Extracting higher l from maps with equidistant rings

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- **Given:**
  a function $f(\vartheta, \varphi)$ on the sphere, band limited at l_{max}, sampled on $l_{\text{max}}+2$ equidistant rings in ϑ (first ring on north pole, last ring on south pole) and $2l_{\text{max}}+2$ equidistant pixels per ring.

- **Wanted:**
  Spherical harmonic coefficients a_{lm} of this map, up to l_{max}.

- **Approach:**
  - Direct numerical quadrature is not possible, since we need at least $2l_{\text{max}}+2$ equidistant rings to use quadrature weights (Clenshaw-Curtis quadrature).
  - **But:** Our pixelization has $2l_{\text{max}}+2$ meridians. Each of these meridians can be continued over the south pole back to the north pole, and this "full" meridian has $2l_{\text{max}}+2$ pixels on it.
  - Since we assume f to be band limited, the ϑ-dependent function along each of these full meridians is a sum of associated Legendre polynomials of degrees up to l_{max} in $\cos(\vartheta)$. In other words, it can be expressed as a Fourier series $\sum_{k=0}^{l_{\text{max}}} A_k \cos(k\vartheta)$. Such a function in turn is completely determined by $2l_{\text{max}}+1$ equidistant samples in the range $\vartheta = [0; 2\pi[$, i.e. one less than we actually have.
    $\Rightarrow$ by means of FFT, zero-padding, and inverse FFT we can upsample the information on each meridian to any desired number of pixels without loss of information.
  - we use this to upsample our number of rings to $2l_{\text{max}}+2$ (again with the first ring on the north pole and the last on the south pole), and can now apply Clenshaw-Curtis quadrature weights to analyze the resulting map accurately up to and including l_{max}.

- **Remarks**

- Requiring an even number of pixels in azimuthal direction (as was done above) is not really necessary, but makes the discussion of the algorithm simpler, since it allows trivial continuation of meridians over the south pole.
- It is not actually necessary to carry out the map analysis on a grid with $2l_{\max}+2$ rings; by means of some additional FFT-shifting this can be reduced to a grid of only $l_{\max}+2$ rings again (the absolute minimum is $l_{\max}+1$ rings), reducing the time for the SHT by roughly a factor of 2. At high band limits (roughly 1000), the cost for the additional FFTs becomes negligible.
  To illustrate why this works, consider the map analysis operator as a sequence of a diagonal operator applying the quadrature weights and the adjoint of the map synthesis operator. As shown before, the map synthesis operator can be broken down into the classic `alm2map` operation from a_{lm} onto a grid with the minimum number of equidistant rings, in our case $l_{\max}+2$, followed by an FFT upsampling operator. The adjoint of this is an FFT downsampling operator followed by a `adjoint_alm2map` operator working again on the minimal grid.
- Analogous derivations can be made if the initial grid does not start and end at the poles, but also if the first and/or last ring are half a ring's width away from their respective pole. (If only one of them is at a pole, we obtain the McEwen-Wiaux pixelization.)