Appendix B

Wigner 3-j Symbol

In this appendix, we summarize basic properties of the Wigner 3-j symbol, following Rotenberg et al. (1959). The Wigner 3-j symbol characterizes geometric properties of the angular bispectrum.

B.1 Triangle conditions

The Wigner 3-j symbol,

$$\left(\begin{array}{ccc} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{array}\right), \tag{B.1}$$

is related to the Clebsh–Gordan coefficients which describe coupling of two angular momenta in the quantum mechanics. In the quantum mechanics, l is the eigenvalue of the angular momentum operator, $\mathbf{L} = \mathbf{r} \times \mathbf{p}$: $\mathbf{L}^2 Y_{lm} = l(l+1)Y_{lm}$. m is the eigenvalue of the z-direction component of the angular momentum, $L_z Y_{lm} = mY_{lm}$.

The symbol such as

$$(-1)^{m_3}\left(\begin{array}{ccc} l_1 & l_2 & l_3 \\ m_1 & m_2 & -m_3 \end{array}\right) \tag{B.2}$$

describes coupling of two angular-momentum states, $\mathbf{L}_1$ and $\mathbf{L}_2$, forming a coupled state, $\mathbf{L}_3 = \mathbf{L}_1 + \mathbf{L}_2$. It follows from $\mathbf{L}_1 + \mathbf{L}_2 - \mathbf{L}_3 = 0$ that $m_1 + m_2 - m_3 = 0$; thus, the Wigner 3-j symbol (B.1) describes three angular momenta forming a triangle, $\mathbf{L}_1 + \mathbf{L}_2 + \mathbf{L}_3 = 0$, and satisfies $m_1 + m_2 + m_3 = 0$.

Since $\mathbf{L}_1$, $\mathbf{L}_2$, and $\mathbf{L}_3$ form a triangle, they have to satisfy the triangle conditions, $|L_i - L_j| \leq L_k \leq L_i + L_j$, where $L_i \equiv |\mathbf{L}_i|$. Hence, l_1, l_2, and l_3 also satisfy the triangle conditions,

$$|l_i - l_j| \leq l_k \leq l_i + l_j; \tag{B.3}$$

otherwise, the Wigner 3-j symbol vanishes. The triangle conditions also include $m_1 + m_2 + m_3 = 0$. These properties may regard (l, m) as vectors, $\mathbf{l}$, which satisfy $\mathbf{l}_1 + \mathbf{l}_2 + \mathbf{l}_3 = 0$. Note that, however, $\mathbf{L} \neq \mathbf{l}$.

For $l_1 = l_2$ and $m_3 = 0$, the Wigner 3-j symbol reduces to

$$(-1)^m \begin{pmatrix} l & l & l' \\ m & -m & 0 \end{pmatrix} = \frac{(-1)^l}{\sqrt{2l+1}}\delta_{l'0}. \tag{B.4}$$

In chapter 3, we have used this relation to reduce the covariance matrix of the angular bispectrum and trispectrum. We have also used this relation to reduce the angular trispectrum for $L = 0$ (see Eq.(3.27)).

B.2 Symmetry

The Wigner 3-j symbol is invariant under even permutations,

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} l_3 & l_1 & l_2 \\ m_3 & m_1 & m_2 \end{pmatrix} = \begin{pmatrix} l_2 & l_3 & l_1 \\ m_2 & m_3 & m_1 \end{pmatrix}, \tag{B.5}$$

while it changes the phase for odd permutations if $l_1 + l_2 + l_3 =$ odd,

$$(-1)^{l_1+l_2+l_3} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \tag{B.6}$$

$$= \begin{pmatrix} l_2 & l_1 & l_3 \\ m_2 & m_1 & m_3 \end{pmatrix} = \begin{pmatrix} l_1 & l_3 & l_2 \\ m_1 & m_3 & m_2 \end{pmatrix} = \begin{pmatrix} l_3 & l_2 & l_1 \\ m_3 & m_2 & m_1 \end{pmatrix}. \tag{B.7}$$

The phase also changes under the transformation of $m_1 + m_2 + m_3 \to -(m_1 + m_2 + m_3)$, if $l_1 + l_2 + l_3 =$ odd,

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{l_1+l_2+l_3} \begin{pmatrix} l_1 & l_2 & l_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix}. \tag{B.8}$$

If there is no z-direction component of the angular momenta in the system, i.e., $m_i = 0$, then the Wigner 3-j symbol of the system,

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}, \tag{B.9}$$

is non-zero only if $l_1 + l_2 + l_3 =$ even. This symbol is invariant under any permutations of l_i.

In chapter 4, we have frequently used the Gaunt integral, $\mathcal{G}_{l_1 l_2 l_3}^{m_1 m_2 m_3}$, defined by

$$\begin{aligned} \mathcal{G}_{l_1 l_2 l_3}^{m_1 m_2 m_3} &\equiv \int d^2\hat{\mathbf{n}} Y_{l_1 m_1}(\hat{\mathbf{n}}) Y_{l_2 m_2}(\hat{\mathbf{n}}) Y_{l_3 m_3}(\hat{\mathbf{n}}) \\ &= \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} \\ &\quad \times \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}, \end{aligned} \tag{B.10}$$

to calculate the angular bispectrum. By definition, the Gaunt integral is invariant under both the odd and the even permutations, and non-zero only if $l_1 + l_2 + l_3 =$ even, $m_1 + m_2 + m_3 = 0$, and $|l_i - l_j| \le l_k \le l_i + l_j$. In other words, the Gaunt integral describes fundamental geometric properties of the angular bispectrum such as the triangle conditions.

