Ellipsoidal Collapse and the environment of low-mass halos

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The spherical top hat

$$\frac{d^2R}{dt^2} = -\frac{GM}{R^2} = -\frac{4\pi G}{3}\overline{\rho}(1+\overline{\delta})R$$

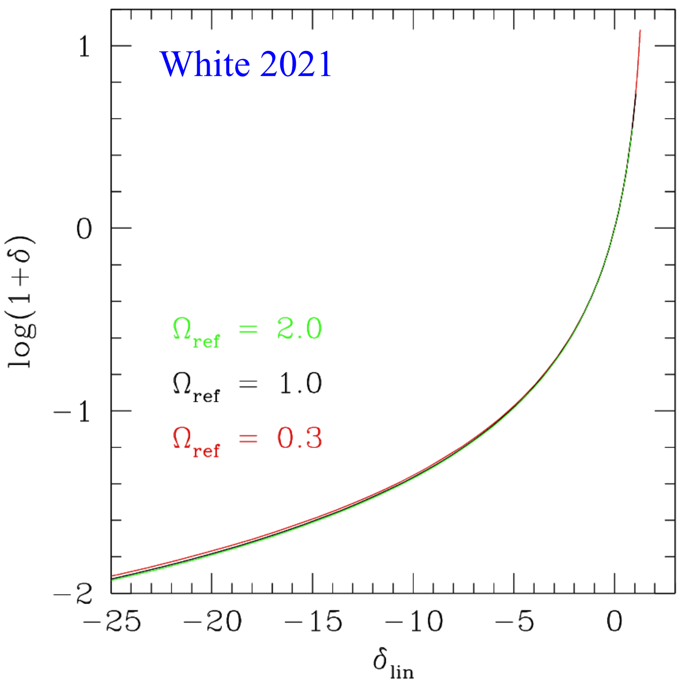
A spherically symmetric perturbation evolves like a separate universe

$$R/R_m = \frac{1}{2}(1-\cos\eta); \quad t/t_m = (\eta - \sin\eta)/\pi$$

Overdense regions collapse in finite time..

$$\delta_{lin,\,coll} = \overline{\delta}(2t_m) = \frac{3}{20}(12\pi)^{2/3} = 1.686$$

..when their extrapolated <u>linear</u> overdensity is 1.69 (almost) independent of Ω

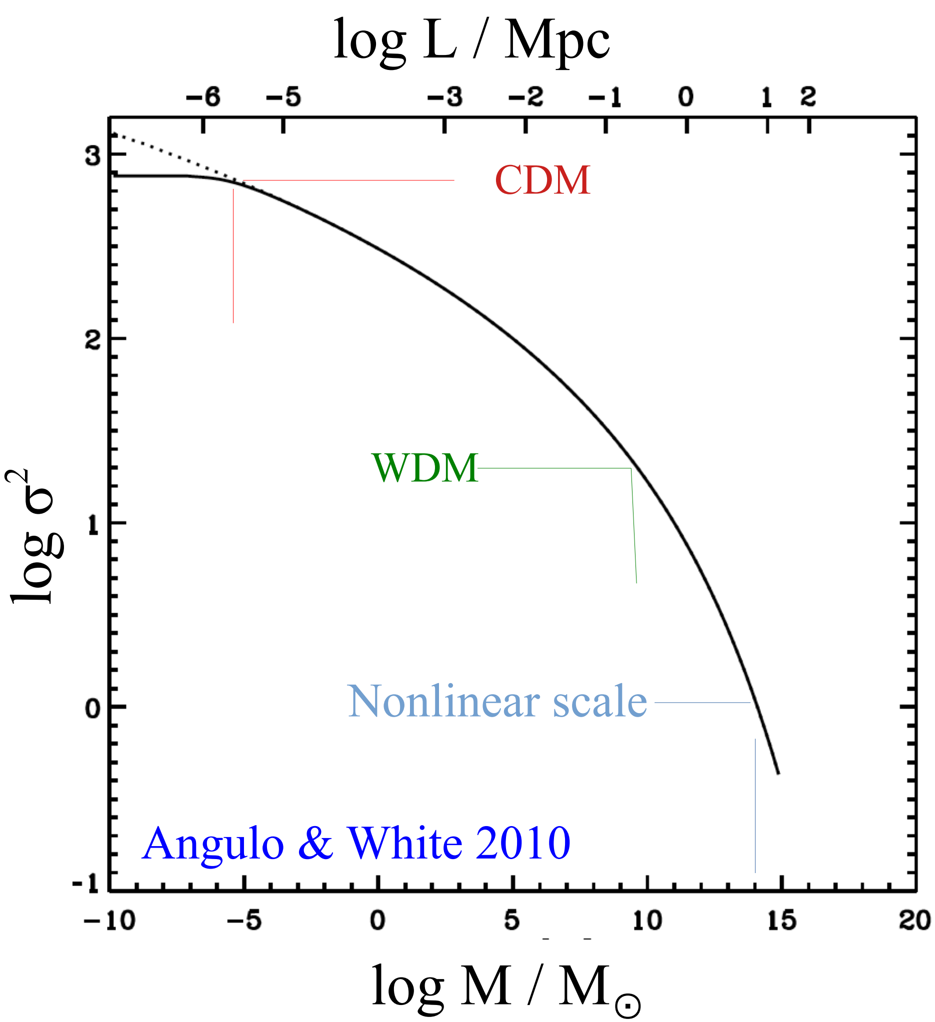


Until the moment of collapse, the nonlinear density, $\rho / \bar{\rho}$, of a spherical over- or underdensity is approximately a **cosmology-independent** function of its extrapolated linear overdensity

Variance of density fluctuations as a function of scale

Smoothing the ΛCDM linear density field with a sharp k-space cutoff, $k_{s,}$ gives an extrapolated field at time t,

$$\delta_s(\mathbf{x}, t; k_s) = \int_{|\mathbf{k}|<k_s} \delta_{\mathbf{k}}(t)\, e^{-i\mathbf{k}.\mathbf{x}}\, d^3k,$$



