Chapter 2

Perturbation Theory in Inflation

2.1 Inflation—Overview

During inflation, the universe expands exponentially. It implies the Hubble parameter, $H(t) = d\ln a/dt$, the expansion rate of the universe, being nearly constant in time, and the expansion scale factor, $a(t)$, given by

$$a(t) = a(t_0)\exp\left(\int_{t_0}^{t} H(t')dt'\right) \approx a(t_0)\exp\left[H(t)\,(t-t_0)\right]. \tag{2.1}$$

The exponential expansion drives the observable universe spatially flat, for as the universe expands rapidly, a small section on a surface of a three-sphere of the universe approaches flat (we live on the section). Thus, inflation predicts flatness of the universe, and recent CMB experiments have confirmed the prediction (Miller et al., 1999; de Bernadis et al., 2000; Hanany et al., 2000).

What makes the exponential expansion possible? One finds that neither matter nor radiation can make it; on the contrary, their energy density, ρ, and pressure, p, make the universe decelerate. Since the universe accelerates only when $\rho + 3p < 0$, one needs a negative pressure component dominating the universe. How can it be possible?

A spatially homogeneous scalar field, ϕ, with a potential, $V(\phi)$, provides negative pressure, making the exponential expansion possible. The energy density is $\rho_\phi = \frac{1}{2}(d\phi/dt)^2 + V(\phi)$, while the pressure is $p_\phi = \frac{1}{2}(d\phi/dt)^2 - V(\phi)$, giving

$$\rho_\phi + 3p_\phi = 2\left[(d\phi/dt)^2 - V(\phi)\right]. \tag{2.2}$$

Hence, one finds that $(d\phi/dt)^2 < V(\phi)$ suffices to accelerate the universe. This *slowly-rolling* scalar field is a key ingredient of inflation; by assuming a slowly-rolling scalar field dominant in early universe, the universe expands exponentially.

While what is ϕ and how it comes to dominate the universe are still in debate, a simple model sketched in figure 2.1 works well. In the phase (a), ϕ rolls down on $V(\phi)$ slowly, driving the universe to expand exponentially. In the phase (b), ϕ oscillates rapidly, terminating inflation. After inflation ends, interactions of ϕ with other particles lead ϕ to decay with a decay rate of Γ_ϕ,

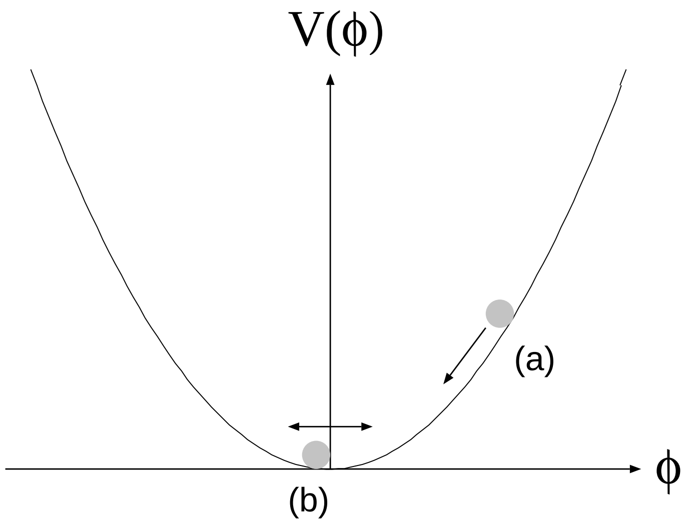

Figure 2.1: Classical Evolution of Scalar Field

Classical evolution of a scalar field, ϕ, in a potential, $V(\phi)$. (a) ϕ rolls down on $V(\phi)$ slowly, driving the universe to expand exponentially. (b) ϕ oscillates rapidly about $\phi = 0$, terminating inflation; it then decays into particles and radiation because of interactions, reheating the universe.

producing particles and radiation. This is called a *reheating* phase of the universe, as ϕ converts its energy density into heat by the particle production. A reheating temperature amounts to on the order of $(\Gamma_\phi m_{\rm pl})^{1/2}$. While a precise value of reheating temperature depends upon models, it is typically on the order of 10^{14-16} GeV $\sim 10^{27-29}$ K. Note that the smallness of the observed CMB anisotropy implies that ϕ is coupled to other particles only very weakly, i.e., $\Gamma_\phi < H$, giving lower reheating temperature. After the reheating, radiation dominates the universe, and the Big-bang scenario describes the rest of the cosmic history.

A class of inflation models with the potential sketched in figure 2.1 is called the *chaotic inflation*, for which $V(\phi) \propto \phi^n$ (Linde, 1983). Until now, this model has remained the most successful realization of inflation with broad applications (Linde, 1990).

2.2 Quantum Fluctuations

Inflation predicts emergence of quantum fluctuations in early universe. As soon as the fluctuations emerge from a vacuum, the exponential expansion stretches the proper wavelength of the fluctuations out of the Hubble-horizon scale, H^{-1}. After leaving the horizon, the fluctuation amplitude does not change in time; on the contrary, it stays constant in time with characteristic r.m.s. amplitude, $|\phi|_{\rm rms} \sim H/(2\pi)$.

After inflation, as the universe decelerates, the fluctuations reenter the Hubble horizon, seeding matter and radiation fluctuations in the universe. Figure 2.2 summarizes the evolution of characteristic length scales: the Hubble-horizon scale (H^{-1}), the *COBE* DMR-scale fluctuation wavelength, and the galaxy-scale fluctuation wavelength.

We estimate H^{-1} during inflation as follows. The scalar-field fluctuations produce CMB fluctuations of order $H/m_{\rm pl}$. Using the DMR measurement (Smoot et al., 1992), $\Delta T/T \sim 10^{-5}$, we obtain $H \sim 10^{-5} m_{\rm pl}$, or $H^{-1} \sim 10^5 m_{\rm pl}^{-1} \sim 10^{-28}$ cm. Since H^{-1} stays nearly constant in time during inflation, this value represents the horizon scale throughout inflation approximately. After inflation, H^{-1} grows as $H^{-1}(a) \propto a^2$ in the radiation era, and $\propto a^{3/2}$ in the matter era.

DMR probes a present-day fluctuation wavelength on the order of 3 Gpc $\sim 10^{28}$ cm. By comparing the reheating temperature, $\sim 10^{27-29}$ K, with the present-day CMB temperature, 2.73 K, one finds that the universe has expanded by a factor of $a_0/a_{\rm rh} \sim 3 \times 10^{26-28}$ since the reheating; thus, the DMR scale corresponds to a proper wavelength of $0.3 - 30$ cm at the reheating epoch (the number could be more uncertain). Here, a_0 is the present-day scale factor, while $a_{\rm rh}$ is the reheating epoch.

The galaxy-scale fluctuations have the linear comoving wavelength on the order of 1 Mpc $\sim 3 \times 10^{24}$ cm. The galaxy-scale fluctuations have left the horizon later than the DMR-scale fluctuations: $a_{\rm rh}/a_{\rm gal} \sim 10^{25}$, while $a_{\rm rh}/a_{\rm dmr} \sim 10^{28}$. Here, $a_{\rm gal}$ and $a_{\rm dmr}$ are the scale factors at which the galaxy- and DMR-scale fluctuations leave the horizon, respectively.

These ratios are often calculated with e-folding numbers, $N \equiv \ln(a_{\rm rh}/a)$. For the galaxy- and DMR-scale fluctuations, we have $N_{\rm gal} = \ln(a_{\rm rh}/a_{\rm gal}) \sim 58$, and $N_{\rm dmr} = \ln(a_{\rm rh}/a_{\rm dmr}) \sim 64$. Moreover, using equation (2.1), we obtain $t_{\rm gal} - t_{\rm dmr} = H^{-1}(N_{\rm dmr} - N_{\rm gal}) \sim 10^{-38}$ s; thus, inflation generates the fluctuations on the DMR scales down to the galaxy scales almost instantaneously.

2.2.1 Quantization in de Sitter spacetime

A basic idea behind quantum-fluctuation generation in inflation is well described by the second quantization of a massive-free scalar field in unperturbed de Sitter spacetime, for which $a(t) = a_0 e^{H(t-t_0)}$ with H independent of t. In this system, the problem is exactly solvable, and finite mass captures an essential point of generating a "tilted" fluctuation spectrum, as we will show in the next subsection. In this subsection, we describe the quantization procedure in de Sitter spacetime, following Birrell and Davies (1982).

Quantization in curved spacetime is generally complicated, as there is no unique vacuum state to define a ground state of quanta, even for inertial observers who detect no particles in the Minkowski vacuum (the vacuum state for the quantum field theory in Minkowski spacetime). Fortunately, in the spatially-flat Robertson-Walker metric, there is a prescription for quantization, largely because of the metric being *conformal* to the Minkowski metric, $g_{\mu\nu} = a^2 \eta_{\mu\nu}$, or more specifically

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j = a^2(\tau)\left(-d\tau^2 + \delta_{ij}dx^i dx^j\right), \tag{2.3}$$

where $\tau \equiv \int^t a^{-1}(t')dt'$ is called the conformal time. In this metric, there is a reasonable vacuum state for particles whose comoving frequencies, ω_k, are higher than the conformal expansion rate,

