Appendix D

Angular Trispectrum in Closed Hyperbolic Universe

In this appendix, we derive the angular connected trispectrum in a closed hyperbolic universe. If topology of our universe is infinite flat, then we can expand the harmonic coefficients of the CMB anisotropy into Fourier series,

$$a_{lm} = 4\pi(-1)^l \int \frac{d^3\mathbf{k}}{(2\pi)^3} \Phi(\mathbf{k}) g_{\mathrm{T}l}(k) Y^*_{lm}(\hat{\mathbf{k}}), \tag{D.1}$$

where $\Phi(\mathbf{k})$ is the primordial curvature perturbation, and $g_{\mathrm{T}l}(k)$ is the radiation transfer function. Hence, statistical properties of a_{lm} take over statistical properties of Φ directly; if Φ is Gaussian, then a_{lm} is also Gaussian.

If topology of our universe is closed hyperbolic, then the relation between a_{lm} and Φ becomes (Inoue, 2001b)

$$a_{lm} = \sum_{\nu} \Phi(\nu) g_{\mathrm{T}l}(\nu) \xi_{lm}(\nu), \tag{D.2}$$

where $\nu = \sqrt{k^2 - 1}$ is a discrete wavenumber, and new expansion coefficients, $\xi_{lm}(\nu)$, describe eigenmodes in a closed hyperbolic geometry. The complex conjugate of ξ_{lm} is given by $\xi^*_{lm}(\nu) = (-1)^m \xi_{l-m}(\nu)$. Since a closed hyperbolic universe is globally inhomogeneous, $\xi_{lm}(\nu)$ is also a function of observer's positions and orientations in the universe. Inoue (2001b) have shown that this property makes $\xi_{lm}(\nu)$ behave as if it were a Gaussian random number with the covariance matrix diagonal,

$$\left\langle \xi_{lm}(\nu) \xi_{l'm'}(\nu') \right\rangle = \left\langle \left|\xi_{lm}(\nu)\right|^2 \right\rangle (-1)^{m'} \delta_{ll'} \delta_{m-m'} \delta_{\nu\nu'}, \tag{D.3}$$

and $\left\langle \left|\xi_{lm}(\nu)\right|^2 \right\rangle \propto \nu^{-2}$, where the bracket denotes the ensemble average over observer's positions and orientations in the universe. Hence, in addition to Φ, statistical properties of ξ_{lm} also affect statistical properties of a_{lm}. If Φ is Gaussian, then it follows from the relation among a_{lm}, Φ, and ξ_{lm} (Eq.(D.2)) that a_{lm} comprises two independent Gaussian random numbers.

The angular power spectrum, C_l, is given by

$$\langle C_l \rangle = \left\langle |a_{lm}|^2 \right\rangle = \sum_\nu P_\Phi(\nu) g_{\mathrm{T}l}^2(\nu) \left\langle |\xi_{lm}(\nu)|^2 \right\rangle, \tag{D.4}$$

where $P_\Phi(\nu)\delta_{\nu\nu'} = \langle \Phi(\nu)\Phi^*(\nu') \rangle$. The ensemble average for Φ is taken over initial conditions. We should keep in mind that the ensemble average for ξ_{lm} is taken over all possible observer's positions and orientations in the universe. Hence, there exist those observers who measure C_l different from $\langle C_l \rangle$ substantially. In other words, there exists theoretical uncertainty in $\langle C_l \rangle$ arising from uncertainty as to where we are.

We calculate the angular trispectrum from equation (D.2) as follows. We begin with

$$\begin{aligned} \langle a_{l_1m_1} a_{l_2m_2} a_{l_3m_3} a_{l_4m_4} \rangle &= \sum_{\text{all } \nu} g_{\mathrm{T}l_1}(\nu_1) g_{\mathrm{T}l_2}(\nu_2) g_{\mathrm{T}l_3}(\nu_3) g_{\mathrm{T}l_4}(\nu_4) \\ &\quad \times \langle \Phi(\nu_1)\Phi(\nu_2)\Phi(\nu_3)\Phi(\nu_4) \rangle \\ &\quad \times \langle \xi_{l_1m_1}(\nu_1)\xi_{l_2m_2}(\nu_2)\xi_{l_3m_3}(\nu_3)\xi_{l_4m_4}(\nu_4) \rangle . \end{aligned} \tag{D.5}$$

If Φ is Gaussian, then this equation yields 9 terms. First, we reduce the second line to

$$\begin{aligned} \langle \Phi(\nu_1)\Phi(\nu_2)\Phi(\nu_3)\Phi(\nu_4) \rangle &= P_\Phi(\nu_1)P_\Phi(\nu_3)\delta_{\nu_1\nu_2}\delta_{\nu_3\nu_4} \\ &\quad + P_\Phi(\nu_1)P_\Phi(\nu_2)\left(\delta_{\nu_1\nu_3}\delta_{\nu_2\nu_4} + \delta_{\nu_1\nu_4}\delta_{\nu_2\nu_3}\right). \end{aligned} \tag{D.6}$$