with variance

$$\sigma_s^2(k_s) = \int_{|\mathbf{k}|<k_s} P_0(k)\, d^3k \le \sigma_t^2,$$

so the smoothed density has pdf

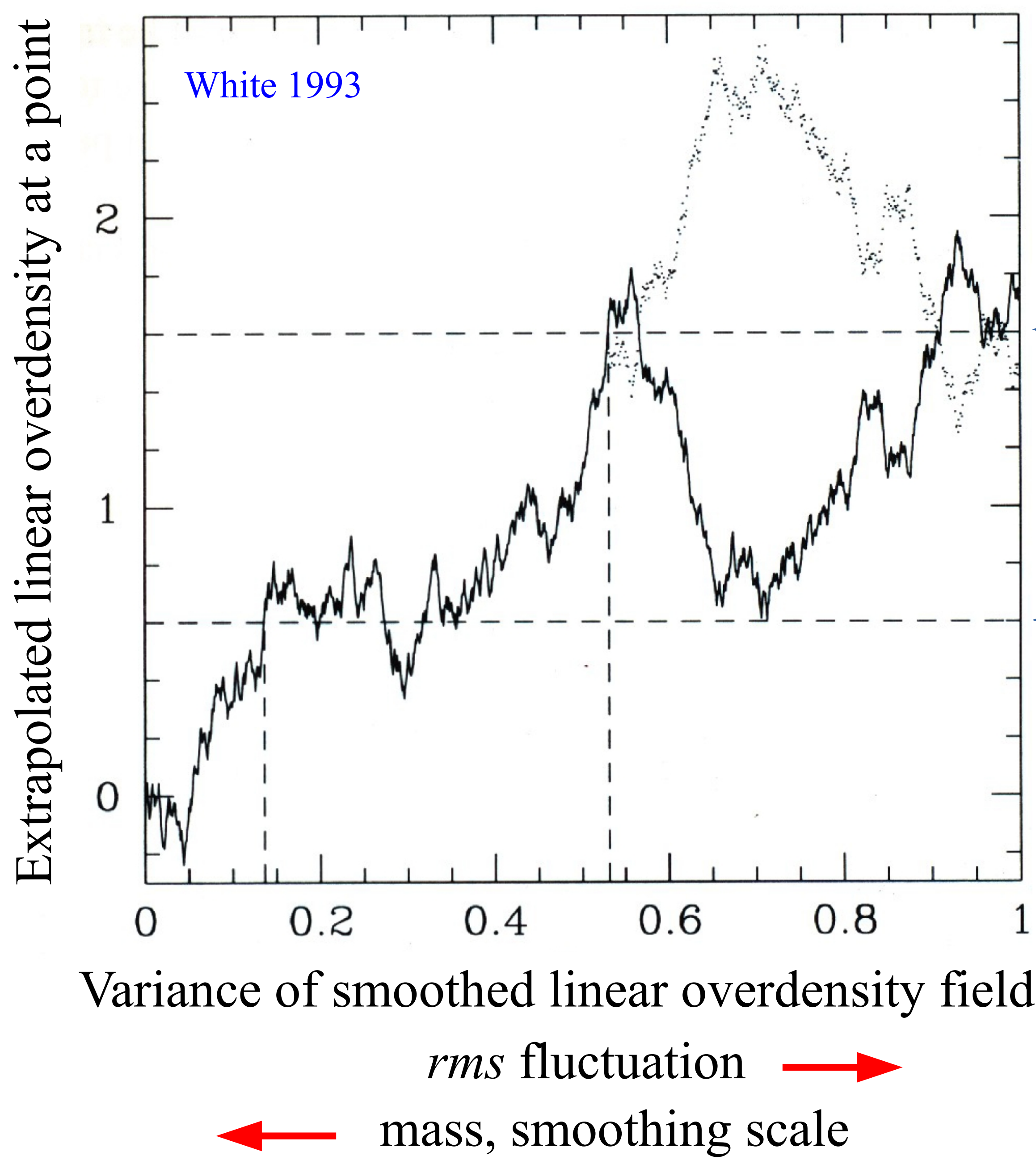
$$f(\delta_s)d\delta_s = \frac{d\delta_s}{\sqrt{2\pi}\sigma_s} \exp\left(-\delta_s^2/2\sigma_s^2\right).$$

with corresponding mass/length scales

$$L_s(k_s) = \pi/k_s; \quad M_s(k_s) = 6\pi^2\rho_0 k_s^{-3},$$

As k_s ↗, σ_s ↗, M_s ↘, L_s ↘ and $\delta_s(\mathbf{x}; k_s)$ executes a *Markov* random walk

1-D excursion set model for structure formation



Markov random walk

Threshold crossing sets the halo mass of the element

Smoothed overdensity on larger scale sets environment density on that scale

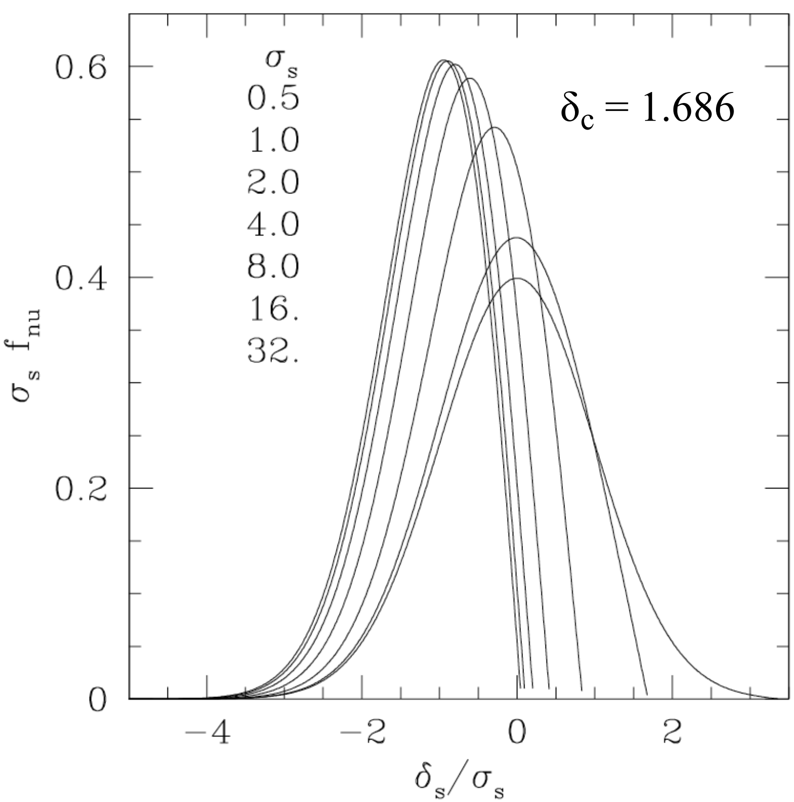
This model determines the statistics of halo properties directly from the (gaussian) initial density field.

The density distribution of uncollapsed regions

For such a random walk the δ_s distribution of mass elements yet to cross the barrier by scale k_s is

$$f_{\rm nu}(\delta_s)d\delta_s = \frac{d\delta_s}{\sqrt{2\pi}\sigma_s}\left[\exp\frac{-\delta_s^2}{2\sigma_s^2} - \exp\frac{-(2\delta_c-\delta_s)^2}{2\sigma_s^2}\right], \quad \delta_s < \delta_c,$$

so the distribution of δ_s/σ_s has the single parameter δ_c/σ_s, and integration over δ_s gives the fraction of all mass which is not part of any halo more massive than M_s