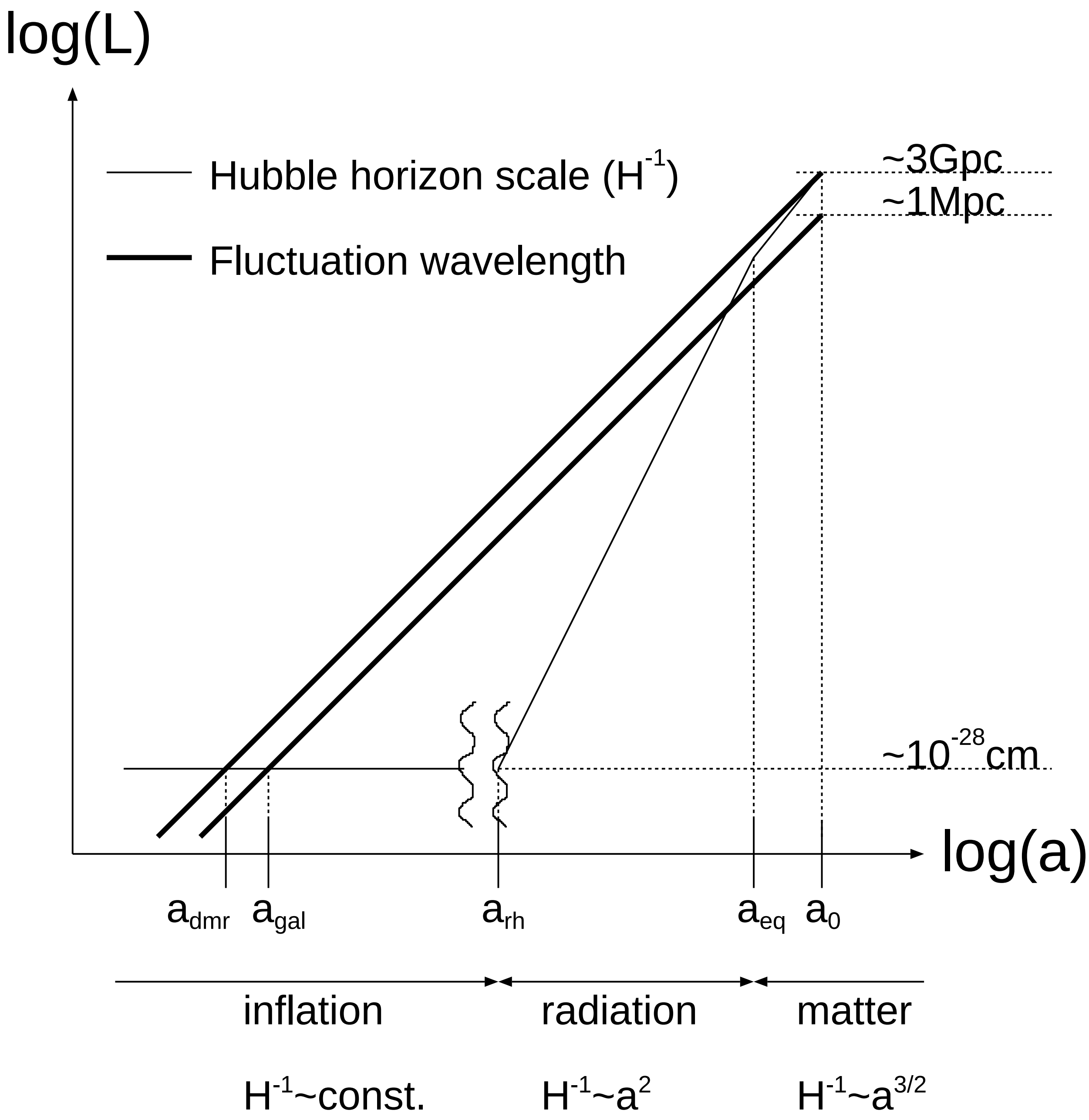

Figure 2.2: Physical Length Scales in Cosmic Evolution

Representative physical length-scale evolution in the cosmic history. The thick lines draw the evolution of fluctuation wavelengths, $2\pi a/k$, while the thin line draws the evolution of the Hubble-horizon scale, $H^{-1}(a)$, where a is the expansion scale factor. $H^{-1}(a)$ stays constant during inflation, grows as a^2 in the radiation era, and grows as $a^{3/2}$ in the matter era. There are characteristic scale factors: a_0 is the present-day, $a_{\rm eq}$ is the matter-radiation equality, and $a_{\rm rh}$ is the reheating epoch. The DMR-scale fluctuations leave the horizon at $a_{\rm dmr}$, while the galaxy-scale fluctuations leave at $a_{\rm gal}$. These scale factors are related to each other roughly as follows: $a_0/a_{\rm eq} \sim 10^4$, $a_{\rm eq}/a_{\rm rh} \sim 10^{24}$, $a_{\rm rh}/a_{\rm gal} \sim 10^{25}$ (e^{58}), and $a_{\rm rh}/a_{\rm dmr} \sim 10^{28}$ (e^{64}).

$d\ln a/d\tau$. We should differ the conformal expansion rate, $d\ln a/d\tau$, from the expansion rate, $H = d\ln a/dt$. They are related through $d\ln a/d\tau = aH$. In de Sitter spacetime, we obtain

$$\tau = \tau_0 + \frac{1 - \exp\left[-H(t - t_0)\right]}{a_0 H} = \left(\tau_0 + \frac{1}{a_0 H}\right) - \frac{1}{a(\tau)H}. \tag{2.4}$$

For simplicity, we set the first term vanishing, so that $\tau = -\left[a(\tau)H\right]^{-1}$; thus, τ lies in $-\infty < \tau < 0$ for $-\infty < t < \infty$ ($0 < a < \infty$). From now on, we will use dots for conformal-time derivatives: $\dot{x} \equiv \partial x/\partial\tau$.

We start quantizing a scalar field, $\phi(\mathbf{x}, \tau)$, by expanding it into creation, $\hat{a}^\dagger_{\mathbf{k}}$, and annihilation, $\hat{a}_{\mathbf{k}}$, operators, which satisfy the commutation relation, $\left[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}\right] = \delta^{(3)}(\mathbf{k} - \mathbf{k}')$. We have

$$\phi(\mathbf{x}, \tau) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \left[\hat{a}_{\mathbf{k}}\varphi_k(\tau)e^{i\mathbf{k}\cdot\mathbf{x}} + \hat{a}^\dagger_{\mathbf{k}}\varphi^*_k(\tau)e^{-i\mathbf{k}\cdot\mathbf{x}}\right]. \tag{2.5}$$

The canonical commutation relation between ϕ and the conjugate momentum, $\pi_\phi = a^2(\tau)\dot{\phi}$:

$$\left[\phi(\mathbf{x}, \tau), \pi_\phi(\mathbf{x}', \tau)\right] = a^2(\tau)\left[\phi(\mathbf{x}, \tau), \dot{\phi}(\mathbf{x}', \tau)\right] = i\delta^{(3)}(\mathbf{x} - \mathbf{x}'), \tag{2.6}$$

gives a normalization condition on $\varphi_k(\tau)$, $a^2\left(\varphi_k\dot{\varphi}^*_k - \varphi^*_k\dot{\varphi}_k\right) = i$.

The normalization condition motivates our using a new mode function given by $\chi_k \equiv a\varphi_k$, which satisfies a new normalization condition, $\chi_k\dot{\chi}^*_k - \chi^*_k\dot{\chi}_k = i$. If χ_k has a positive frequency mode with respect to the conformal timelike Killing vector ($\partial/\partial\tau$ for our metric), i.e., $\dot{\chi}_k = -i\omega_k\chi_k$, then the condition gives $\chi_k(\tau) = (2\omega_k)^{-1/2}e^{-i\omega_k\tau}$, a ground state in the Minkowski vacuum. Since χ_k gives the closest analogy to the Minkowski vacuum state, we will use χ_k more frequently than φ_k.

The Klein–Gordon equation for a massive-free scalar field, $g^{\alpha\beta}\phi_{,\alpha;\beta} = m^2\phi^2$, gives equation of motion for $\chi_k(\tau)$,

$$\ddot{\chi}_k(\tau) + \left[k^2 + m^2_\chi(\tau)\right]\chi_k(\tau) = 0, \tag{2.7}$$

where $m^2_\chi(\tau)$ is the time-dependent effective mass,

$$m^2_\chi(\tau) \equiv (m^2 - 2H^2)a^2(\tau) = m^2a^2(\tau) - \frac{2}{\tau^2}. \tag{2.8}$$

We thus find that the Hubble parameter effectively reduces m^2 by $2H^2$. This time-dependent, negative contribution to $m^2_\chi(\tau)$ is the effect of de Sitter spacetime, which is not Minkowski but curved.

Fortunately, there is an exact solution to the Klein–Gordon equation (2.7):

$$\chi_k(\tau) = \sqrt{-\tau}\left[C_1H^{(1)}_\nu(-k\tau) + C_2H^{(2)}_\nu(-k\tau)\right], \tag{2.9}$$

where C_1 and C_2 are integration constants, and $\nu^2 = 9/4 - m^2/H^2$. $H^{(1)}_\nu(x)$ is a Hankel function of the first kind; $H^{(2)}_\nu(x) = \left[H^{(1)}_\nu(x)\right]^*$. We have negative sign in front of τ to recall that τ lies in $-\infty < \tau < 0$.

How do we determine the integration constants, C_1 and C_2? In other words, how do we normalize our mode function properly? Since we know how to quantize a scalar field in the Minkowski vacuum, we should find a mode function that matches the Minkowski positive frequency mode, $\chi_k(\tau) = (2\omega_k)^{-1/2}e^{-i\omega_k\tau}$; however, we cannot find an unique positive frequency mode valid throughout inflation, as the time-dependent spacetime creates particles.

Instead, we define a vacuum state in the *in* state, the remote past, $\tau \to -\infty$. Using an asymptotic form of the Hankel function,

$$H_\nu^{(1)}(x \gg 1) \approx \sqrt{\frac{2}{\pi x}} \exp\left[i\left(x - \nu\frac{\pi}{2} - \frac{\pi}{4}\right)\right], \tag{2.10}$$

we obtain χ_k in the in state ($\tau \to -\infty$) from equation (2.9),

$$\chi_k(\tau \to -\infty) \longrightarrow \sqrt{\frac{2}{\pi k}}\left(C_1 e^{-ik\tau} + C_2 e^{ik\tau}\right). \tag{2.11}$$

Here, we have neglected the contribution from m^2/H^2 compared with $-k\tau$ in the exponent. The second term has a negative frequency, so that $C_2 = 0$. The first term with $C_1 = \sqrt{\pi}/2$ gives $\chi_k(\tau) = (2k)^{-1/2}e^{-ik\tau}$, the Minkowski positive frequency mode with $\omega_k = k$, and thus in the in state, the solution describes a ground state of a massless field in the Minkowski vacuum.

Using the solution for $\chi_k(\tau)$, we obtain a solution for $\varphi_k(\tau)$,

$$\varphi_k(\tau) = \frac{\sqrt{-\pi\tau}}{2a(\tau)} H_\nu^{(1)}(-k\tau), \tag{2.12}$$

and $\phi(\mathbf{x}, \tau)$ becomes

$$\phi(\mathbf{x}, \tau) = \frac{\sqrt{-\pi\tau}}{2a(\tau)} \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \left[\hat{a}_\mathbf{k} H_\nu^{(1)}(-k\tau)\, e^{i\mathbf{k}\cdot\mathbf{x}} + \hat{a}_\mathbf{k}^\dagger H_\nu^{(2)}(-k\tau)\, e^{-i\mathbf{k}\cdot\mathbf{x}}\right]. \tag{2.13}$$

Since all the k modes in the integral are independent of each other, the nearly infinite sum of those modes makes ϕ obey Gaussian statistics almost exactly, because of the central limit theorem; thus, the two-point statistics specify all the statistical properties of ϕ. This is a generic property of the ground-state quantum fluctuations.

