from equation (16). The redshift-mass degeneracy is shown in the small inset: the redshift of the lens and the NFW virial mass expected from equation (16) are marked by the dotted lines. We see that there is effectively a degeneracy between mass and redshift, but it has a more complicated shape than what is found by comparing the deflection angles (Figure 6). Moreover, the uncertainty in redshift is \(\Delta z \approx 0.15\) at a 3\(\sigma\) level and it does not span the whole redshift space between the observer and the source, meaning that not all the configurations given by equations (15) and (17) are effectively equivalent. Using the surface brightness and modelling the lens and the source at the same time adds an additional level of information, with respect to the deflection angles alone, allowing to restrict the degeneracy range. Nevertheless, if we force a particular \(z \neq z_l\) for the NFW perturber, the degeneracy curve from equation (15) still approximates quite well the recovered mass. In the opposite case, where a NFW line-of-sight halo placed at a certain redshift along the line-of-sight is
Figure 9. Projected number of line-of-sight haloes for three lenses. In each panel we consider a mass $m_{\text{low}} = (10^6, 10^7, 10^8) \, M_{\odot}$ that corresponds to the minimum subhalo mass that can be detected. Using the degeneracy curve in Figure 7, we exclude from the line-of-sight mass function all the structures that cannot be detected: the resulting effective perturber mass functions are shown in all panels by the dashed and dotted lines, while the solid line shows the uncorrected one. The black and red lines stand respectively for the CDM and WDM mass functions. In the top and middle panels we assume that subhaloes and line-of-sight have the same profile (NFW), while in the bottom panels we treat subhaloes as PJ profiles and line-of-sight haloes as NFW - we use equation 16 to rescale between the two masses. In the top panels we show the total number of projected line-of-sight haloes as a function of redshift, in order to highlight the redshift regimes in which the mass function is more affected, i.e. moving away from $z_l$ in both directions. The redshift of the lens is marked by the vertical gray line. The middle and bottom panels show instead the integrated line-of-sight mass function, together with the subhalo mass function (blue points for CDM and yellow for WDM). Comparing the two rows of plots we can notice the effect of assuming different profiles for the two populations: following equation 16, for any PJ subhalo the equivalent NFW is roughly one order of magnitude more massive, imposing a sharper cut in the line of sight mass function.
Number density expectations - dark matter only

<table>
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<tr>
<th>$z_l$</th>
<th>$z_s$</th>
<th>$M_{\text{low}}$</th>
<th>$n_{\text{sub}}$ (CDM)</th>
<th>$n_{\text{los}}$ (CDM)</th>
<th>$n_{\text{los}}$ (CDM)-PJ</th>
<th>$n_{\text{sub}}$ (WDM)</th>
<th>$n_{\text{los}}$ (WDM)</th>
<th>$n_{\text{los}}$ (WDM)-PJ</th>
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</thead>
<tbody>
<tr>
<td>0.2</td>
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<td>$10^6$</td>
<td>0.758</td>
<td>6.112</td>
<td>1.620</td>
<td>0.0033</td>
<td>0.214</td>
<td>0.185</td>
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<td></td>
<td></td>
<td>$10^7$</td>
<td>0.0952</td>
<td>0.905</td>
<td>0.161</td>
<td>0.0025</td>
<td>0.165</td>
<td>0.086</td>
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<td></td>
<td></td>
<td>$10^8$</td>
<td>0.0118</td>
<td>0.123</td>
<td>0.017</td>
<td>0.0012</td>
<td>0.074</td>
<td>0.015</td>
</tr>
<tr>
<td>0.58</td>
<td>2.403</td>
<td>$10^6$</td>
<td>0.758</td>
<td>4.173</td>
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<td>2.309</td>
<td>1.240</td>
</tr>
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<td>0.445</td>
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</tbody>
</table>

Table 2. Expected projected number density of subhaloes and line-of-sight haloes (per arcsec$^{-2}$). We count all the (sub)haloes more massive than $M_{\text{low}}$: the lower detectable subhalo mass is listed in the third column, while the corresponding value for the line-of-sight halo is calculated from equation 17. We calculate the number of line of sight haloes both for the case in which the two profiles are the same (columns 5 and 8) and for the case in which they differ (columns 6 and 9). The first table shows the results for the dark-matter-only subhalo mass function, while the lower one includes the effect of baryons, which further reduce the number of subhaloes. As a WDM model, we choose the 3.3 keV sterile neutrino, described in Lovell et al. (2016); we apply the same WDM correction to the dark-matter-only and hydro mass functions, since the effects due to baryons are expected to be very similar on CDM and sterile neutrinos (Lovell et al. 2016). A generalized version of these results, spanning a wide range of both source and lens redshift, is shown in Figure 10.