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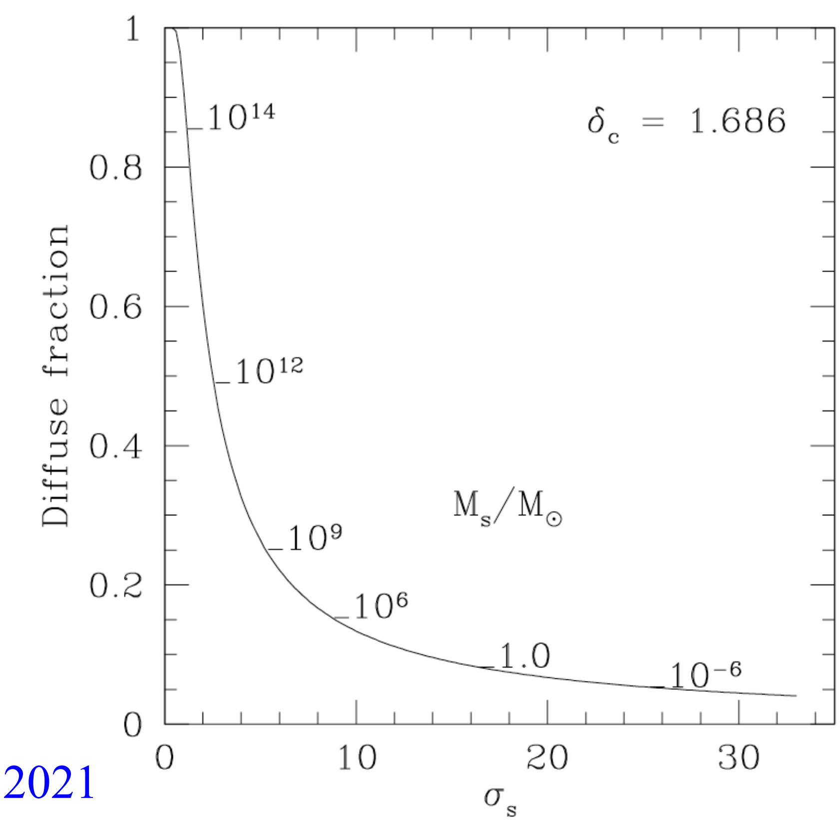
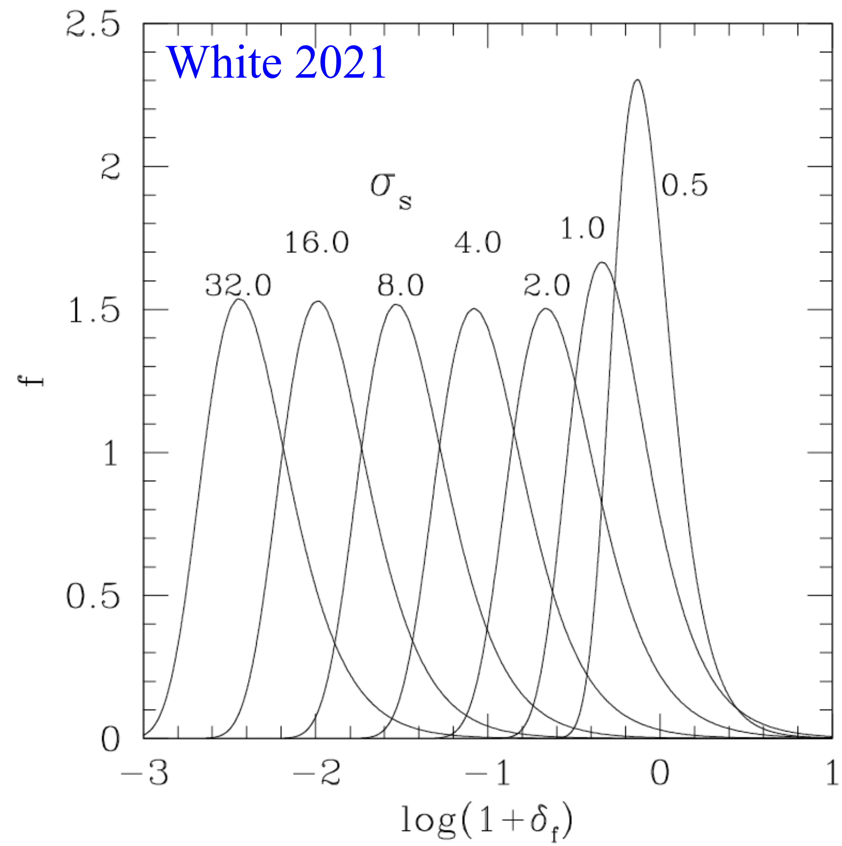
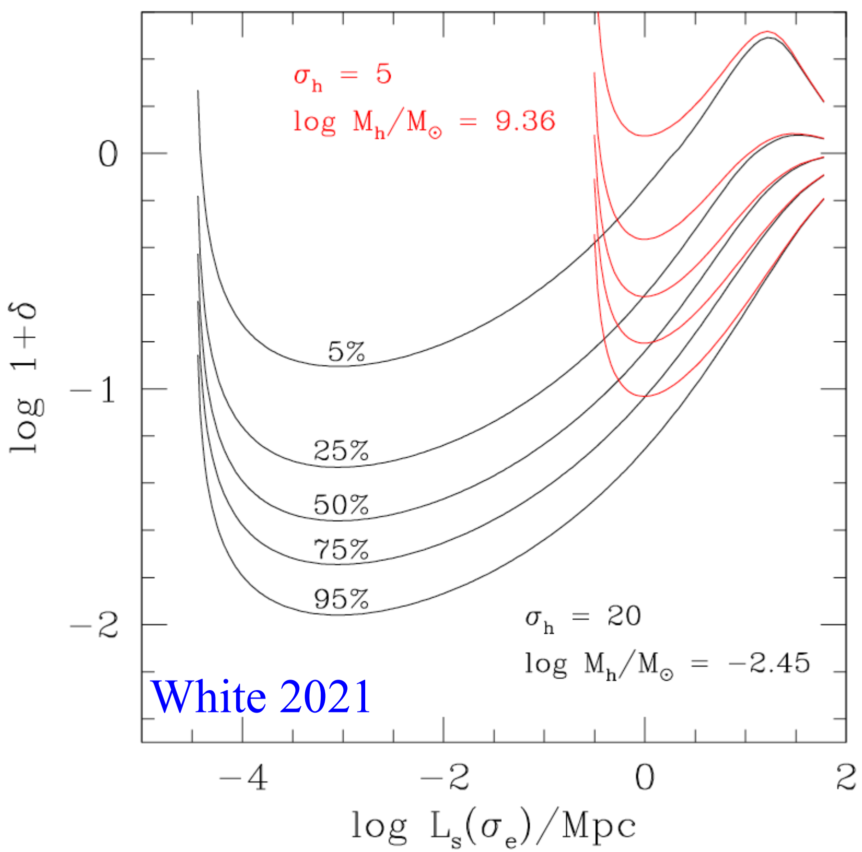


The nonlinear density of uncollapsed regions

The spherical top hat model allows $f(\delta_s)$ to be converted to $f(1+\delta_f = \rho/\bar{\rho})$, giving the z=0 density distribution of regions of Lagrangian scale $L_s(\sigma_s)$





Using P(k) for ΛCDM, this allows, for example, the calculation of the distribution of mean density in regions of scale $L_s(\sigma_e)$ surrounding halos of mass $M_s(\sigma_h)$

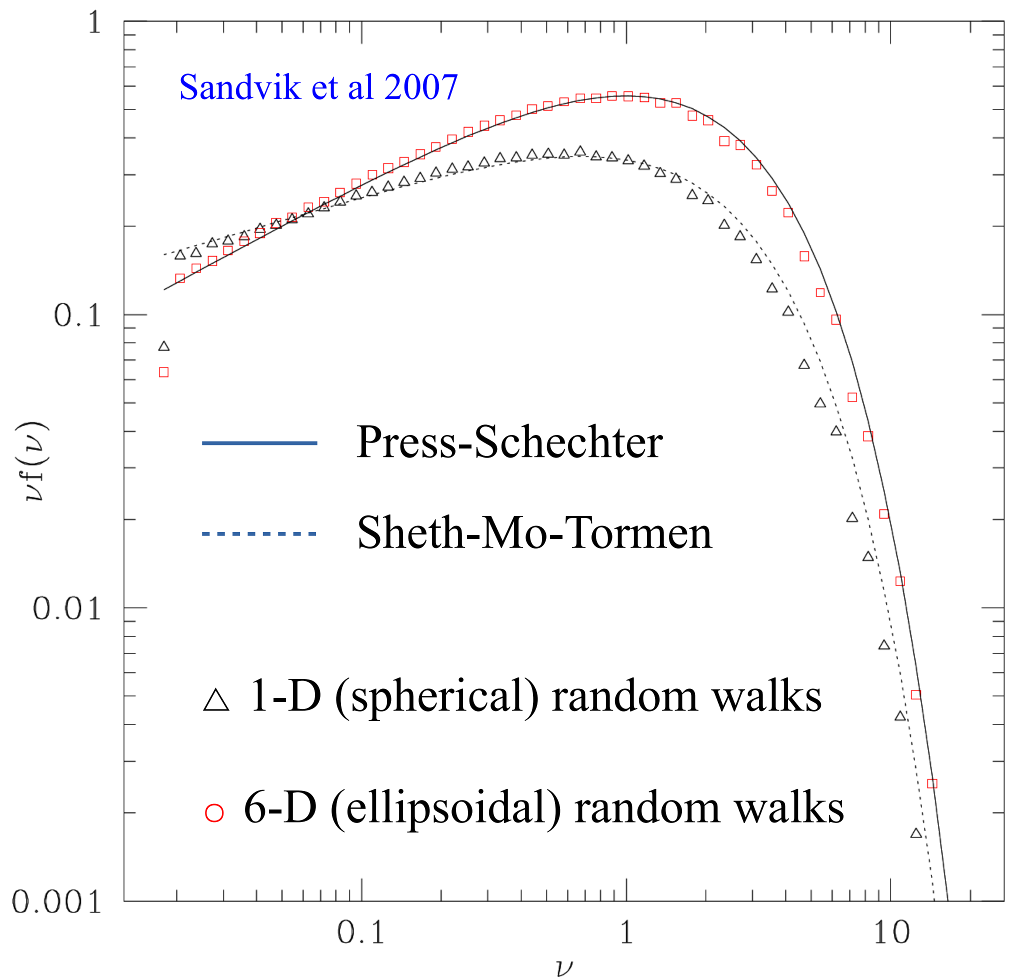
For low-mass halos, these densities are $\ll \bar{\rho}$

Limitations of the spherical excursion set model

- Both theory (e.g. the Zeldovich approximation) and simulation show the early evolution of mass elements to be highly **an**isotropic
- Collapse typically occurs first to a sheet and then into filaments and then finally halos
- The threshold for halo formation thus depends on local anisotropy as well as on linear overdensity
- A **non**spherical model is needed to link the nonlinear density of uncollapsed regions to their linearly extrapolated overdensity
- Regions which have not collapsed into a halo may nevertheless already have collapsed into a pancake and so no longer be diffuse