The annihilation operator, $\hat{a}_\mathbf{k}$, annihilates the vacuum state defined in the in state: $\hat{a}_\mathbf{k}|\,0_{\rm in}\rangle = 0$. In this vacuum, we calculate amplitude of ground-state ϕ fluctuations as

$$\left\langle 0_{\rm in}\left|\phi^\dagger(\mathbf{x},\tau)\phi(\mathbf{x},\tau)\right|0_{\rm in}\right\rangle = \int_0^\infty \frac{k^2 dk}{2\pi^2}\left|\varphi_k(\tau)\right|^2 = \frac{-\tau}{8\pi a^2(\tau)}\int_0^\infty k^2 dk \left|H_\nu^{(1)}(-k\tau)\right|^2. \tag{2.14}$$

Since we probe a limited range of k observationally, we also use the fluctuation spectrum in a logarithmic k range, $\Delta^2(k)$, which represents variance of fluctuations at a given comoving wavelength $2\pi k^{-1}$,

$$\Delta_\phi^2(k) \equiv \frac{k^3}{2\pi^2}\left|\varphi_k(\tau)\right|^2 = \frac{-k^3\tau\left|H_\nu^{(1)}(-k\tau)\right|^2}{8\pi a^2(\tau)} = \frac{H^2}{8\pi}(-k\tau)^3\left|H_\nu^{(1)}(-k\tau)\right|^2, \tag{2.15}$$

where we have used $a(\tau) = -(H\tau)^{-1}$ in the last equality. Let us recall that $\nu^2 = 9/4 - m^2/H^2$, and τ lies in $-\infty < \tau < 0$.

Here, we have the quantization of ϕ completed, and formally calculated the fluctuation spectrum. These results are exact, and valid on all scales. In the next subsection, we study the solution on super-horizon scales, where inflation produces observationally relevant fluctuations.

2.2.2 Scale-invariant fluctuations on super-horizon scales

Equation (2.13) describes a quantum massive-free scalar field, ϕ, in the unperturbed de Sitter spacetime on all scales. As the universe expands exponentially, $-k\tau = k/(aH)$ quickly becomes very small; the mode leaves the Hubble-horizon scale, H^{-1}. Figure 2.2 shows that the fluctuations on the observationally relevant scales should have left the horizon during inflation. Hence, the behavior of ϕ on super-horizon scales is practically important. In this subsection, we study the ϕ fluctuation spectrum on super-horizon scales.

In equation (2.15), using an asymptotic form of the Hankel function,

$$H_\nu^{(1)}(x \ll 1) \approx -i\frac{\Gamma(\nu)}{\pi}\left(\frac{x}{2}\right)^{-\nu}, \tag{2.16}$$

we obtain the ϕ fluctuation spectrum on super-horizon scales,

$$\Delta_\phi^2(k) \approx \left(\frac{H}{2\pi}\right)^2 2^{2\nu-3}\left[\frac{\Gamma(\nu)}{\Gamma(3/2)}\right]^2\left(\frac{k}{aH}\right)^{3-2\nu}, \tag{2.17}$$

where $\nu^2 = 9/4 - m^2/H^2$. We have used $\tau = -(aH)^{-1}$ and $\pi = \Gamma^2(1/2) = 4\Gamma^2(3/2)$.

One finds that $\nu = 3/2$ is a special point, for which the spectrum is independent of k, i.e., scale invariant, $\Delta_\phi^2(k) = H^2/(2\pi)^2$. This happens when we assume $m^2/H^2 \ll 1$, and thus $\nu = 3/2 - m^2/(3H^2) + \mathcal{O}(m^4/H^4)$, which gives

$$\Delta_\phi^2(k) \approx \left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{2m^2/(3H^2)}, \tag{2.18}$$

or the spectral index,

$$\frac{d\ln\Delta_\phi^2}{d\ln k} = \frac{2m^2}{3H^2}. \tag{2.19}$$

Since $m^2 \ll H^2$, the spectrum is almost scale invariant, giving the ϕ fluctuations characteristic r.m.s. amplitude, $|\phi|_{\rm rms} = H/(2\pi)$. Finite mass makes the spectrum slightly "blue", the power of k being positive.

The above assumption, $m^2 \ll H^2$, offers long-lasting inflation that makes the observable universe flat and homogeneous; otherwise, ϕ rolls down to a potential minimum too quickly, terminating inflation too early. In the inflationary regime, the Friedmann equation gives

$$\frac{m^2}{H^2} = \frac{3}{4\pi}\frac{m_{\rm pl}^2}{\phi^2} \sim 10^{-2}. \tag{2.20}$$

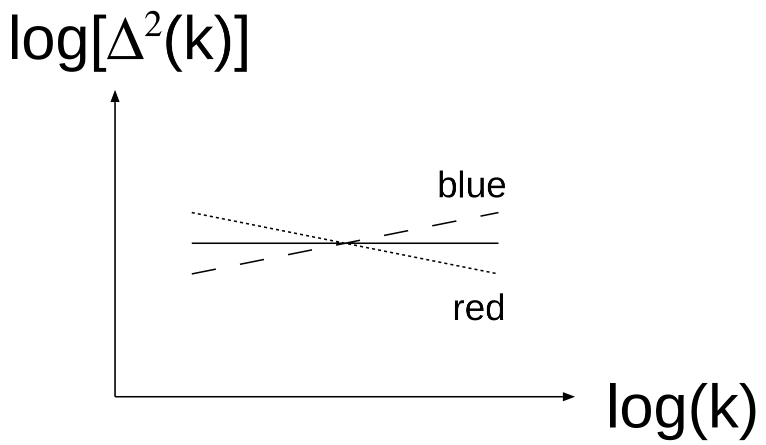

Figure 2.3: Color of Spectrum

A sketch of the fluctuation spectrum, $\Delta^2(k)$, which represents the fluctuation power in a logarithmic k range. The solid line plots a scale-invariant spectrum, the dotted line plots a "red" spectrum whose spectral index is negative, and the dashed line plots a "blue" spectrum whose spectral index is positive.

Hence, for a massive-free scalar field to drive inflation, the mass cannot be comparable to the Hubble parameter, and the fluctuation spectrum is almost exactly scale invariant.

The argument until now has assumed the exact de Sitter spacetime in which H is constant in time, and neglected perturbations in the metric. As a result, we have obtained a blue spectrum whose spectral index is $0 < 2m^2/3H^2 \ll 1$. In realistic inflation models, however, none of the above assumptions apply: H decreases slowly in time, and the metric is perturbed. In the next subsection, we will show that both the effects give a tilted "red" spectrum, for which the power of k is negative of order -10^{-2}. Figure 2.3 sketches what blue, scale-invariant, and red spectra look like.

2.2.3 Tilted "red" spectrum

If H decreases in time, and the metric is linearly perturbed, a scalar field acquires a negative effective mass-squared, $\Delta m^2 < 0$. It also modifies the mass of χ_k, $m_\chi(\tau)$ (Eq.(2.8)), as

$$m_\chi^2(\tau) \longrightarrow \left(m^2 + \Delta m^2\right) a^2(\tau) - \frac{2}{\tau^2}. \tag{2.21}$$

We thus expect that Δm^2 modifies the fluctuation spectral index, for the mass determines the spectral index through equation (2.19). This parameterization, Δm^2, may be useful to understand what physical effect is responsible for the spectral index.

How large is Δm^2? The derivation of Δm^2 is quite involved, but Mukhanov et al. (1992) show that the exact form of $m_\chi^2(\tau)$

$$m_\chi^2(\tau) = -\frac{H}{\dot{\phi}}\frac{d^2(\dot{\phi}/H)}{d\tau^2} \approx \left(\frac{d^2V}{d\phi^2} + 9\frac{dH}{dt}\right)a^2(\tau) - \frac{2}{\tau^2} = m_{\rm eff}^2 a^2(\tau) - \frac{2}{\tau^2}. \tag{2.22}$$

Here, since H and ϕ change in time t only very slowly, we have retained the terms on the order of $-dH/dt = 4\pi m_{\rm pl}^{-2}(d\phi/dt)^2$ or larger. This approximation is called the slow-roll approximation, and we evaluate the approximation explicitly in appendix A. For a massive-free field, $V(\phi) = m^2\phi^2/2$, by comparing equation (2.22) with (2.21), we find $\Delta m^2 = 9dH/dt < 0$. One can show that the 3 of the 9 comes from the effect of H changing in time, and the 6 comes from the effect of the metric perturbations.

By using the first-order slow-roll approximation, we obtain a relation between dH/dt and the potential slope, $dV/d\phi$, $dH/dt \approx -\frac{1}{6V}(dV/d\phi)^2 = -\frac{1}{3}m^2$. It thus follows that the total effective mass-squared becomes negative, $m_{\rm eff}^2 = m^2 + \Delta m^2 \approx -2m^2 < 0$, and we have a "red" spectrum or a negative spectral index,

$$\frac{d\ln\Delta_\phi^2}{d\ln k} = \frac{2m_{\rm eff}^2}{3H^2} \approx -\frac{4m^2}{3H^2} < 0. \tag{2.23}$$

Let us summarize what has made the spectral index negative. The spectral index is determined by the mass, or the effective mass, of a scalar field. In inflation, the intrinsic mass, m^2, is over-compensated by the induced mass from gravitational effects, Δm^2, which is negative: $m^2 + \Delta m^2 < 0$. As a result, the fluctuation spectrum becomes "red". Actually, any power-law potential of the form $V(\phi) = A\phi^n$ with A positive gives a negative $m_{\rm eff}^2$ through $m_{\rm eff}^2 = -(1+n/2)nA\phi^{n-2}$. Here, we have used $dH/dt \approx -\frac{1}{6V}(dV/d\phi)^2 = -\frac{1}{6}n^2A\phi^{n-2}$.

For a generic scalar field with an arbitrary potential, we find

$$m_{\rm eff}^2 = \frac{d^2V}{d\phi^2} + 9\frac{dH}{dt} = 6\frac{dH}{dt} - 3H\frac{d^2\phi/dt^2}{d\phi/dt}, \tag{2.24}$$

and the spectral index

$$\frac{d\ln\Delta_\phi^2}{d\ln k} = 4\frac{dH/dt}{H^2} - 2\frac{d^2\phi/dt^2}{H(d\phi/dt)}, \tag{2.25}$$

which agrees with Liddle and Lyth (1992).

CMB experiments have shown that a scale-invariant fluctuation spectrum fits the data well (de Bernadis et al., 2000; Hanany et al., 2000). Combining all the CMB experiments to date, Wang et al. (2001) show that a slightly red spectrum, the power of k being ~ -0.07, fits the data even better, while the error of the fit is still of order 0.1. Accurate measurement of the spectral index constrains the shape of $V(\phi)$ through equation (2.25), as $V(\phi)$ determines the time variation of H and ϕ; thus, it potentially discriminates between different inflation models.

