Appendix C

Angular Bispectrum from Isocurvature Fluctuations

In this appendix, we derive the angular bispectrum from isocurvature fluctuations generated in inflation. The mechanism of generating isocurvature fluctuations we consider here is that of Linde and Mukhanov (1997): a massive-free scalar field, σ, oscillating about $\sigma = 0$ during inflation.

Quantum fluctuations of σ produce Gaussian fluctuations, $\delta\sigma$. Since there is no mean σ-field because of the oscillation about $\sigma = 0$ in the model, the model predicts density fluctuations which are *quadratic* in $\delta\sigma$:

$$\delta\rho_\sigma \sim m^2\sigma\delta\sigma + m^2(\delta\sigma)^2 = m^2(\delta\sigma)^2. \tag{C.1}$$

$\delta\rho_\sigma$ is thus non-Gaussian. Moreover, since the energy density of σ does not dominate the universe during inflation, $\delta\rho_\sigma$ does not perturb the spatial curvature, being isocurvature density fluctuations.

After inflation, $\delta\rho_\sigma$ may produce the spatial curvature perturbations in the Newtonian gauge, Φ, through the evolution. If $\delta\rho_\sigma$ becomes dominant in the universe at some point, then the linear perturbation theory gives $\Phi = \frac{1}{8}(\delta\rho_\sigma/\rho_\sigma)(a/a_{\rm eq})$ in the radiation era, and $\Phi = \frac{1}{5}(\delta\rho_\sigma/\rho_\sigma)$ in the matter era (Kodama and Sasaki, 1984). Φ then produces CMB fluctuations through the Sachs–Wolfe effect, $\Delta T/T = -2\Phi$. Since $\delta\rho_\sigma$ is non-Gaussian, Φ is also non-Gaussian, so is $\Delta T/T$.

Our goal in this appendix is to calculate the CMB angular bispectrum from the Φ-field bispectrum. We start with writing Φ as a Gaussian-variable-squared in real space, $\Phi(\mathbf{x}) = \eta^2(\mathbf{x}) - \langle\eta^2(\mathbf{x})\rangle$, where η is Gaussian. We have subtracted the mean from Φ, ensuring $\langle\Phi(\mathbf{x})\rangle = 0$. Transforming $\Phi(\mathbf{x})$ into Fourier space, we obtain

$$\Phi(\mathbf{k}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\, \eta(\mathbf{k}+\mathbf{p})\eta^*(\mathbf{p}) - (2\pi)^3\delta^{(3)}(\mathbf{k}) \left\langle \eta^2(\mathbf{x}) \right\rangle. \tag{C.2}$$

Using the η power spectrum, $P_\eta(k)$, we write the Φ power spectrum, $P_\Phi(k)$, as

$$P_\Phi(k) = 2\int \frac{d^3\mathbf{p}}{(2\pi)^3}\, P_\eta(p) P_\eta\left(|\mathbf{k}+\mathbf{p}|\right). \tag{C.3}$$

If we use a conventional power-law spectrum, $P_\Phi(k) \propto k^{n-4}$, then we find $P_\eta(k) \propto k^{(n-7)/2}$. On the other hand, quantum fluctuations of a massive-free scalar field give $P_\sigma(k) \propto k^{-3+2m^2/(3H^2)}$, where H is the Hubble parameter during inflation (chapter 2). It then follows from $\Phi \propto \delta\rho_\sigma$ that $P_\eta(k) \propto P_\sigma(k) \propto k^{-3+2m^2/(3H^2)}$, and $n = 1 + 4m^2/(3H^2)$; thus, the model predicts a tilted "blue" spectrum (Linde and Mukhanov, 1997).

The Φ-field bispectrum is

$$\begin{aligned}\langle \Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)\Phi(\mathbf{k}_3)\rangle &= (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\frac{8}{3}\int \frac{d^3\mathbf{p}}{(2\pi)^3}\, P_\eta(p) \\ &\times \left[P_\eta\left(|\mathbf{k}_1+\mathbf{p}|\right)P_\eta\left(|\mathbf{k}_2-\mathbf{p}|\right)\right. \\ &+P_\eta\left(|\mathbf{k}_2+\mathbf{p}|\right)P_\eta\left(|\mathbf{k}_3-\mathbf{p}|\right) \\ &\left.+P_\eta\left(|\mathbf{k}_3+\mathbf{p}|\right)P_\eta\left(|\mathbf{k}_1-\mathbf{p}|\right)\right]. \end{aligned} \tag{C.4}$$