→ A triaxial generalisation of the top hat model

Excursion set mass functions



1-D excursion set theory leads to the Press-Schechter (1974) mass function

SMT (2001) show (approximate) ellipsoidal collapse fits simulations better

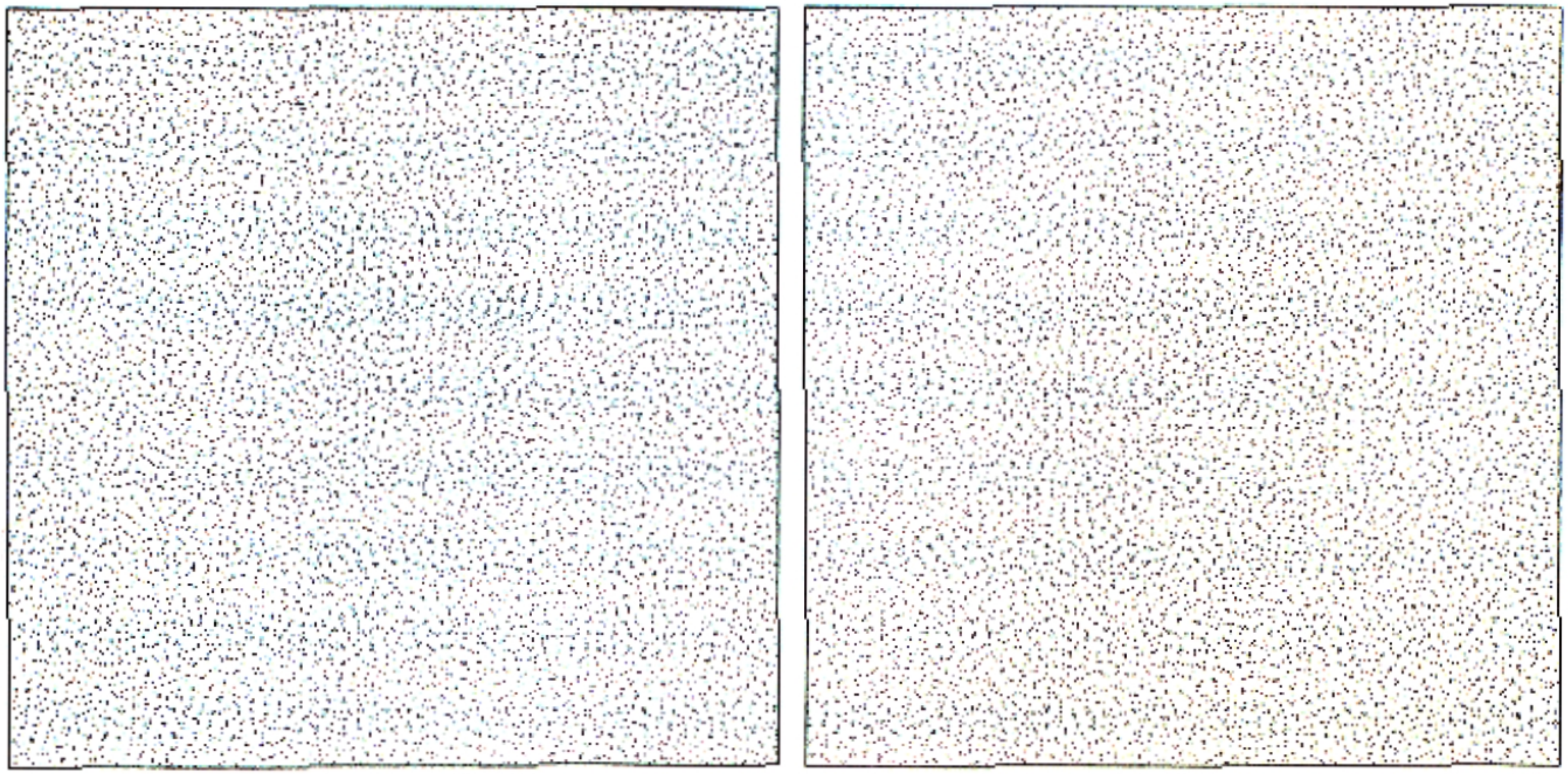
SMLW (2007) reproduce this with 6-D random walks + ellipsoidal collapse