Although the CMB experiments almost exclude possibility of the massive-free scalar field with sizable $m^2/H^2 \sim 1$ dominating the matter and radiation fluctuations in the universe (in terms of the spectrum index), the field may produce some of the fluctuations that are non-Gaussian. In this case, ϕ does not drive inflation, so that m^2 can be comparable to H^2; the field rolls down to a potential minimum quickly, and oscillates about the minimum, producing non-Gaussian isocurvature density fluctuations, $\delta\rho \propto \phi^2$ (Linde and Mukhanov, 1997). Even if the isocurvature fluctuations are sub-dominant in the universe, they could produce non-Gaussian temperature fluctuations in CMB. Measuring non-Gaussianity in CMB thus potentially probes particle physics in inflation.

2.2.4 Emergence of classical fluctuations

Until now, we have considered generation of quantum fluctuations in inflation, and derived a fluctuation spectrum on super-horizon scales. We then expect that the fluctuations seed observed CMB anisotropies and large-scale structures in the universe; but, how? Inflation generates *quantum* fluctuations, not *classical* fluctuations. How can the quantum fluctuations make classical objects like galaxies seen today?

Generation of classical fluctuations, or quantum-to-classical transition of fluctuations, during inflation, has been in debate (Calzatta and Hu, 1995; Matacz, 1997a,b; Kiefer et al., 1998). In this subsection, we describe a possible mechanism to produce classical fluctuations in inflation. Our approach is partly close to Kiefer et al. (1998).

A basic idea behind our classical-to-quantum transition mechanism is to approximate a field, ϕ (Eq.(2.5)), with the sum of the long-wavelength (super-horizon) modes and the short-wavelength (sub-horizon) modes. This mode separation is unique, as the comoving Hubble-horizon scale, $(aH)^{-1}$, has been a characteristic length scale in the solutions for mode functions, $\varphi_k(\tau)$ (Eq.(2.12)).

In the super-horizon limit, $k \ll aH$, equations (2.12) and (2.16) give $\varphi^*_{k\ll aH} = -\varphi_{k\ll aH}$. Using this in equation (2.5), we obtain

$$\begin{aligned}\phi(\mathbf{x},\tau) \quad \approx \quad & \int_{k<aH} d^3\mathbf{k} \left(\hat{a}_{\mathbf{k}} - \hat{a}^\dagger_{-\mathbf{k}}\right) \varphi_{k\ll aH}(\tau) e^{i\mathbf{k}\cdot\mathbf{x}} \\ & + a^{-1}(\tau) \int_{k>aH} \frac{d^3\mathbf{k}}{(2\pi)^{3/2}\sqrt{2k}} \left(\hat{a}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x} - ik\tau} + \hat{a}^\dagger_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x} + ik\tau}\right). \end{aligned} \tag{2.26}$$

We name the first term $\phi_{\rm CL}$, the second term $\phi_{\rm QM}$. The corresponding conjugate momenta are $\pi_{\rm CL} = a^2\dot{\phi}_{\rm CL}$ and $\pi_{\rm QM} = a^2\dot{\phi}_{\rm QM}$, respectively. The first term, $\phi_{\rm CL}$, becomes just the Fourier transform, while the second term, $\phi_{\rm QM}$, becomes an ordinary ground state in the Minkowski vacuum except for $a^{-1}(\tau)$ in the front. To see what happens to these terms in the context of the quantum field theory, we calculate the canonical commutation relations of $\phi_{\rm CL}$ and $\phi_{\rm QM}$.

For $\phi_{\rm QM}$, we find

$$[\phi_{\rm QM}(\mathbf{x},\tau), \pi_{\rm QM}(\mathbf{x}',\tau)] = i\int_{k>aH} \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} = i\delta^{(3)}(\mathbf{x}-\mathbf{x}') \tag{2.27}$$

for $|\mathbf{x}-\mathbf{x}'| < (aH)^{-1}$; thus, $\phi_{\rm QM}$ is a quantum field inside the horizon, and $a\phi_{\rm QM}$ is identical to a quantum field in the Minkowski vacuum. For $\phi_{\rm CL}$, we find the commutation relation vanishing,

$$[\phi_{\rm CL}(\mathbf{x},\tau), \pi_{\rm CL}(\mathbf{x}',\tau)] = 0; \tag{2.28}$$

thus, $\phi_{\rm CL}$ is no longer a quantum field, but a *classical* field. In other words, once smoothing out the fluctuations inside the horizon, we are left with the classical fluctuations. Notice that there has been no explicit decoherence mechanism in the system. The exact solution to the Klein–Gordon equation in the exact de Sitter spacetime naturally yields the classical fluctuations on super-horizon scales.

Without smoothing out the sub-horizon fluctuations, however, ϕ remains quantum. The commutation relation of ϕ (Eq.(2.6)) is exact as long as all the k modes equally contribute to ϕ. If nothing happens to the sub-horizon fluctuations, then the quantum, not classical, fluctuations will reenter the horizon after inflation. Yet, practically speaking, this argument may not be relevant for our actual observations because of the following reason. Consider a fluctuation wavelength just leaving the horizon at the end of inflation, at which the proper horizon size is $H^{-1} \sim 10^{-28}$ cm. As the universe has expanded by a factor of order 10^{27} since the end of inflation, we may find the fluctuation wavelength to be ~ 0.1 cm in the present universe. Now we ask: "does this-size fluctuation affects classicality of galaxy-scale fluctuations?" We may answer "no", if we assume that these substantially different-scale fluctuations have undergone different physics; if so, quantum coherence should have disappeared.

Even right after inflation, the super-horizon modes and the sub-horizon modes have undergone different physics. At the end of inflation, the reheating begins, and scalar fields decay into particles and radiation, thermalizing the universe. By causality, the thermalization process occurs only inside the horizon; thus, the reheating affects the sub-horizon fluctuations differently from the super-horizon fluctuations, breaking quantum nature of ϕ. In other words, in equation (2.26), the second term may have disappeared during the reheating, while the first term may remain unaffected. As a result, the reheating effectively smoothes out the sub-horizon scale fluctuations, and makes a quantum-to-classical transition possible.

2.3 Linear Perturbation Theory in Inflation

In the previous section, we have followed generation of scalar-field fluctuations in the unperturbed de Sitter spacetime, i.e., no perturbations in the metric. Scalar-field fluctuations, however, perturb the stress-energy tensor, and produce metric perturbations. Since the metric perturbations regulate the matter and radiation fluctuations that we observe today, we must include the metric perturbations in the analysis, and follow the evolution. In this section, we explore the *linear* perturbation theory in inflation, which includes perturbations to the metric and a scalar field. Our notation follows Bardeen (1980).

We use a linearly perturbed conformal Robertson–Walker metric of the form,

$$ds^2 = a^2(\tau)\left\{-(1+2AQ)d\tau^2 - 2BQ_i d\tau dx^i + \left[(1+2H_{\rm L}Q)\,\delta_{ij} + 2H_{\rm T}Q_{ij}\right]dx^i dx^j\right\}. \tag{2.29}$$

Here, all the metric perturbations, A, B, $H_{\rm L}$, and $H_{\rm T}$, are $\ll 1$, and functions of τ. The spatial coordinate dependence of the perturbations is described by the scalar harmonic eigenfunctions, Q, Q_i, and Q_{ij}, that satisfy $\delta^{ij}Q_{,ij} = -k^2 Q$, $Q_i = -k^{-1}Q_{,i}$, and $Q_{ij} = k^{-2}Q_{,ij} + \frac{1}{3}\delta_{ij}Q$. Note that Q_{ij} is traceless: $\delta^{ij}Q_{ij} = 0$. Kodama and Sasaki (1984) use different symbols, Y, Y_i, and Y_{ij}, for Q, Q_i, and Q_{ij}, respectively.

The four metric-perturbation variables are not entirely free, but some of which should be fixed to fix our coordinate system before we analyze the perturbations. The choice of coordinate system is often called the choice of *gauge*, or the *gauge transformation*; we will describe it later.

2.3.1 Fluid representation of scalar field

The metric perturbations enter into the stress-energy tensor perturbations, δT^μ_ν. We expand a scalar field into its homogeneous mean field, $\phi(\tau)$, and fluctuations about the mean, $\delta\phi(\tau)Q(\mathbf{x})$. The energy density and pressure fluctuations are given by

$$\delta\rho_\phi Q \equiv -\delta T^0_0 = \left[a^{-2}\left(\dot\phi\delta\dot\phi - A\dot\phi^2\right) + V_{,\phi}\delta\phi\right]Q, \tag{2.30}$$

$$\delta p_\phi Q \equiv \frac{\delta T^k_k}{3} = \left[a^{-2}\left(\dot\phi\delta\dot\phi - A\dot\phi^2\right) - V_{,\phi}\delta\phi\right]Q. \tag{2.31}$$

The energy flux, T^0_i, gives the velocity field, $v_\phi Q_i$,

$$\left(\rho_\phi + p_\phi\right)\left(v_\phi - B\right)Q_i \equiv T^0_i = \left(\frac{\dot\phi}{a^2}k\delta\phi\right)Q_i. \tag{2.32}$$

Using $\rho_\phi + p_\phi = a^{-2}\dot\phi^2$, we obtain $v_\phi - B = k\dot\phi^{-1}\delta\phi$; thus, $\delta\phi$ is directly responsible for the fluid's peculiar motion. The anisotropic stress, $T^i_j - p_\phi\delta^i_j$, is a second-order perturbation variable for a scalar field, being negligible.

When we choose our coordinate system so as $B \equiv v_\phi$ (a fluid element is comoving with the origin of the spatial coordinate), we have $\delta\phi$ vanishing, $\delta\phi \equiv 0$. This coordinate is called the comoving gauge, and we write the scalar-field fluctuations in this gauge as $\delta\phi_{\rm com} \equiv 0$.

Since we have only one degree of freedom, a scalar field, in the system, $\delta\rho_\phi$, δp_ϕ, and v_ϕ are not independent of each other. Nevertheless, this fluid representation is useful, as the cosmological linear perturbation theory has been developed as the general relativistic fluid dynamics. We can plague these fluid variables into the well-established general relativistic fluid equations, and see what happens to the metric perturbations. While we do not use those fluid equations explicitly in the following, but solve equation of motion for a scalar field (Klein–Gordon equation) directly, the fluid equations give the same answer.