Figure 10. Excess of projected line-of-sight structures with respect to the number of substructures in the lens (per arcsec$^{-2}$). The colour scale represents the excess (following the color bar of each panel), for each combination of lens (x-axis) and source (y-axis) redshift. The left column shows the results for the CDM case, while the right one for the WDM one; we chose $M_{\text{low}} = 10^8, 10^6 M_\odot$ and we applied the redshift dependent cut from equation 15 in order to calculate $M_{\text{low}}(z)$ for the line of sight haloes. The location in the $(z_l, z_s)$ plane of the lenses considered in this work are marked by the black circles.
Figure 11. Same as Figure 10, but allowing for different density profiles for substructures and line-of-sight haloes. In this case $M_{\text{low}} = M_{NJ} = 10^8$, $10^6 M_\odot$ is the total mass of the Pseudo-Jaffe profile which describes the subhalo. The correspondent $M_{\text{low}}(z)$ for the line-of-sight mass function is calculated using equation 17. For the NFW haloes we have $M_{\text{low}}(z = z_l) = 8 \times 10^8$, $5 \times 10^6 M_\odot$ respectively for the two cases. As we can see comparing these results with those from Figure 10, the mass definition plays an important role.

modeled as a PJ subhalo in the lens, gives complementary results.

The width and exact shape of the contours also depend on the particular properties of each lens, and in particular on the complexity of the source and image surface brightness distribution. In order to prove this point, in Figure 13 we show the parameter posterior probability distributions for the reference case of the SIS lens at $z = 0.2$; also in this case, a $10^9 M_\odot$ PJ subhalo has been added to the lens model and it is modeled as a NFW line-of-sight halo, allowing both mass and redshift to change; we show only the contours relative to the perturber mass, position and redshift. We see that in this case mass and redshift are completely degenerate and that, even if the true position is recovered quite well by the peak of the distribution, the uncertainties are large, spanning almost half of the image plane within $3\sigma$. This is due to the fact that the Einstein ring in perfectly smooth and symmetrical; the width and the rounder shape of the contours also explains why for this lens results coming from different positions of the perturber are equivalent (Figure 5). In general, uncertainty on mass and redshift depends on the chosen position of the perturber and in particular the inferred quantities may be less precise for perturbers located where the surface brightness or its gradient is lower: in Figure 14 we show the constrains derived by inserting a $10^9 M_\odot$ subhalo in 2 different positions, where position 2 is the one from the previous Figures while position 1 is located at $(x, y) = (0.8, 0.8)$ where the surface brightness and thus the sensitivity to substructures is lower. Finally, Figure 15 shows the probability contours for different subhalo masses, all located in the same point, for the system based on the SLACS lens J0946+1006. The sensitivity function in the chosen pixel sets the minimum detectable mass to $4 \times 10^8 M_\odot$: we see how the contours are larger when the inserted PJ subhalo is only slightly more massive than this limit ($5 \times 10^8 M_\odot$ - grey contours), but shrink for higher mass values, becoming more and more precise.

Even though we ran our lensing code on mock images for all lenses, we only show a representative subset of contour plots. Nevertheless, Figure 16 summarizes our results showing how well the lensing code recovers the PJ input mass (top panel) and how precise is the estimate of NFW mass (middle panel) and redshift (bottom panel) when both parameters are let free to vary. We conclude that the redshift is well recovered mostly within $1\sigma$ and that the corresponding NFW virial mass is consistent with the expectations, even thought its exact value depends on the exact best fit redshift and on the image resolution. Nevertheless, we are able to restrict the degeneracy region: not all the combinations of mass and redshift that are mapped as equivalent in Figure 6 are effectively indistinguishable. On average the redshift uncertainty is $\Delta z \simeq 0.15$, thus restricting the number of possible combinations calculated from equation (15).