The ellipsoidal top hat

$$a_1 : a_2 : a_3 = 1 : 1.25 : 1.5$$

$$\delta_{lin} = 0.1$$

White 1993

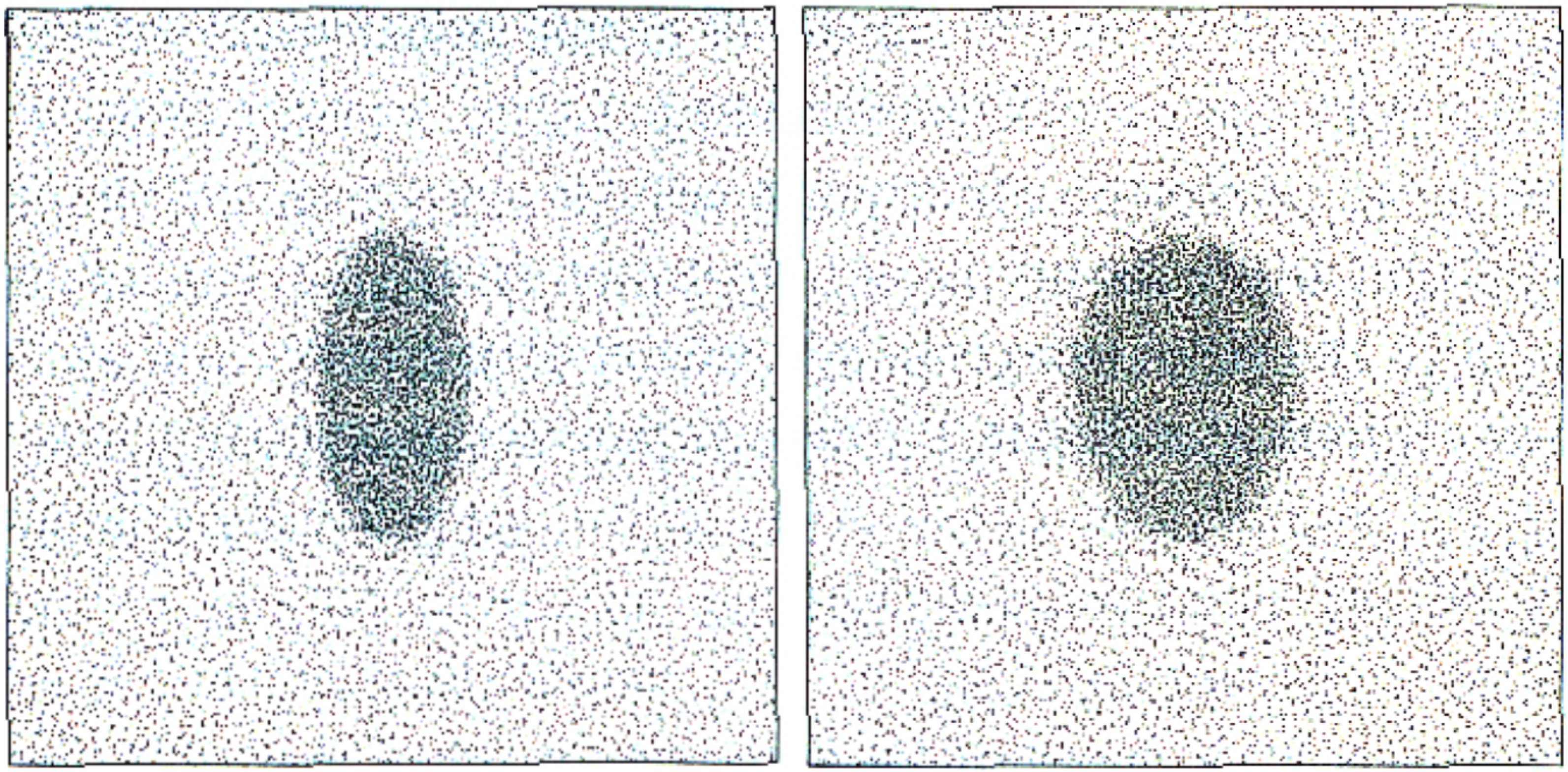


The ellipsoidal top hat

$$a_1 : a_2 : a_3 = 1 : 1.25 : 1.5$$

$$\delta_{lin} = 1.0$$

White 1993

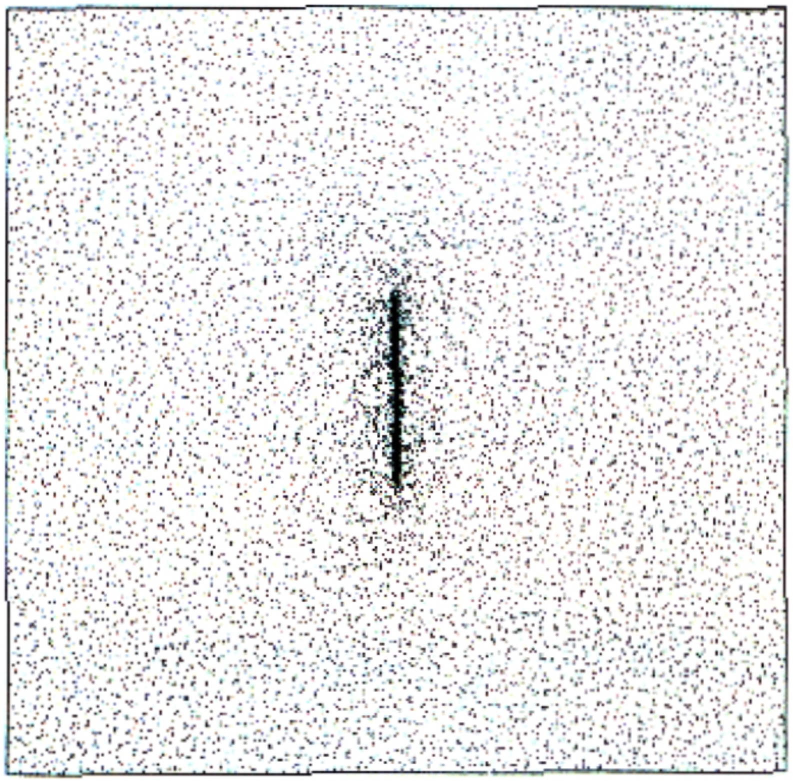


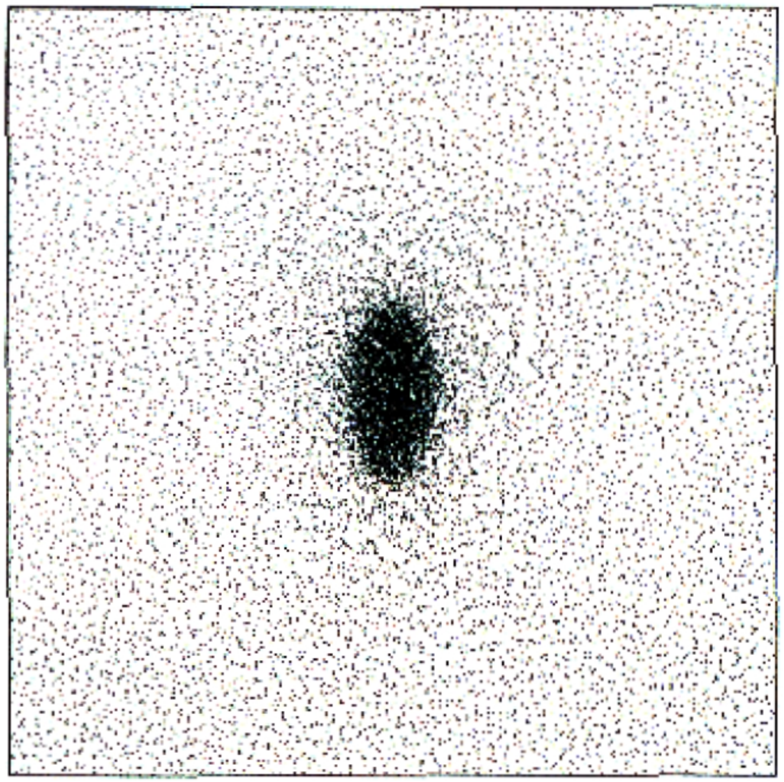
The ellipsoidal top hat

$$a_1 : a_2 : a_3 = 1 : 1.25 : 1.5$$

$$\delta_{lin} = 1.6$$

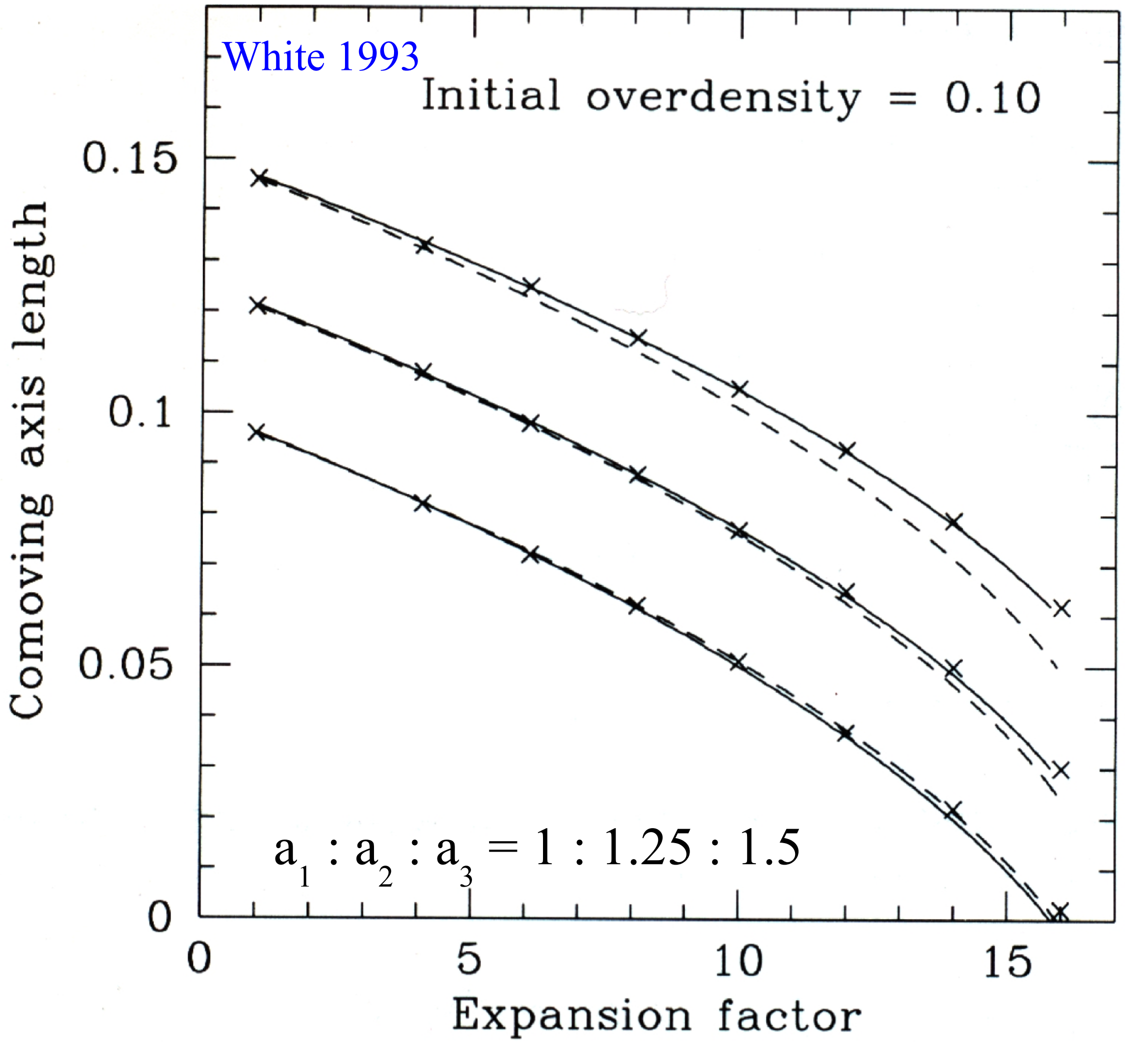
White 1993





The ellipsoidal top hat

Assuming that the exterior density remains uniform (true to linear order)



$$\frac{d^2 a_i}{dt^2} = -2\pi G[\rho_e \alpha_i + (\tfrac{2}{3} - \alpha_i)\rho_b]a_i \,,$$

$$\frac{d^2 R_b}{dt^2} = -\frac{4\pi}{3} G\rho_b R_b \,,$$

$$\rho_e a_1 a_2 a_3 = \text{const.} \,,$$

$$\rho_b R_b^3 = \text{const.} \,,$$

$$\alpha_i = a_1 a_2 a_3 \int_0^\infty (a_i^2 + \lambda)^{-1} \prod_{j=1}^{3} (a_j^2 + \lambda)^{-1/2} d\lambda \,,$$