2.3.2 Gauge-invariant perturbations

In the previous subsection, we have seen that scalar-field fluctuations vanish in the comoving gauge in which $B \equiv v_\phi$; thus, a choice of gauge defines perturbations. For the scalar-type perturbations that we are considering, the gauge transformation is

$$\tau \longrightarrow \tau' = \tau + T(\tau)Q(\mathbf{x}), \tag{2.33}$$

$$x^i \longrightarrow x'^i = x^i + L(\tau)Q_i(\mathbf{x}), \tag{2.34}$$

where T and L are $\ll 1$. Accordingly, scalar-field fluctuations, $\delta\phi$, transform as

$$\delta\phi(\tau) \longrightarrow \widetilde{\delta\phi}(\tau') = \delta\phi(\tau) - \dot\phi(\tau)T(\tau). \tag{2.35}$$

Hence, if we choose $T = \dot{\phi}^{-1}\delta\phi$, then we obtain $\widetilde{\delta\phi} = 0$. This choice of T defines the comoving gauge, $\delta\phi_{\rm com} = 0$. In this way, we find different values for the perturbation variables in different gauges.

So, what gauge should we use? Unfortunately, there is no answer to the question: "what gauge *should* we use?". Although there is no best gauge in the world, depending on a problem that we intend to solve, we may find that one gauge is more *useful* than the other, or vice versa. As long as we fix the gauge uniquely, and understand what gauge we are working on clearly, no problems occur.

In practice, however, problems occur when one author understands its own gauge, but does not understand the other author's gauge. Since there is no best gauge in the world, different authors may use different gauges, and may disagree with each other because of their misunderstanding of the gauges. In other words, one author's calculation on amplitude of $\delta\phi$ may disagree with the other's calculation, if they are using different gauges. The author using the comoving gauge sees $\delta\phi = 0$, but others may see $\delta\phi \neq 0$.

One way to overcome this undesirable property is to make perturbation variables *invariant* under the gauge transformation, and let them represent *gauge-invariant* perturbations. As an example, consider a new perturbation variable (Mukhanov et al., 1992),

$$u \equiv \delta\phi - \frac{\dot{\phi}}{aH}\left(H_{\rm L} + \frac{1}{3}H_{\rm T}\right). \tag{2.36}$$

One can prove this variable gauge invariant, $\widetilde{u} = u$, using equation (2.35) and

$$\widetilde{H}_{\rm L} + \frac{1}{3}\widetilde{H}_{\rm T} = H_{\rm L} + \frac{1}{3}H_{\rm T} - aHT. \tag{2.37}$$

Actually, $H_{\rm L} + \frac{1}{3}H_{\rm T}$ represents perturbations in the intrinsic spatial curvature, $\mathcal{R}$, as it is the scalar potential of the 3-dimension Ricci scalar: $\delta^{(3)}R = a^{-2}k^2\mathcal{R}Q$, where $\mathcal{R} \equiv H_{\rm L} + \frac{1}{3}H_{\rm T}$. While u reduces to $\delta\phi$ in the spatially flat gauge ($\mathcal{R} \equiv 0$), or to $-(\dot{\phi}/aH)\mathcal{R}$ in the comoving gauge ($\delta\phi \equiv 0$), its value is invariant under any gauge transformation. Any authors should agree upon the value of u.

For the physical interpretation of u, we may name u "scalar-field fluctuations in the spatially flat gauge" or "intrinsic spatial curvature perturbations in the comoving gauge". Either name describes the physical meaning of u correctly. The physical meaning of u depends upon what gauge we are using; however, the most important point is that the value of u is independent of a gauge choice. In this sense, u can be a "common language" among different authors.

Bardeen et al. (1983) use a similar gauge-invariant variable to u,

$$\zeta \equiv -\frac{aH}{\dot{\phi}}u = \mathcal{R} - \frac{aH}{\dot{\phi}}\delta\phi, \tag{2.38}$$

that reduces to $\mathcal{R}$ in the comoving gauge, or to $-(aH/\dot{\phi})\delta\phi$ in the spatially flat gauge. This variable helps our perturbation analysis not only because of being gauge invariant, but also being *conserved* on super-horizon scales throughout the cosmic evolution. We will show this property in the next subsection.

Using gauge invariance of u or ζ, we obtain a relation between $\delta\phi$ in the spatially flat gauge, $\delta\phi_{\rm flat}$, and $\mathcal{R}$ in the comoving gauge, $\mathcal{R}_{\rm com}$, as

$$\mathcal{R}_{\rm com} = -\frac{aH}{\dot{\phi}}\delta\phi_{\rm flat}. \tag{2.39}$$

It is derived from $u_{\rm com} = u_{\rm flat}$, or $\zeta_{\rm com} = \zeta_{\rm flat}$. As we have seen in the previous section, $\delta\phi$ obeys Gaussian statistics to very good accuracy because of the central limit theorem, that is, $\delta\phi$ is the sum of the nearly infinite number of independent modes (Eq.(2.13)). Since $\mathcal{R}_{\rm com}$ is linearly related to $\delta\phi_{\rm flat}$, $\mathcal{R}_{\rm com}$ also obeys Gaussian statistics in the linear order; however, as we will show in the next section, non-linear correction to this linear relation makes $\mathcal{R}_{\rm com}$ weakly non-Gaussian.

The spatial curvature perturbation, $\mathcal{R}$, is more relevant for the structure formation in the universe than the scalar-field fluctuation, $\delta\phi$, itself, as $\mathcal{R}$ regulates the matter density and velocity perturbations through the Poisson equation. Actually, $\mathcal{R}$ reduces to the Newtonian potential inside the horizon. Since the quantum fluctuations generate $\delta\phi$, we expect it to generate $\mathcal{R}$ through $\mathcal{R}_{\rm com} = -(aH/\dot{\phi})\delta\phi_{\rm flat}$. This is naively true, but may sound tricky. In the next subsection, we will show how to calculate $\mathcal{R}$ generated in inflation more rigorously.

2.3.3 Generation of spatial curvature perturbations

To calculate the intrinsic spatial curvature perturbation, $\mathcal{R} = H_{\rm L} + \frac{1}{3}H_{\rm T}$, that is generated in inflation, we need to track its evolution equation, and figure out how it is related to $\delta\phi$. We will show in this subsection that $\mathcal{R}$ is actually more than related to $\delta\phi$; it is almost *equivalent* to $\delta\phi$. We can track the evolution of $\delta\phi$ and $\mathcal{R}$ simultaneously, using the gauge-invariant variable, u (Eq.(2.36)).

Mukhanov et al. (1992) show that $au = a\delta\phi - (\dot{\phi}/H)\mathcal{R}$ obeys the same Klein–Gordon equation as we have used in the previous section (Eq.(2.7)),

$$\ddot{\chi}_k + \left[k^2 + m_\chi^2(\tau)\right]\chi_k = 0, \tag{2.40}$$

where χ_k is the mode function that expands au (see Eq.(2.5)). It thus follows that $a\delta\phi$ and $(\dot{\phi}/H)\mathcal{R}$ obey the same equation, and our argument on the quantum-fluctuation generation during inflation in the previous section applies to u as well. u being quantum fluctuations means that it also obeys Gaussian statistics very well because of the central limit theorem (see Eq.(2.13) and the text after equation).

We consider the Klein–Gordon equation for au on super-horizon scales. As equation (2.5), we expand au into the mode functions, $\chi_k = a\varphi_k$, where χ_k and φ_k are exactly the same functions that we have used in the previous section.

We give a slightly different expression for the solution, emphasizing its time dependence on super-horizon scales. Taking the long-wavelength limit, $k^2 \ll m_\chi^2(\tau)$, and using the exact form of

$m_\chi^2(\tau)$ (Eq.(2.22)) (Mukhanov et al., 1992), we obtain the Klein–Gordon equation on super-horizon scales,

$$\ddot{\chi}_k - \frac{H}{\dot{\phi}}\frac{d^2(\dot{\phi}/H)}{d\tau^2}\chi_k = 0. \tag{2.41}$$

There is an exact solution to this equation,

$$\frac{H}{\dot{\phi}}\chi_k = \frac{aH}{\dot{\phi}}\varphi_k = C_1 + C_2 \int \frac{H^2}{\dot{\phi}^2}d\tau, \tag{2.42}$$

where C_1 and C_2 are integration constants independent of τ. The second term is a decaying mode as $\int \dot{\phi}^{-2}H^2 d\tau = \int a^{-3}(d\phi/dt)^{-2}H^2 dt$, and thus $(H/\dot{\phi})\chi_k$ remains constant in time on super-horizon scales. This implies that $\zeta = -(aH/\dot{\phi})u$ also remains constant in time on super-horizon scales. Note that ζ obeys Gaussian statistics in the linear order, as it is related to a Gaussian variable, u, linearly; however, as we will show in the next section, non-linear correction to this linear relation makes ζ weakly non-Gaussian. This statement is equivalent to that we have made on $\mathcal{R}_{\rm com}$.

The solution obtained here for ζ is valid throughout the cosmic history regardless of whether a scalar field, radiation, or matter dominates the universe; thus, once created and leaving the Hubble horizon during inflation, ζ remains constant in time throughout the subsequent cosmic evolution until reentering the horizon. The amplitude of ζ, i.e., C_1, is fixed by the quantum-fluctuation amplitude derived in the previous section (Eq.(2.18)),

$$\frac{k^3}{2\pi^2}|C_1|^2 = \left(\frac{aH}{\dot{\phi}}\right)^2 \Delta_\phi^2(k) \approx \left(\frac{aH^2}{2\pi\dot{\phi}}\right)^2 = \left[\frac{H^2}{2\pi(d\phi/dt)}\right]^2. \tag{2.43}$$

This is the spectrum of ζ, $\Delta_\zeta^2(k)$, on super-horizon scales. While we have neglected the k dependence of the spectrum here, the spectral index of ζ is the same as of ϕ (Eq.(2.25)),

$$\frac{d\ln\Delta_\zeta^2}{d\ln k} = 4\frac{dH/dt}{H^2} - 2\frac{d^2\phi/dt^2}{H(d\phi/dt)}. \tag{2.44}$$

$\Delta_\zeta^2(k)$ gives the *primordial curvature-perturbation spectrum.* This is very important prediction of inflation, as it directly predicts the observables such as the CMB anisotropy spectrum and the matter fluctuation spectrum. Strictly speaking, $\Delta_\zeta^2(k)$ reduces to the curvature-perturbation spectrum in the comoving gauge.

To summarize, the quantum fluctuations generate the gauge-invariant perturbation, u, that reduces to either $\delta\phi_{\rm flat}$ or $(\dot{\phi}/aH)\mathcal{R}_{\rm com}$ depending on which gauge we use, either the spatially flat gauge or the comoving gauge. Hence, $\delta\phi_{\rm flat}$ and $(\dot{\phi}/aH)\mathcal{R}_{\rm com}$ are essentially equivalent to each other. The benefit of u is that it relates these two variables unambiguously, simplifying the transformation between $\delta\phi_{\rm flat}$ and $\mathcal{R}_{\rm com}$. This is a virtue of the linear perturbation theory; we do not have this simplification when dealing with non-linear perturbations for which we have to find non-linear transformation between $\delta\phi_{\rm flat}$ and $\mathcal{R}_{\rm com}$. The non-linear transformation actually makes $\mathcal{R}_{\rm com}$ weakly non-Gaussian, even if $\delta\phi_{\rm flat}$ is exactly Gaussian. We will see this in the next section.