The delta function has appeared in the formula to satisfy the triangle condition, $\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3 = 0$. Since we will work in harmonic space eventually, it may be useful to expand $P_\eta\left(|\mathbf{k}\pm\mathbf{p}|\right)$ into harmonic space,

$$P_\eta\left(|\mathbf{k}\pm\mathbf{p}|\right) = \sum_{LM}\widetilde{P}^{(\pm)}_{\eta L}(k,p)Y_{LM}(\hat{\mathbf{k}})Y^*_{LM}(\hat{\mathbf{p}}). \tag{C.5}$$

The next step is to project the 3-dimensional Φ-field bispectrum onto the 2-dimensional CMB angular bispectrum. We write the harmonic coefficients of the CMB anisotropy, a_{lm}, with Φ and the radiation transfer function, $g_{\mathrm{T}l}(k)$, as

$$a_{lm} = 4\pi(-1)^l\int\frac{d^3\mathbf{k}}{(2\pi)^3}\Phi(\mathbf{k})g_{\mathrm{T}l}(k)Y^*_{lm}(\hat{\mathbf{k}}). \tag{C.6}$$

On large angular scales, the Sachs–Wolfe effect gives $g_{\mathrm{T}l}(k) = -2j_l\left[k(\tau_0-\tau_{\mathrm{dec}})\right]$, where τ_0 and τ_{dec} are the present-day conformal time and the decoupling-epoch conformal time, respectively. On small angular scales, we need the full radiation transfer function for isocurvature fluctuations.

We then calculate $\langle a_{l_1m_1}a_{l_2m_2}a_{l_3m_3}\rangle$. A key point on calculations is to expand the delta function in equation (C.4) into harmonic space with the Rayleigh's formula. After lengthy calculations, we obtain the CMB angular bispectrum from isocurvature fluctuations in inflation

$$\begin{aligned}\langle a_{l_1m_1}a_{l_2m_2}a_{l_3m_3}\rangle &= \mathcal{G}^{m_1m_2m_3}_{l_1l_2l_3}\frac{8}{3}\int_0^\infty r^2dr\int_0^\infty p^2dp\; P_\eta(p)\left[\sum_{l_1'l_2'L}\mathcal{F}^{l_1l_2l_3}_{l_2'l_1'L}\right. \\ &\times\frac{2}{\pi}\int_0^\infty k_1^2dk_1\;\widetilde{P}^{(+)}_{\eta L}(k_1,p)g_{\mathrm{T}l_1}(k_1)j_{l_1'}(k_1r)(-i)^{l_1-l_1'} \\ &\times\frac{2}{\pi}\int_0^\infty k_2^2dk_2\;\widetilde{P}^{(-)}_{\eta L}(k_2,p)g_{\mathrm{T}l_2}(k_2)j_{l_2'}(k_2r)(-i)^{l_2-l_2'} \\ &\times\frac{2}{\pi}\int_0^\infty k_3^2dk_3\; g_{\mathrm{T}l_3}(k_3)j_{l_3}(k_3r) \\ &\left.+\,(1\leftrightarrow 2, 2\leftrightarrow 3)+(1\leftrightarrow 3, 2\leftrightarrow 1)\right], \end{aligned} \tag{C.7}$$

where $\mathcal{G}_{l_1 l_2 l_3}^{m_1 m_2 m_3}$ and $\mathcal{F}_{l_2' l_1' L}^{l_1 l_2 l_3}$ represent geometric structures of the bispectrum. They are written with the Wigner 3-j and 6-j symbols,

$$\mathcal{G}_{l_1 l_2 l_3}^{m_1 m_2 m_3} \equiv \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}, \tag{C.8}$$

$$\begin{aligned} \mathcal{F}_{l_2' l_1' L}^{l_1 l_2 l_3} \equiv & \frac{(2l_1'+1)(2l_2'+1)(2L+1)}{4\pi} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}^{-1} \begin{Bmatrix} l_1 & l_2 & l_3 \\ l_2' & l_1' & L \end{Bmatrix} \\ & \times \begin{pmatrix} l_1' & l_2' & l_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_1' & L \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_2 & l_2' & L \\ 0 & 0 & 0 \end{pmatrix} (-1)^{l_1'+l_2'+L}. \end{aligned} \tag{C.9}$$