White & Silk 1979

an ellipsoidal top hat stays uniform and ellipsoidal as it evolves. The axis ratios become more extreme as it collapses

Ellipsoidal dynamics of peaks

Bond & Myers (1996) approximate the dynamics of a peak in a gaussian random field by an homogeneous ellipsoid in an external tidal field by using equations,

$$\frac{d^2}{dt^2} X_i = -4\pi G \bar{\rho}_{\rm nr} X_i \left[\frac{1}{3} + \frac{\delta_{\rm nr}}{3} + \frac{b'_i}{2} \delta_{\rm nr} + \lambda'_{vi}(t) \right]$$

Converting to a consistent notation

$$\frac{d^2 R}{dt^2} = -\frac{4\pi}{3} G \overline{\rho} R$$

Homogeneous and isotropic universe (Friedmann 1922)

$$\frac{d^2 R_i}{dt^2} = -\frac{4\pi}{3} G \overline{\rho} R_i [1 + \frac{3}{2} \alpha_i \delta]$$

Ellipsoidal top hat (White & Silk 1979)

$$\frac{d^2 R_i}{dt^2} = -\frac{4\pi}{3} G \overline{\rho} R_i [1 + \frac{3}{2} \alpha_i \delta + T_{{\rm ext},i}]$$

Peak collapse (Bond & Myers 1996)
$T_{{\rm ext},i}$ are e-values of the external tidal field and align with the ellipsoid (??)

Geodesic Deviation Equation – ellipsoidal collapse

Stücker et al 2017

$$\dot{\mathbf{x}} = a^{-2}\mathbf{p}, \quad \dot{\mathbf{p}} = -\nabla\phi, \quad \nabla^2\phi = 4\pi G a^2 \bar{\rho}\delta = 4\pi G a^{-1}\rho_0\delta,$$

Equations of motion of a particle trajectory

$$\dot{D}_{ij} = a^{-2}P_{ij}, \quad \dot{P}_{ij} = -T_{ik}D_{kj}\,, \quad \bar{\rho}(t)/|\det(\underline{\underline{D}})| = \bar{\rho}(1+\delta),$$

Geodesic deviation equation for neighboring trajectories

where

$$D_{ij} = \frac{\partial x_i}{\partial q_j}, \quad P_{ij} = \frac{\partial p_i}{\partial q_j}, \quad S_{ij} = \frac{\partial s_i}{\partial q_j} \text{ and } T_{ij} = \frac{\partial^2\phi}{\partial x_i \partial x_j}.$$

In linear theory D_{ij}, P_{ij} and T_{ij} all share the same unchanging principal axes. As long as this remains true, their e-values satisfy the "ellipsoidal" equations

$$\dot{X}_i = a^{-2}P_i, \quad \dot{P}_i = -T_i X_i, \quad 1+\delta = 1/X_1X_2X_3$$

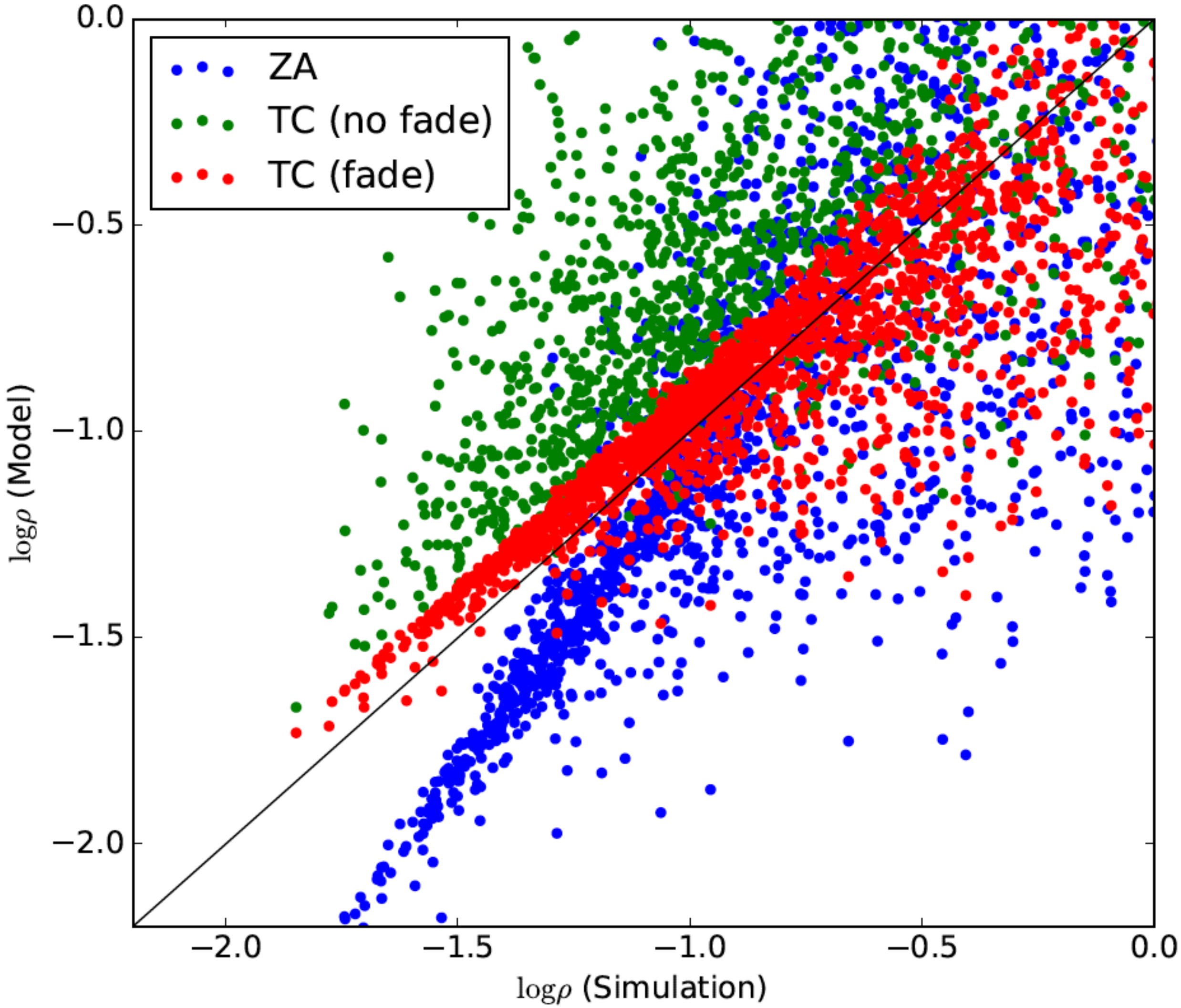
$$\dot{X}_i = a^{-2}P_i, \quad \dot{P}_i = -\left(\frac{4\pi G\rho_0}{3a}\delta + T_{\mathrm{ext},i}\right)X_i.$$

Anisotropy evolution is driven purely by the *external* tide – there is **no** local ellipsoidal contribution.

We need a model for the external tide – linear theory *à la* Bond+Myers96?