Here, we have the generation of the primordial spatial curvature perturbations completed. In the next subsection, we will derive the CMB anisotropy spectrum.

2.3.4 Generation of primary CMB anisotropy

The metric perturbations perturb CMB, producing the CMB anisotropy on the sky. Among the metric perturbation variables, the curvature perturbations play a central role in producing the CMB anisotropy.

As we have shown in the previous subsection, the gauge-invariant perturbation, ζ, does not change in time on super-horizon scales throughout the cosmic evolution regardless of whether a scalar field, radiation, or matter dominates the universe. The intrinsic spatial curvature perturbation, $\mathcal{R}$, however, does change when equation of state of the universe, $w \equiv p/\rho$, changes. Since ζ remains constant, it may be useful to write the evolution of $\mathcal{R}$ in terms of ζ and w; however, $\mathcal{R}$ is *not* gauge invariant itself, but ζ is gauge invariant, so that the relation between $\mathcal{R}$ and ζ may look misleading.

Bardeen (1980) has introduced another gauge-invariant variable, Φ (or $\Phi_{\rm H}$ in the original notation), which reduces to $\mathcal{R}$ in the zero-shear gauge, or the Newtonian gauge, in which $B \equiv 0 \equiv H_{\rm T}$. Φ is given by

$$\Phi \equiv \mathcal{R} - \frac{aH}{k}\left(-B + \frac{\dot{H}_{\rm T}}{k}\right). \tag{2.45}$$

Here, the terms in the parenthesis represent the shear, or the anisotropic expansion rate, of the τ = constant hypersurfaces. While Φ represents the curvature perturbations in the zero-shear gauge, it also represents the shear in the spatially flat gauge in which $\mathcal{R} \equiv 0$. Using Φ, we may write ζ as

$$\zeta = \mathcal{R} - \frac{aH}{\dot{\phi}}\delta\phi = \Phi - \frac{aH}{k}\left(v_\phi - \frac{\dot{H}_{\rm T}}{k}\right), \tag{2.46}$$

where the terms in the parenthesis represent the gauge-invariant fluid velocity.

Why use Φ? We use Φ because it gives the closest analogy to the Newtonian potential, for Φ reduces to $\mathcal{R}$ in the zero-shear gauge (or the Newtonian gauge) in which the metric (Eq.(2.29)) becomes just like the Newtonian limit of the general relativity. It thus gives a natural connection to the ordinary Newtonian analysis.

The gauge-invariant velocity term, $v - k^{-1}\dot{H}_{\rm T}$, differs ζ from Φ. In other words, the velocity and Φ share the value of ζ. Since a fraction of sharing depends upon equation of state of the universe, $w = p/\rho$, the velocity and Φ change as w changes. ζ is independent of w.

The general relativistic cosmological linear perturbation theory gives the evolution of Φ on super-horizon scales (Kodama and Sasaki, 1984),

$$\Phi = \frac{3+3w}{5+3w}\zeta, \tag{2.47}$$

for adiabatic fluctuations, and hence $\Phi = \frac{2}{3}\zeta$ in the radiation era ($w = 1/3$), and $\Phi = \frac{3}{5}\zeta$ in the matter era ($w = 0$). Φ then perturbs CMB through the so-called (static) Sachs–Wolfe effect (Sachs and Wolfe, 1967).

The Sachs–Wolfe effect predicts that CMB that resides in a Φ potential well initially has an initial adiabatic temperature fluctuation of $\Delta T/T = [2/3(1+w)]\Phi$, and it further receives an

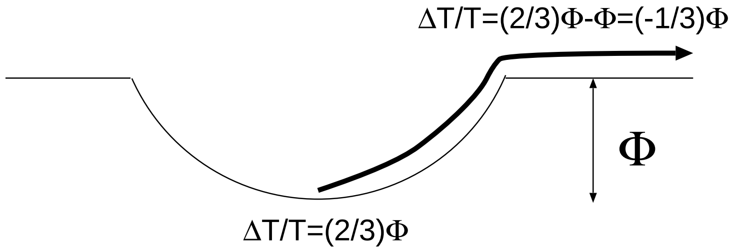

Figure 2.4: The Static Sachs–Wolfe Effect

The static Sachs–Wolfe effect predicts that CMB that resides in a Φ potential well has an initial adiabatic temperature fluctuation of $\Delta T/T = \frac{2}{3}\Phi$ in the matter era. It further receives an additional fluctuation of $-\Phi$ when climbing up the potential at the decoupling epoch. In total, we observe $\Delta T/T = -\frac{1}{3}\Phi$.

additional fluctuation of $-\Phi$ when climbing up the potential at the decoupling epoch. In total, the CMB temperature fluctuations that we observe today amount to

$$\frac{\Delta T}{T} = \frac{2}{3(1+w)}\Phi - \Phi = -\frac{1+3w}{3+3w}\Phi = -\frac{1+3w}{5+3w}\zeta. \tag{2.48}$$

Figure 2.4 sketches the static Sachs–Wolfe effect.

For isocurvature fluctuations, initial temperature fluctuations in a potential well are given by $-\Phi$ in both the radiation era and the matter era; thus, total temperature fluctuations amount to $\Delta T/T = -\Phi - \Phi = -2\Phi$. By definition of the isocurvature fluctuations, Φ is initially zero, but there exist non-vanishing initial entropy fluctuations. As the universe evolves, the entropy fluctuations create Φ, and hence the temperature fluctuations.

At the decoupling epoch, the universe has already been in the matter era in which $w = 0$, so that we observe adiabatic temperature fluctuations of $\Delta T/T = -\frac{1}{3}\Phi = -\frac{1}{5}\zeta$, and the CMB fluctuation spectrum of the Sachs–Wolfe effect, $\Delta^2_{\rm SW}(k)$, is

$$\Delta^2_{\rm SW}(k) = \frac{1}{9}\Delta^2_{\Phi}(k) = \frac{1}{25}\Delta^2_{\zeta}(k) = \left[\frac{H^2}{10\pi(d\phi/dt)}\right]^2, \tag{2.49}$$

where H is the Hubble parameter during inflation. While we have not shown the k dependence of the spectrum here, the spectral index is given by equation (2.44). By projecting the 3-dimension CMB fluctuation spectrum, $\Delta^2_{\rm SW}(k)$, on the sky, we obtain the angular power spectrum, C_l (Bond and Efstathiou, 1987),

$$C_l^{\rm SW} = 4\pi \int_0^\infty \frac{dk}{k}\, \Delta^2_{\rm SW}(k) j_l^2\left[k(\tau_0 - \tau_{\rm dec})\right] = C_2^{\rm SW} \frac{\Gamma\left[(9-n)/2\right]\Gamma\left[l + (n-1)/2\right]}{\Gamma\left[(n+3)/2\right]\Gamma\left[l + (5-n)/2\right]}, \tag{2.50}$$

where τ_0 and $\tau_{\rm dec}$ denote the present day and the decoupling epoch, respectively, and $n \equiv 1 + \left[d\ln\Delta^2(k)/d\ln k\right]$ is a spectral index which is conventionally used in the literature. If the spectrum is exactly scale invariant, $n = 1$, then we obtain $C_l^{\rm SW} = \left[l(l+1)\right]^{-1} 6C_2^{\rm SW}$.

Equation (2.50) provides a simple yet good fit to the CMB power spectrum measured by *COBE* DMR. The spectrum comprises two parameters, $C_2^{\rm SW}$ and n. Bennett et al. (1996) find $n = 1.2 \pm 0.3$ and $C_2^{\rm SW} = 7.9^{+2.0}_{-1.4} \times 10^{-11}$. When fixing $n = 1$, they find $C_2^{\rm SW|n=1} = (9.3 \pm 0.8) \times 10^{-11}$. The measured C_l is thus consistent with CMB being scale invariant, supporting inflation. Moreover, it implies that $H^2/(d\phi/dt) \sim \left(C_2^{\rm SW}\right)^{1/2} \sim 10^{-5}$, constraining amplitude of the Hubble parameter during inflation.

On the angular scales smaller than the DMR angular scales, the Sachs–Wolfe approximation breaks down, and the acoustic physics in the photon-baryon fluid system modifies the primordial radiation spectrum (Peebles and Yu, 1970; Bond and Efstathiou, 1987). To calculate the modification, we have to solve the Boltzmann photon transfer equation together with the Einstein equations. The modification is often described by the *radiation transfer function*, $g_{\rm T\it l}(k)$, which can be calculated numerically with the Boltzmann code such as CMBFAST (Seljak and Zaldarriaga, 1996). Using $g_{\rm T\it l}(k)$, we write the CMB power spectrum, C_l, as

$$C_l = 4\pi \int_0^\infty \frac{dk}{k}\ \Delta_\Phi^2(k) g_{\rm T\it l}^2(k). \tag{2.51}$$

Note that for the static Sachs–Wolfe effect, adiabatic fluctuations give $g_{\rm T\it l}(k) = -\frac{1}{3} j_l\left[k(\tau_0 - \tau_{\rm dec})\right]$, while isocurvature fluctuations give $g_{\rm T\it l}(k) = -2 j_l\left[k(\tau_0 - \tau_{\rm dec})\right]$. Here, we have used $\Delta_\Phi^2(k)$ rather than $\Delta_{\rm SW}^2(k)$ or $\Delta_\zeta^2(k)$, following the literature. The literature often uses the Φ power spectrum, $P_\Phi(k)$, to replace $\Delta_\Phi^2(k)$; the relation is $\Delta_\Phi^2(k) = (2\pi^2)^{-1} k^3 P_\Phi(k)$. $\Delta_\Phi^2(k)$ is called the dimensionless power spectrum.

If Φ were exactly Gaussian, then C_l would specify all the statistical properties of Φ, which are equivalent to those of ζ. Since ζ is related to a Gaussian variable, u, through $\zeta = -(aH/\dot\phi)u$, in the linear order ζ also obeys Gaussian statistics; however, the relation between ζ and u becomes *non-linear* when we take into account non-linear perturbations. As a result, ζ, and hence Φ, becomes non-Gaussian even if u is exactly Gaussian, yielding non-Gaussian CMB anisotropies. In the next section, we will analyze non-linear perturbations in inflation.

Using the second-order gravitational perturbation theory, Pyne and Carroll (1996) derive the second-order correction to the relation between ΔT and Φ (Eq.(2.48)). It gives $\Delta T/T = -\frac{1}{3}\Phi + \mathcal{O}(1)\Phi^2$; thus, even if Φ is Gaussian, ΔT becomes weakly non-Gaussian.