More in details, from this analysis we have found that: (i) when a PJ subhalo is modelled as such, we recover its mass and projected position with a precision of 0.2 dex.
and 0.1" respectively, similarly for a NFW subhalo - note that 0.1" means an uncertainty of ≃ 2-3 pixel depending on the sample; (ii) \( \Delta M, \Delta r = (0.4 \text{dex}, 0.1") \) when a PJ subhalo is modelled as a NFW subhalo - here \( \Delta M \) in intended with respect to the NFW mass expected from equation 16; (iii) when a NFW subhalo is modelled as a PJ subhalo, the results are consistent with the previous case, with a reversed ordering in mass; (iv) when a PJ subhalo is modelled as a NFW line-of-sight halo, meaning that both mass and redshift of the perturber are let free to vary, we get \( \Delta M, \Delta z, \Delta r = (0.6 \text{dex}, 0.1, 0.2") \) - here the error on the position is calculated by taking into account the two planes rescaling and the \( \beta \) factor; (v) when a NFW line-of-sight halo is modelled as a PJ subhalo we recover the expected PJ mass within 0.2 dex.

We stress that these are the largest uncertainties that we have found for realistic lenses (thus excluding the SIS+gaussian source case), but as discussed above they decrease with increasing data complexity and are larger for idealized smooth and symmetric cases. They also decrease with increasing mass of the perturber, as seen in Figure 15, since the surface brightness distribution becomes more and more different from the smooth one. In all cases, we find that the main lens parameters and source regularization adapt themselves to partly accommodate the presence of the perturber and when necessary for the wrong choice of perturber mass profile. These changes are at the 3 percent at most. With the shear strength and the source regularization being the more sensitive parameters.

Figures 16 and 17 summarize the results of this analysis, showing the recovered masses (PJ, NFW at the redshift of the lens and NFW along the line of sight) and redshifts for the mock images of all our cases, where the perturber inserted in the lens system is in all cases a PJ subhalo with a mass of \( 10^9 M_\odot \).

5 CONCLUSIONS

In this paper we have studied the relative gravitational effect of substructures within lens galaxies and haloes along their line-of-sight on the surface brightness distribution of lensed arcs and Einstein rings. The main goal was to quantify the relative contribution to the total number of detectable objects, as well as to provide an interpretation of detections in terms of these two populations.

Using a set of idealised and realistic lensing observations we have derived an analytic mass-redshift relation that allows us to rescale the substructure detection threshold (i.e. the smallest detectable substructure mass) into a line-of-sight detection threshold as a function of redshift. For line-of-sight haloes in the foreground with masses much smaller than the mass of the main lens, non-linear effects arising from the double-lens-plane configuration are essentially negligible, and the above expression provides therefore a precise way to quantify the amount of detectable objects. For line-of-sight haloes in the background this relation is strictly only valid in an average sense, instead. In particular, we find that departures from the average relations increase with increasing asymmetries in the lensing systems, either due to ellipticity in the main lens mass distribution or the presence of strong external shear. This translates into a small under-estimation on the total number of detectable background line-of-sight haloes per arc second squared of \( \lesssim 4\% \).

For a given substructure mass detection threshold we have then used the respective sensitivity functions to quantify the line-of-sight contribution to the total number of small mass haloes that can be detected. We use the mass-redshift relations of equations (15) and (17) to rescale the lowest detectable mass as a function of redshift and then integrate the line-of-sight halo mass function. We found that varying the lowest integration limit \( M_{\text{low}} \) allows us to take into account only the line-of-sight structures that would have a significant effect and strongly reduces the number of detectable objects. In particular, we found that WDM predictions are less affected by the sensitivity function as often the cut happens at masses below or close to the regime where the mass function is already suppressed.

Moreover, we highlighted the role of the density profile and the importance of comparing subhalo and line-of-sight masses meaningfully. As Pseudo-Jaffe profiles are commonly used to described subhaloes, the total masses of observationally detected substructures are calculated under this assumption. These truncated profiles are steeper in the center than NFW profiles, which are the best candidate to describe isolated and dark line of sight haloes. Thus, it is important to take into account that a NFW halo that is more massive than a certain PJ subhalo in the lens is needed to produce a similar effect, as shown by equation 16. When the difference between these two profiles is taken into account, the number of NFW line-of-sight haloes that can effectively contribute is further reduced.