Simulation tests of GDE ellipsoidal collapse

Stücker et al 2017



Model density vs GDE simulation results for 1000 random "uncollapsed" particles in a high-resolution WDM simulation

Blue points assume linear theory (Zeldovich approx.) dynamics

Green points assume GDE model with linear T_{ext} evolution.

Red points assume tidal evolution weakens for nonlinear densities

$$T_{ext} = T_{ext,lin} / (1 + \sigma)$$

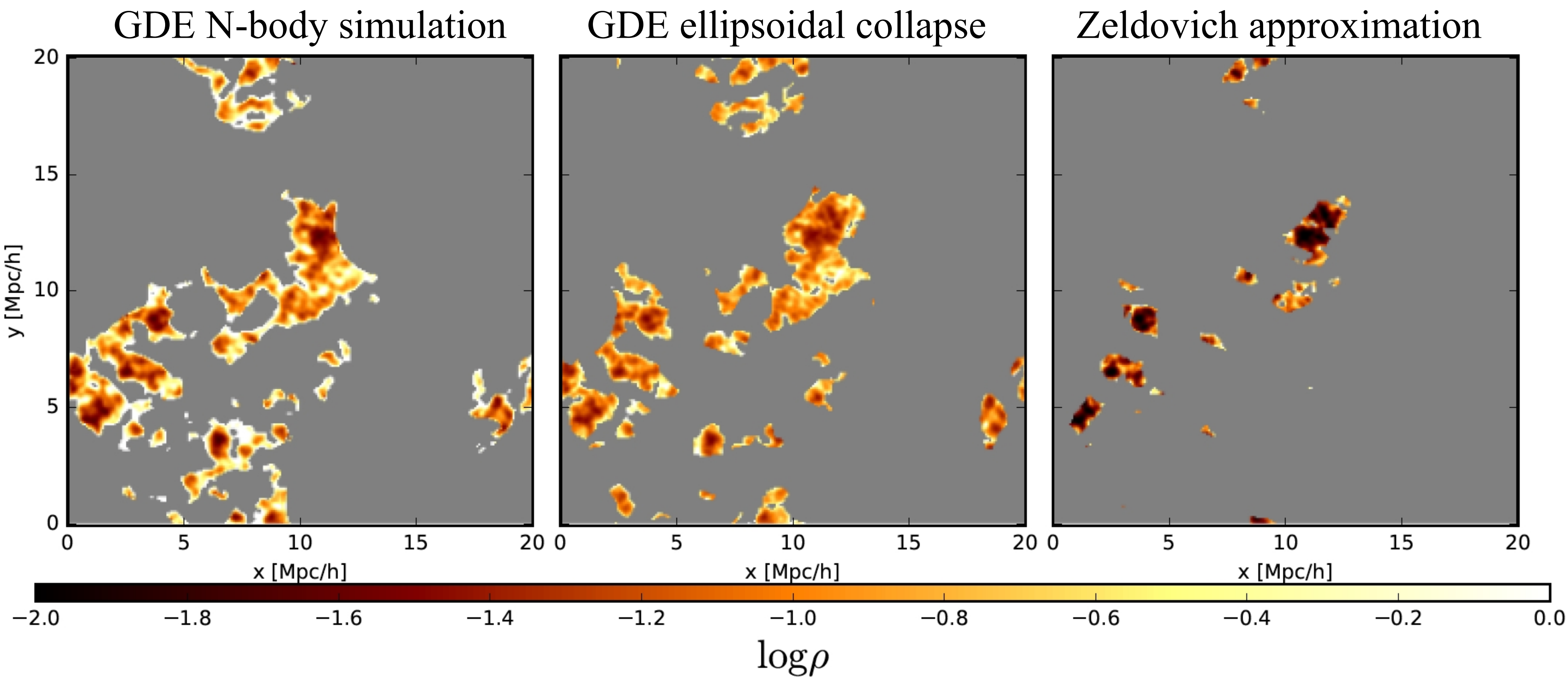
rms (extrapolated) linear overdensity

Simulation tests of GDE ellipsoidal collapse

Stücker et al 2017

Densities of uncollapsed particles plotted in Lagrangian (initial position) space

Grey regions are collapsed in 1D so are part of sheets, filaments or halos



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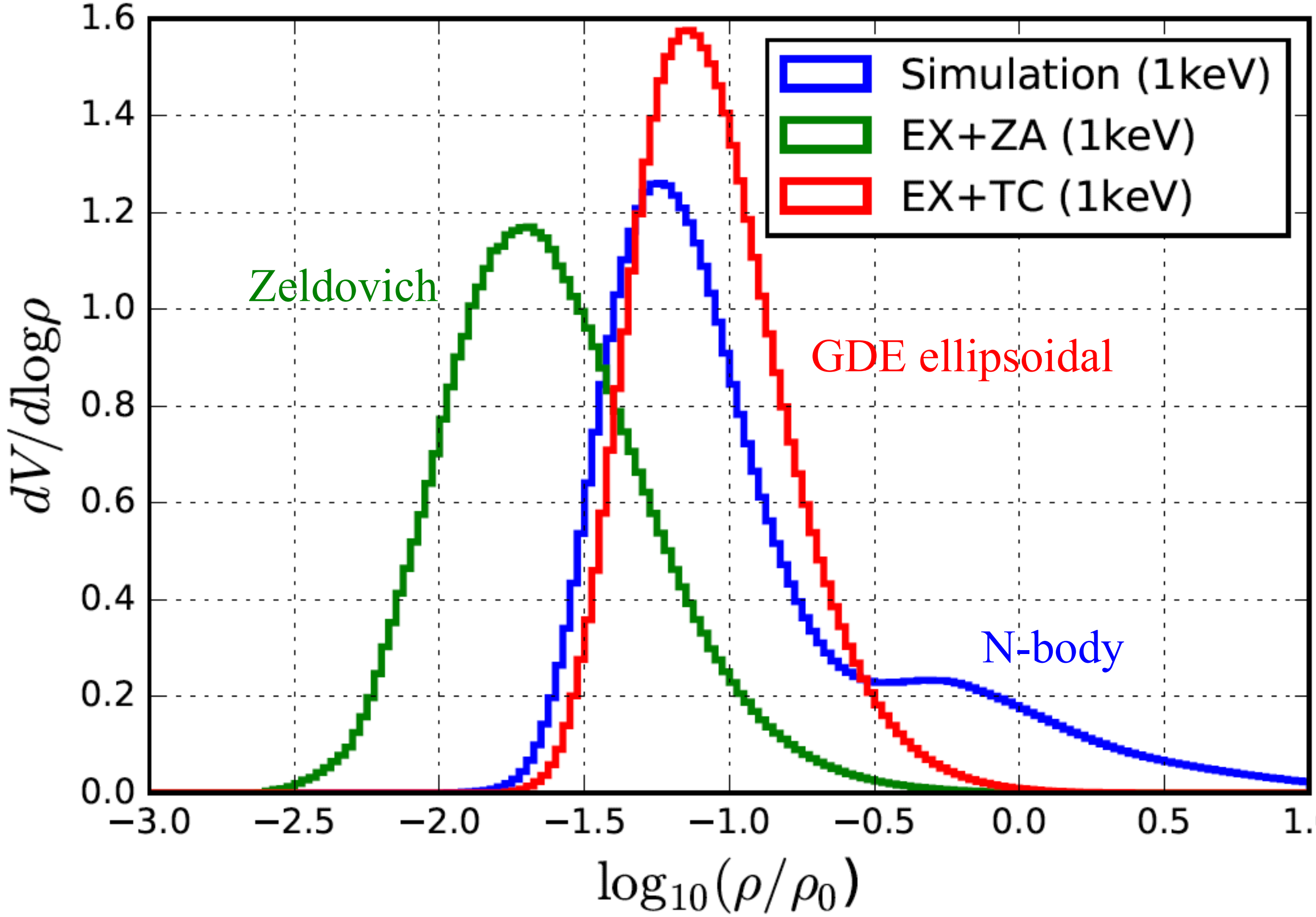