2.4 Non-linear Perturbations in Inflation

In the previous section, we have shown that the quantum fluctuations generate the gauge-invariant perturbation, $u = \delta\phi - (\dot\phi/aH)\mathcal{R}$, and u obeys Gaussian statistics very well because of the central limit theorem. Another gauge-invariant variable, $\zeta = -(aH/\dot\phi)u = \mathcal{R} - (aH/\dot\phi)\delta\phi$, which remains constant in time outside the horizon, also obeys Gaussian statistics in the linear order, as it is related to a Gaussian variable, u, linearly.

In the non-linear order, however, the situation may change. In the relation between ζ and u, the factor in front of u, $aH/\dot\phi$, is also a function of ϕ, and it may produce additional fluctuations like $[\partial(aH/\dot{\phi})/\partial\phi]\delta\phi$. Suppose that $\mathcal{R}$ in the comoving gauge ($\delta\phi_{\rm com} \equiv 0$), $\mathcal{R}_{\rm com}$, is an arbitrary function of a scalar field: $\mathcal{R}_{\rm com} = f(\phi)$. Note that $\mathcal{R}_{\rm com}$ is equivalent to ζ. By perturbing ϕ as $\phi = \phi_0 + \delta\phi_{\rm flat}$, where $\delta\phi_{\rm flat}$ is a scalar-field fluctuation in the spatially flat gauge ($\mathcal{R}_{\rm flat} \equiv 0$), we have

$$\begin{aligned} \mathcal{R}_{\rm com} &= f(\phi_0 + \delta\phi_{\rm flat}) \\ &= f(\phi_0) + \left(\frac{\partial f}{\partial\phi}\right)\delta\phi_{\rm flat} + \frac{1}{2}\left(\frac{\partial^2 f}{\partial\phi^2}\right)\delta\phi^2_{\rm flat} + \mathcal{O}\left(\delta\phi^3_{\rm flat}\right). \end{aligned} \tag{2.52}$$

By comparing this equation with the linear-perturbation result (Eq.(2.39)), $\mathcal{R}_{\rm com} = -(aH/\dot{\phi})\delta\phi_{\rm flat}$, we find $f(\phi_0) = 0$, $\partial f/\partial\phi = -(aH/\dot{\phi})$, and

$$\mathcal{R}_{\rm com} = -\frac{aH}{\dot{\phi}}\delta\phi_{\rm flat} - \frac{1}{2}\frac{\partial}{\partial\phi}\left(\frac{aH}{\dot{\phi}}\right)\delta\phi^2_{\rm flat} + \mathcal{O}\left(\delta\phi^3_{\rm flat}\right); \tag{2.53}$$

thus, even if $\delta\phi_{\rm flat}$ is exactly Gaussian, $\mathcal{R}_{\rm com}$, and hence ζ, becomes weakly non-Gaussian because of $\delta\phi^2_{\rm flat}$ or the higher-order terms. While the treatment here may look rather crude, we will show in this section that the solution (Eq.(2.53)) actually satisfies a more proper treatment of non-linear perturbations in inflation.

In the linear regime, we have the gauge-invariant perturbation variable that characterizes the curvature perturbations as well as the scalar-field fluctuations, and the single equation that describes the perturbation evolution on all scales. In the non-linear regime, however, we cannot make such great simplification. Since the Einstein equations are highly non-linear, fully analyzing non-linear problems is technically very difficult. Hence, we need a certain approximation.

2.4.1 Gradient expansion of Einstein equations

In inflation, there is an useful scheme of approximation, the so-called anti-Newtonian approximation (Tomita, 1975, 1982; Tomita and Deruelle, 1992), or later called the long-wavelength approximation (Kodama and Hamazaki, 1998; Sasaki and Tanaka, 1998) or the *gradient expansion method* (Salopek and Bond, 1990; Salopek and Stewart, 1992; Nambu and Taruya, 1996, 1998).

The approximation neglects higher-order spatial derivatives in the Einstein equations as well as in the equations of motion for matter fields, and is equivalent to taking a long-wavelength limit of the system. The equation system is further simplified if we set the shift vector zero in the metric. Once neglecting higher-order spatial derivatives and the shift vector, one finds that the shear decays away very rapidly.

We use the metric of the form

$$ds^2 = -N^2 d\tau^2 + {}^{(3)}g_{ij}dx^i dx^j, \tag{2.54}$$

where N is the Lapse function, and ${}^{(3)}g_{ij}$ describes the 3-metric. We have set the shift vector zero; it corresponds to $B \equiv 0$ in the linearized metric (2.29). We perturb N and ${}^{(3)}g_{ij}$ non-linearly. While the full treatment of non-linear evolution of the system is highly complicated, by neglecting higher-order spatial derivatives, we reduce the Einstein equations to rather simplified forms (Salopek and Bond, 1990):

$$\begin{aligned} H^2 - \frac{1}{3}\sigma^{ij}\sigma_{ij} &= \frac{8\pi G}{3}\left[\frac{1}{2N^2}\dot{\phi}^2 + V(\phi)\right], &(2.55)\\ H_{,i} - \frac{1}{2}\sigma^k_{i,k} &= -4\pi G\frac{\dot{\phi}}{N}\phi_{,i}, &(2.56)\\ \frac{2}{N}\dot{H} + 3H^2 + \sigma^{ij}\sigma_{ij} &= -8\pi G\left[\frac{1}{2N^2}\dot{\phi}^2 - V(\phi)\right], &(2.57)\\ \frac{1}{N}\dot{\sigma}^i_j + 3H\sigma^i_j &= 0. &(2.58)\end{aligned}$$

H is the inhomogeneous Hubble parameter which defines the isotropic expansion rate of the $\tau =$ constant hypersurfaces,

$$H \equiv \frac{1}{6N}{}^{(3)}g^{ij\,(3)}\dot{g}_{ij}, \tag{2.59}$$

or ${}^{(3)}\dot{g}_{ij} = 2NH^{(3)}g_{ij}$. By introducing the inhomogeneous scale factor, $a(\mathbf{x},\tau)$, given by $H = \dot{a}/(Na)$, we can write the 3-metric as ${}^{(3)}g_{ij} = a^2(\mathbf{x},\tau)\gamma_{ij}(\mathbf{x})$, where $\gamma_{ij}(\mathbf{x})$ is a function of the spatial coordinate only. This form of ${}^{(3)}g_{ij}$ satisfies equation (2.59). We have obtained the simplified form for the 3-metric because of setting the shift vector zero.

σ^i_j is the shear which quantifies the anisotropic expansion rate,

$$\sigma^i_j \equiv \frac{1}{2N}{}^{(3)}\dot{g}_{ij} - H^{(3)}g_{ij}. \tag{2.60}$$

It follows from the traceless-part equation (2.58) and $NH = \dot{a}/a$ that the shear decays rapidly as the universe expands: $\sigma^i_j \propto a^{-3}$; thus, we neglect the shear terms in the Einstein equations henceforth.

By neglecting the shear term in the trace-part equation (2.57), and substituting the Friedmann equation (2.55) for H^2, we obtain $\dot{H} = -4\pi GN^{-1}\dot{\phi}^2$. Comparing this equation with the momentum constraint equation (2.56) without the shear term, $H_{,i} = -4\pi GN^{-1}\dot{\phi}\phi_{,i}$, we find $H(\mathbf{x},\tau) = H\left(\phi(\mathbf{x},\tau)\right)$, and the scalar-field momentum

$$\frac{\dot{\phi}}{N} = -\frac{1}{4\pi G}\left(\frac{\partial H}{\partial\phi}\right). \tag{2.61}$$

Substituting this for the kinetic term in the Friedmann equation (2.55), and neglecting the shear term, we finally obtain a closed evolution equation for $H(\phi)$,

$$H^2(\phi) = \frac{1}{12\pi G}\left(\frac{\partial H}{\partial\phi}\right)^2 + \frac{8\pi G}{3}V(\phi). \tag{2.62}$$

From this equation, $H(\phi)$ may be solved as $H(\phi, I)$, where I is an integration constant which parameterizes the initial condition. This equation fully describes the evolution of the system including non-linear perturbations. Note that $H(\phi)$ depends upon the spatial coordinate through $\phi = \phi(\mathbf{x},\tau)$.

A perturbation to H is given by $\delta H = (\partial H/\partial\phi)_I\,\delta\phi + (\partial H/\partial I)_\phi\,\delta I$. For the latter term, by differentiating equation (2.62) with respect to I for a fixed ϕ, we find $(\partial H/\partial I)_\phi \propto a^{-3}$; thus, it decays very rapidly during inflation, giving $\delta H = (\partial H/\partial\phi)_I\,\delta\phi$. Hence, the comoving gauge, $\delta\phi \equiv 0$, coincides with the constant Hubble parameter gauge, $\delta H \equiv 0$. Sasaki and Tanaka (1998) also observe this property from a different point of view, and find that this property holds for multiple scalar-field system as well.

2.4.2 Generation of non-linear curvature perturbations

Our goal in this subsection is to find a non-linear relation between $\mathcal{R}_{\rm com}$ and $\delta\phi_{\rm flat}$, following Salopek and Bond (1990). In the absence of the shear, $\mathcal{R}$ obeys $\dot{\mathcal{R}} = \delta\,(NH) = \delta\,(\partial\ln a/\partial\tau)$, and hence $\mathcal{R}$ is equivalent to fluctuations in the inhomogeneous scale factor, $\delta\ln a$.

We calculate $\mathcal{R}_{\rm com}$ by perturbing $\ln a$ non-linearly in the comoving gauge. Since ϕ is homogeneous in the comoving gauge, we choose ϕ as a time coordinate: $\tau \equiv \phi$. We find the Lapse function

$$N = -\frac{4\pi G}{\partial H/\partial\phi} \tag{2.63}$$

for this time coordinate from equation (2.61) with setting $\dot{\phi} \equiv 1$.

To calculate the scalar-field fluctuations in the spatially flat gauge, $\delta\phi_{\rm flat}$, we need to generate quantum fluctuations first; however, since we are solving the equation system on super-horizon scales only, we cannot calculate quantum fluctuations of $\delta\phi$ within the current framework. Instead, we assume that $\delta\phi$ on super-horizon scales is provided by the small-scale quantum fluctuations that are stretched out of the horizon by inflationary expansion. We use the linear perturbation theory to calculate the fluctuation amplitude at the horizon crossing, and provide $\delta\phi$ as initially linear, Gaussian fluctuations. We then calculate $\delta\phi_{\rm flat}$ on the $\ln a =$ constant hypersurfaces. While this treatment may sacrifice a virtue of the current framework which does not assume linearity of perturbations, non-linearity of the scalar-field fluctuations may also be incorporated into the analysis with the so-called stochastic inflation approach (Starobinsky, 1986; Salopek and Bond, 1991). Using this, Gangui et al. (1994) show that the non-linearity makes $\delta\phi_{\rm\ flat}$ weakly non-Gaussian.