We then evaluate the first three terms in the trispectrum:

$$\begin{aligned} & \sum_{\nu_1\nu_3} g_{\mathrm{T}l_1}(\nu_1) g_{\mathrm{T}l_2}(\nu_1) g_{\mathrm{T}l_3}(\nu_3) g_{\mathrm{T}l_4}(\nu_3) P_\Phi(\nu_1) P_\Phi(\nu_3) \\ & \times \left[\langle \xi_{l_1m_1}(\nu_1)\xi_{l_2m_2}(\nu_1) \rangle \langle \xi_{l_3m_3}(\nu_3)\xi_{l_4m_4}(\nu_3) \rangle \right. \\ & + \langle \xi_{l_1m_1}(\nu_1)\xi_{l_3m_3}(\nu_3) \rangle \langle \xi_{l_2m_2}(\nu_1)\xi_{l_4m_4}(\nu_3) \rangle \\ & \left. + \langle \xi_{l_1m_1}(\nu_1)\xi_{l_4m_4}(\nu_3) \rangle \langle \xi_{l_2m_2}(\nu_1)\xi_{l_3m_3}(\nu_3) \rangle \right] \\ = \; & \langle C_{l_1} \rangle \langle C_{l_3} \rangle (-1)^{m_2+m_4} \delta_{l_1l_2}\delta_{l_3l_4}\delta_{m_1-m_2}\delta_{m_3-m_4} \\ & + F_{l_1l_2}(-1)^{m_3+m_4}\left(\delta_{l_1l_3}\delta_{l_2l_4}\delta_{m_1-m_3}\delta_{m_2-m_4} + \delta_{l_1l_4}\delta_{l_2l_3}\delta_{m_1-m_4}\delta_{m_2-m_3}\right), \end{aligned} \tag{D.7}$$

where

$$F_{l_il_j} \equiv \sum_\nu P_\Phi^2(\nu) g_{\mathrm{T}l_i}^2(\nu) g_{\mathrm{T}l_j}^2(\nu) \left\langle |\xi_{l_im_i}(\nu)|^2 \right\rangle \left\langle \left|\xi_{l_jm_j}(\nu)\right|^2 \right\rangle. \tag{D.8}$$

The first term in the r.h.s. of equation (D.7) is the unconnected term, while the second and third terms are the connected ones. By collecting all the connected terms, we obtain

$$\begin{aligned} & \langle a_{l_1m_1} a_{l_2m_2} a_{l_3m_3} a_{l_4m_4} \rangle_{\mathrm{c}} \\ = \; & F_{l_1l_2}(-1)^{m_3+m_4}\left(\delta_{l_1l_3}\delta_{l_2l_4}\delta_{m_1-m_3}\delta_{m_2-m_4} + \delta_{l_1l_4}\delta_{l_2l_3}\delta_{m_1-m_4}\delta_{m_2-m_3}\right) \\ & + (2 \leftrightarrow 3) + (2 \leftrightarrow 4). \end{aligned}$$

Actually, the functional form of this equation is exactly the same as the unconnected terms, if we replace $2F_{l_il_j}$ with $\langle C_{l_i} \rangle \left\langle C_{l_j} \right\rangle$. Hence, by substituting $2F_{l_il_j}$ for $\langle C_{l_i} \rangle \left\langle C_{l_j} \right\rangle$ in the formula of the unconnected trispectrum (Eq.(3.25)), we find the angular connected trispectrum in a closed hyperbolic universe

$$\left\langle T^{l_1 l_2}_{l_3 l_4}(L) \right\rangle_{\rm c} = (-1)^{l_1+l_3}\sqrt{(2l_1+1)(2l_3+1)}\,(2F_{l_1 l_3})\,\delta_{l_1 l_2}\delta_{l_3 l_4}\delta_{L0} + (2L+1)\,(2F_{l_1 l_2})\left[(-1)^{l_2+l_3+L}\delta_{l_1 l_3}\delta_{l_2 l_4} + \delta_{l_1 l_4}\delta_{l_2 l_3}\right]. \quad \text{(D.9)}$$

Unfortunately, the unconnected terms always contaminate the connected terms produced in a closed hyperbolic universe: there is no non-Gaussian signal in the modes for which the unconnected terms vanish. Since what we measure is the sum of the unconnected terms and the connected terms, $\langle C_{l_i}\rangle \left\langle C_{l_j}\right\rangle + 2F_{l_i l_j}$, we have to subtract $\langle C_{l_i}\rangle \left\langle C_{l_j}\right\rangle$ from measurement to obtain $2F_{l_i l_j}$. Although $F_{l_i l_j}$ is difficult to measure because of the contamination, if we detect non-Gaussian signals that exist only in these limited modes ($L = 0$, or $l_1 = l_2 = l_3 = l_4$), then it may suggest that the universe is closed hyperbolic.

The connected trispectrum for $L = 0$ contributes to the power spectrum covariance matrix (Eq.(3.12)). Using our prediction for the connected trispectrum (Eq.(D.9)) for $L = 0$, we obtain the power spectrum covariance matrix in a closed hyperbolic universe,

$$\langle C_l C_{l'}\rangle - \langle C_l\rangle \langle C_{l'}\rangle = \frac{2}{2l+1}\left(\langle C_l\rangle^2 + 2F_{ll}\right)\delta_{ll'} + 2F_{ll'}. \quad \text{(D.10)}$$

The variance is amplified, and the off-diagonal terms appear in the covariance matrix; these properties may be used to search for a signature of closed hyperbolic geometry in the CMB angular power spectrum covariance.