The cosmic volume fraction occupied by mass elements as a function of density

Collapsed elements have infinite density/zero volume in Zeldovich and GDE cases

Collapse thresholds for the GDE ellipsoidal model

$$\frac{dX_i}{da} = \frac{3}{2} a^{-3/2} (t_0 P_i),$$
$$\frac{d(t_0 P_i)}{da} = -a^{-1/2} \Big(\delta/3 + \alpha(\lambda_i - \delta_0/3) \Big) X_i, \quad (34)$$
$$\delta = (X_1 X_2 X_3)^{-1} - 1, \quad \alpha = a/(1+\sigma_s a), \quad \delta_0 = \sum_i \lambda_i.$$

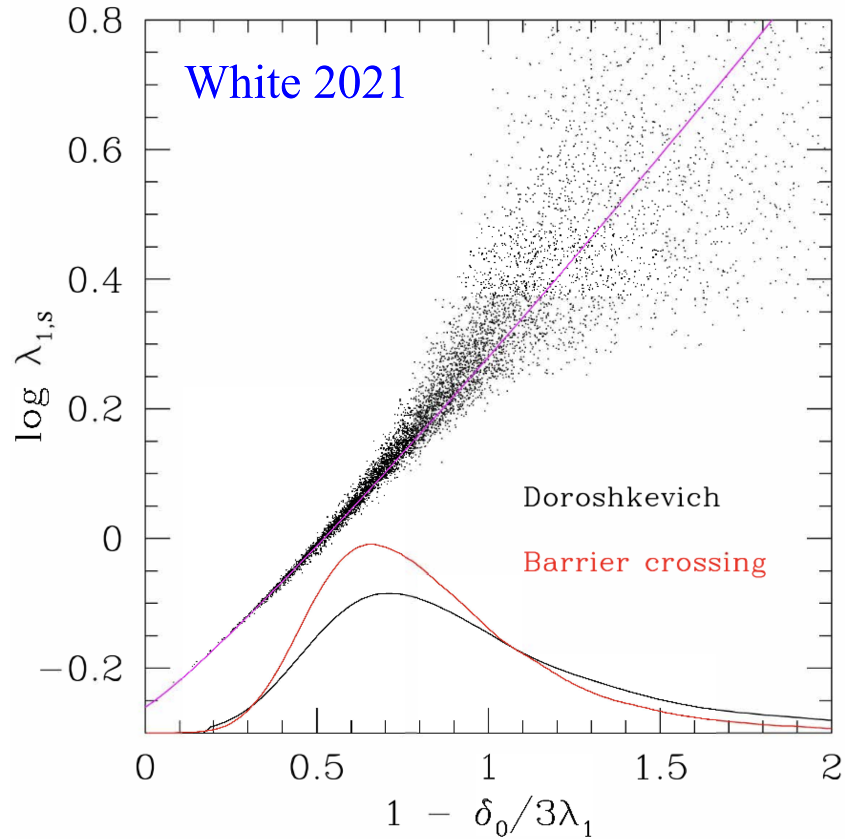
For any chosen set of λ_i, this closed set of equations can be integrated forwards starting from initial conditions given by

$$X_i = 1 - a\lambda_i, \quad t_0 P_i = -\frac{2}{3} a^{3/2} \lambda_i, \quad a \ll 1, \quad (35)$$

and stopping either at the present day ($a = 1$) or when collapse occurs (for which $\lambda_1 > 0$ is a necessary but not sufficient condition).

The equations with fading tidal field require λ_1, the largest e-value of the tidal tensor to be positive for collapse, but the linear overdensity can be negative!

A collapse threshold **cannot** refer to linearly extrapolated overdensity.

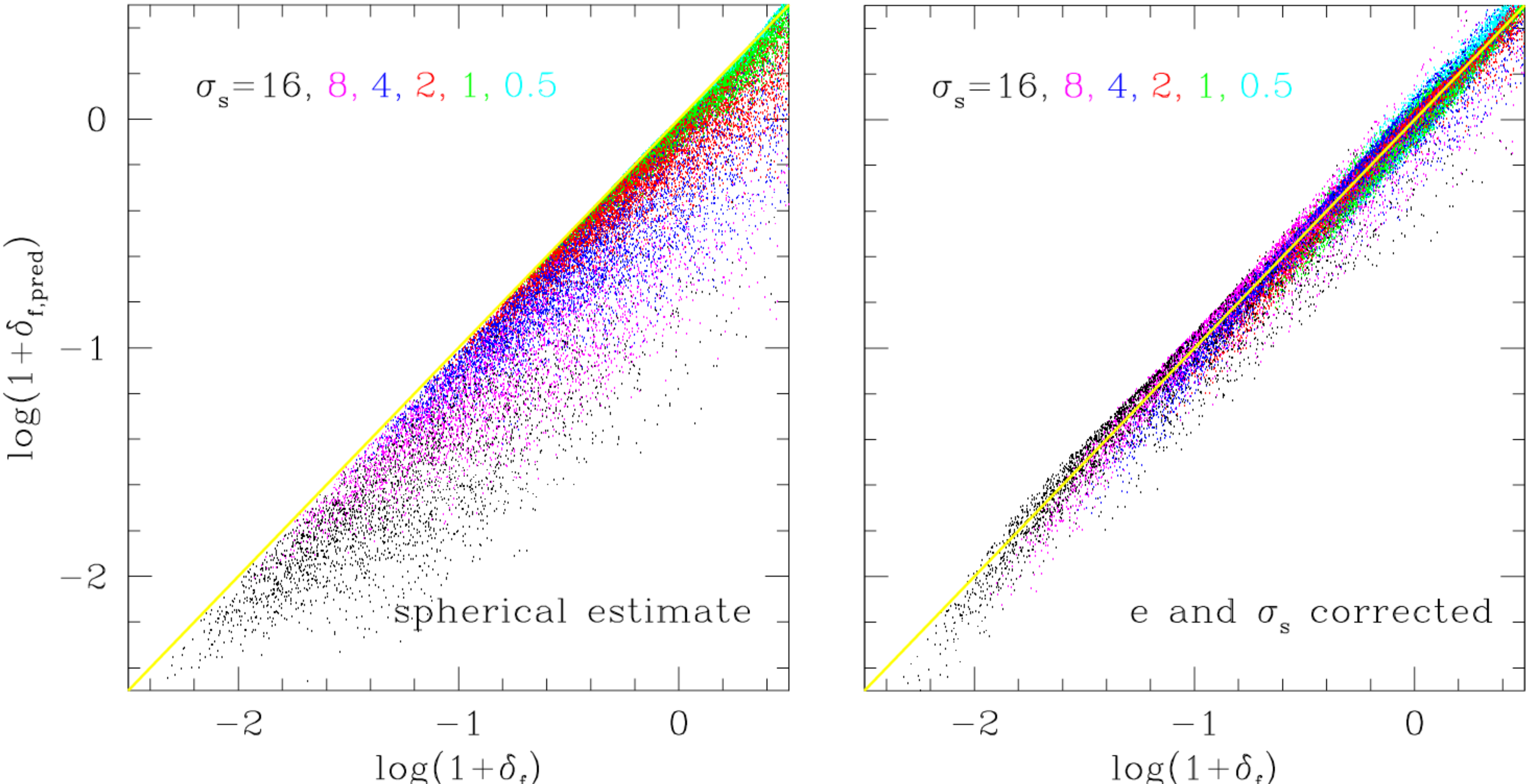


Eventual collapse in 1-D is predicted for 85% of elements selected from a gaussian random field, hence for 70% of *underdense* regions.

Nearly spherical evolution is the exception

Densities of uncollapsed mass elements

White 2021



$1 + \delta_f$ from ellipsoidal collapse for e-values chosen from Doroshkevich (1970)
Analytic predictions from the spherical model and a fit corrected by σ_s and by the shape of the tidal tensor ($e = T_1/T_3$)

Current state of play

- The GDE casts doubt on the model normally used to describe ellipsoidal collapse in excursion set theories of structure formation
- An ellipsoidal model derived from the GDE predicts that many initially underdense regions will undergo tidally driven collapse
- As a result collapse barriers cannot be expressed as a thresholds for the linearly extrapolated overdensity, only for the linearly extrapolated value of λ_1
- This may effect the mass function predictions based on ellipsoidal collapse (e.g. Sheth-Mo-Tormen) but this depends on the validity of models for collapse of sheets to filaments to halos

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This project is not finished so these results may change!