After all, our goal is to relate $\delta\phi(\ln a)$ to $\delta\ln a(\phi)$. In other words, we transform the perturbations on the $\ln a =$ constant hypersurfaces to the ones on the $\phi =$ constant hypersurfaces. We do this as follows (Salopek and Bond, 1990), and figure (2.5) shows the following process schematically.

First, imagine $\phi - \ln a$ plane on which we transform the perturbations. We then mark a point on the plane with $(\phi_0, \ln a_0)$, and draw a short line from this point to $(\phi_0 + \delta\phi, \ln a_0)$ in parallel to the ϕ axis. This line represents $\delta\phi(\mathbf{x}, \ln a_0) \equiv \phi(\mathbf{x}, \ln a_0) - \phi_0$, i.e., ϕ perturbations on a $\ln a = \ln a_0$ hypersurface. Next, using evolution equation of $\ln a$, we evolve the line until it coincides with the $\phi = \phi_0$ line, namely

$$\ln a_0 = \ln\left[a(\mathbf{x}, \phi_0 + \delta\phi)\right] \longrightarrow \ln\left[a(\mathbf{x}, \phi_0)\right]. \tag{2.64}$$

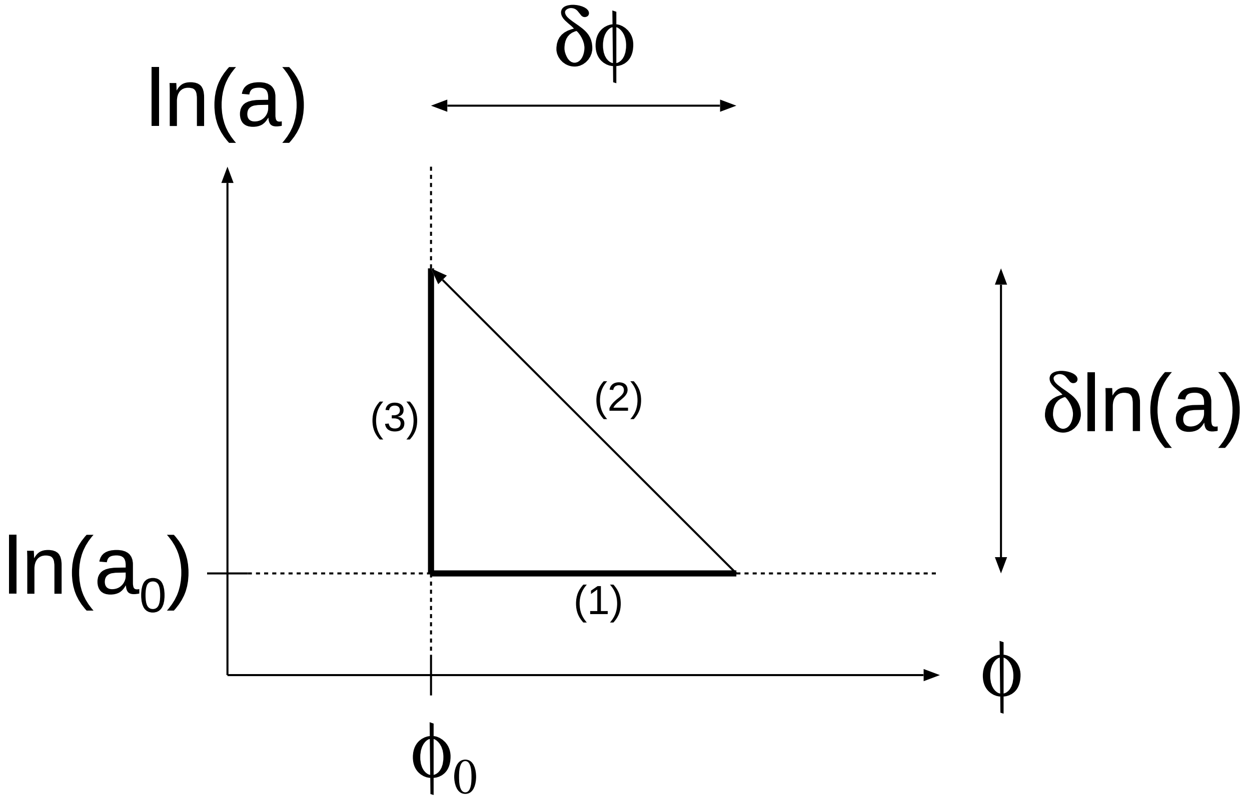

Figure 2.5: Hypersurface Transformation

Hypersurface transformation of the scalar-field fluctuations in the spatially flat gauge, $\delta\phi_{\rm flat} = \delta\phi(\ln a_0)$, into the spatial curvature perturbations in the comoving gauge, $\mathcal{R}_{\rm com} = \delta \ln a(\phi_0)$. (1) To represent $\delta\phi(\ln a_0)$, draw a short line from $(\phi_0, \ln a_0)$ to $(\phi_0 + \delta\phi, \ln a_0)$ in parallel to the ϕ axis. (2) By evolving $\ln a$, evolve this line until it coincides with the $\phi = \phi_0$ line. (3) Measure the line length, $\delta \ln a(\phi_0)$.

The evolution equation of $\ln a$ is $d\ln a/d\tau = NH$. By integrating this equation, we obtain

$$\ln\left[a(\mathbf{x},\phi_0)\right] - \ln a_0 = \int_{\phi_0+\delta\phi(\mathbf{x},\ln a_0)}^{\phi_0} d\phi\, \frac{N(\phi)H(\phi)}{\dot{\phi}}, \tag{2.65}$$

where the Lapse function, N, is given by equation (2.63) for ϕ being the time coordinate. Finally, we measure the line length, $\delta\ln\left[a(\mathbf{x},\phi_0)\right] \equiv \ln\left[a(\mathbf{x},\phi_0)\right] - \ln a_0$, to obtain $\mathcal{R}_{\rm com}$.

By noting $\delta\phi_{\rm flat} = \delta\phi(\mathbf{x},\ln a_0)$, we find a *non-linear* relationship between $\mathcal{R}_{\rm com}$ and $\delta\phi_{\rm flat}$,

$$\mathcal{R}_{\rm com} = -\int_{\phi_0}^{\phi_0+\delta\phi_{\rm flat}} d\phi\, \frac{N(\phi)H(\phi)}{\dot{\phi}} = 4\pi G\int_{\phi_0}^{\phi_0+\delta\phi_{\rm flat}} d\phi \left[\frac{\partial\ln H}{\partial\phi}\right]^{-1}. \tag{2.66}$$

This non-linear relation should be compared with the crude estimate from the linear analysis, equation (2.53). By expanding the non-linear relation into Taylor series with respect to $\delta\phi_{\rm flat}$, we find that the crude estimate agrees with the non-linear result on the term-by-term basis. Note that a in equation (2.53) should be replaced by N, as it has used the conformal time as the time coordinate, i.e., $N = a$.

2.4.3 Generation of weakly non-Gaussian adiabatic fluctuations

We have shown that non-linearity in inflation creates weakly non-Gaussian curvature perturbations outside the horizon from Gaussian quantum fluctuations inside the horizon. By expanding the non-linear relation between $\mathcal{R}_{\rm com}$ and $\delta\phi_{\rm flat}$ up to the second order, we obtain a non-linear curvature perturbation, $\mathcal{R}_{\rm com} = \mathcal{R}_{\rm com}^{\rm L} + \mathcal{R}_{\rm com}^{\rm NL}$, where

$$\mathcal{R}_{\rm com}^{\rm L} \equiv 4\pi G\left(\frac{\partial\ln H}{\partial\phi}\right)^{-1}\delta\phi_{\rm flat}, \tag{2.67}$$

$$\mathcal{R}_{\rm com}^{\rm NL} \equiv -\frac{1}{8\pi G}\left(\frac{\partial^2\ln H}{\partial\phi^2}\right)\left(\mathcal{R}_{\rm com}^{\rm L}\right)^2. \tag{2.68}$$

Using equation (2.46), we obtain a non-linear Newtonian potential, $\Phi = \Phi_{\rm L} + \Phi_{\rm NL}$, where $\Phi_{\rm L} = \frac{2}{3}\mathcal{R}_{\rm com}^{\rm L}$ and $\Phi_{\rm NL} = \frac{2}{3}\mathcal{R}_{\rm com}^{\rm NL}$ in the radiation era, and $\Phi_{\rm L} = \frac{3}{5}\mathcal{R}_{\rm com}^{\rm L}$ and $\Phi_{\rm NL} = \frac{3}{5}\mathcal{R}_{\rm com}^{\rm NL}$ in the matter era. Note that these formulae should be modified when we include non-linear effects of the stochastic inflation approach (Gangui et al., 1994), which add extra terms to the formulae.

In the following chapters, we will focus on measuring $\Phi_{\rm NL}$ using non-Gaussian CMB temperature fluctuations. We parameterize the amplitude of $\Phi_{\rm NL}$ with a *non-linear coupling parameter*, $f_{\rm NL}$, as

$$\Phi_{\rm NL}(\mathbf{x}) = f_{\rm NL}\left[\Phi_{\rm L}^2(\mathbf{x}) - \left\langle\Phi_{\rm L}^2(\mathbf{x})\right\rangle\right]. \tag{2.69}$$

How big is $f_{\rm NL}$? If a scalar field rolls down on a potential, $V(\phi)$, slowly, then $\partial\ln H/\partial\phi \approx \frac{1}{2}\partial\ln V/\partial\phi$. We thus obtain

$$f_{\rm NL} = -\frac{5}{24\pi G}\left(\frac{\partial^2\ln H}{\partial\phi^2}\right) \approx -\frac{5}{48\pi G}\left(\frac{\partial^2\ln V}{\partial\phi^2}\right). \tag{2.70}$$

As an example, consider a power-law potential, $V(\phi) \propto \phi^n$. We find

$$f_{\rm NL} = \frac{5n}{48\pi}\frac{m_{\rm pl}^2}{\phi^2} \sim 10^{-2}. \tag{2.71}$$

On the other hand, we will show in chapter 4 that $|f_{\rm NL}|$ should be larger than of order unity to produce detectable non-Gaussian signals in CMB. Even for the fastest motion of ϕ allowed in slowly-rolling single-field inflation, we find $|f_{\rm NL}| < 5/2$. Therefore, it is quite hard for this class of inflation models to produce detectable CMB non-Gaussianity. In other words, any detection of cosmological non-Gaussian signals in CMB constrains slow-roll inflation models very strongly, or may favor multiple scalar-field models or isocurvature fluctuations.