The other main goal of this paper was to quantify the degeneracy between the redshift and the mass of detected objects. While the results from Section 3 have a statistically relevance, in Section 4 we wanted to determine what we can say about a perturber mass and redshift, once a detection has been made. In order to do so, we used the lens modelling code from Veggetti & Koopmans (2009) to analyse mock observations in which a perturber - that may be either a subhalo or a line-of-sight halo - had been artificially inserted and modelled either as a subhalo or a line-of-sight halo. This analysis confirmed the mass conversion given by equation (16) and the presence of a degeneracy between mass and redshift. However, while equations (15) and (17) and Figure 6 suggested that at each redshift between the observer and the source all masses following the mass-redshift relation would have the same lensing effect and therefore would be indistinguishable, we found instead that the mass-redshift degeneracy is restricted to a smaller redshift range. In particular, from the modelling of realistic lens systems we find that for all combinations of mass-density profiles, we are able to recover the input perturber redshift with an absolute error of at most \( \Delta z \simeq 0.15 \). This is due to the fact that the data surface brightness distribution (which is the input to the lens modelling) contains more information than the deflection angle and allows therefore for a more precise measurement. From this analysis we also find that we are able to recover the perturber true mass with a relative error of at most 0.6 dex. The perturber projected position is also recovered within a few (typically 2-3) pixels. We find that the precision on the redshift, mass and position measurements increases with increasing surface brightness complexity in the data. In particular, for a given system the errors are
Figure 12. Example of parameter posterior probability distributions. We show the result for the HST lens 2, where the mock image is created adding a $10^9 M_\odot$ subhalo. The colored contours show 1, 2 and 3σ levels for three different modelling choices, where we impose that the perturber is (i) a PJ subhalo (blue), (ii) a NFW subhalo (gray), (iii) a NFW line of sight halo - thus optimizing also for its redshift (red). The redshift-degeneracy for this last case is shown in the small inset.

smaller for perturbers located where the surface brightness complexity is larger. The same trend is seen between different lens systems. In particular, we found that for a very regular source surface brightness and perfectly symmetric lens, the degeneracy is stronger, while we are able to reduce it for more complex systems.

To conclude, the contribution from small-mass haloes along the line-of-sight is important for three reasons: (1) as the lensing effect depends on the redshift of the perturber, small line-of-sight haloes that are located at a lower redshift than the lens produce larger perturbations of the lensed images than substructures of the same mass inside the lens-galaxy halo, meaning that the detection threshold is effectively lower for foreground objects; (2) depending on the redshift of the lens and the smallest detectable mass, the number of detectable line-of-sight haloes can be larger or equal to the number of detectable subhaloes. The line-of-sight population represents therefore an important contribu-
Figure 13. Posterior probability distributions for the SIS lens. We again insert a $10^9 M_\odot$ subhalo when creating the mock image and we model it as a NFW line of sight halo. Here we show only the probability contours relative to the perturber mass, position and redshift. The variations in the main lens parameters with respect to the unperturbed model are very small, due to the particularly symmetric configuration of the system. The true subhalo position is $(x, y) = (1.51, 2.11)$ at $z = 0.2$.

Figure 14. Posterior probability distributions for two different perturber positions, again for the HST lens 2: the blue contours correspond to those from Figure 12, while the red ones are for a different position at $(x, y) = (0.8, 0.8)$.

Figure 15. Posterior probability distributions for the SLACS J0946 lens (position 1). We see how the contours restrict and become more precise as we increase the mass; the offset of the recovered position from the true one $(x_t, y_t)$ is of the order of 2 pixels in the worst case. The minimum detectable mass in this pixel, according to the sensitivity function, is $4 \times 10^8 M_\odot$.

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Figure 16. Summary of the posterior probability distributions results for all the lenses, for the modeling of the mock images containing a $10^9M_\odot$ PJ subhalo as such or as a NFW at $z = z_L$. The black dots show the mean value of the posterior distribution and the black(gray) error bars the 1(2)σ uncertainties. The red empty squares show the peak of the distribution for the same case, which does not take into account all the degeneracies.

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