The Gaunt integral for $m_i = 0$ gives the identity for the Legendre polynomials,

$$\int_{-1}^{1} \frac{dx}{2}\, P_{l_1}(x)P_{l_2}(x)P_{l_3}(x) = \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}^2. \tag{B.11}$$

In chapter 3, we have used this identity to derive the bias for the angular bispectrum on the incomplete sky (Eq.(3.41)). Here, we have used

$$Y_{l0}(\hat{\mathbf{n}}) = \sqrt{\frac{4\pi}{2l+1}} P_l(\cos\theta). \tag{B.12}$$

B.3 Orthogonality

The Wigner 3-j symbol has the following orthogonality properties:

$$\sum_{l_3 m_3} (2l_3+1) \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1' & m_2' & m_3 \end{pmatrix} = \delta_{m_1 m_1'} \delta_{m_2 m_2'}, \tag{B.13}$$

and

$$\sum_{m_1 m_2} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3' \\ m_1 & m_2 & m_3' \end{pmatrix} = \frac{\delta_{l_3 l_3'} \delta_{m_3 m_3'}}{2l_3+1}, \tag{B.14}$$

or

$$\sum_{\text{all } m} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}^2 = 1. \tag{B.15}$$

The orthogonality properties are essential for any basic calculations involving the Wigner 3-j symbols. Note that these orthogonality properties are consistent with orthonormality of the angular-momentum eigenstate vectors, and unitality of the Crebsh–Gordan coefficients, by definition.

B.4 Rotation matrix

A finite rotation operator for the Euler angles α, β, and γ, $D(\alpha,\beta,\gamma)$, comprises angular momentum operators,

$$D(\alpha,\beta,\gamma) = e^{-i\alpha L_z} e^{-i\beta L_y} e^{-i\gamma L_z}. \tag{B.16}$$

Since the Wigner 3-j symbol describes coupling of two angular momenta, it also describes coupling of two rotation operators. Using the rotation matrix element, $D^{(l)}_{m'm} = \langle l, m' | D | l, m \rangle$, we have

$$D^{(l_1)}_{m_1' m_1} D^{(l_2)}_{m_2' m_2} = \sum_{l_3} (2l_3+1) \sum_{m_3 m_3'} D^{(l_3)*}_{m_3' m_3} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1' & m_2' & m_3' \end{pmatrix}. \tag{B.17}$$

In chapter 3, we have used this relation to evaluate rotationally invariant harmonic spectra. Note that the rotation matrix is orthonormal,

$$\sum_m D^{(l)*}_{m'm} D^{(l)}_{m''m} = \delta_{m'm''}. \tag{B.18}$$

B.5 Wigner 6-j symbol

The Wigner 6-j symbol,

$$\left\{ \begin{array}{ccc} l_1 & l_2 & l_3 \\ l_1' & l_2' & l_3' \end{array} \right\}, \tag{B.19}$$

describes coupling of three angular momenta. We often encounter the Wigner 6-j symbol, when we calculate the angular bispectrum which has more complicated geometric structures (Goldberg and Spergel, 1999). The angular trispectrum also often includes the Wigner 6-j symbol (Hu, 2001).

The Wigner 6-j symbol is related to the Wigner 3-j symbols through

$$\begin{aligned} &(-1)^{l_1'+l_2'+l_3'} \left\{ \begin{array}{ccc} l_1 & l_2 & l_3 \\ l_1' & l_2' & l_3' \end{array} \right\} \left(\begin{array}{ccc} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{array} \right) \\ &= \sum_{\text{all } m'} (-1)^{m_1'+m_2'+m_3'} \\ &\times \left(\begin{array}{ccc} l_1 & l_2' & l_3' \\ m_1 & m_2' & -m_3' \end{array} \right) \left(\begin{array}{ccc} l_1' & l_2 & l_3' \\ -m_1' & m_2 & m_3' \end{array} \right) \left(\begin{array}{ccc} l_1' & l_2' & l_3 \\ m_1' & -m_2' & m_3 \end{array} \right). \end{aligned} \tag{B.20}$$

In appendix C, we use this relation to derive the angular bispectrum from isocurvature fluctuations in inflation (Eq.(C.7)). By using equation (B.15), we also obtain

$$\begin{aligned} (-1)^{l_1'+l_2'+l_3'} \left\{ \begin{array}{ccc} l_1 & l_2 & l_3 \\ l_1' & l_2' & l_3' \end{array} \right\} &= \sum_{\text{all } mm'} (-1)^{m_1'+m_2'+m_3'} \\ &\times \left(\begin{array}{ccc} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{array} \right) \left(\begin{array}{ccc} l_1 & l_2' & l_3' \\ m_1 & m_2' & -m_3' \end{array} \right) \\ &\times \left(\begin{array}{ccc} l_1' & l_2 & l_3' \\ -m_1' & m_2 & m_3' \end{array} \right) \left(\begin{array}{ccc} l_1' & l_2' & l_3 \\ m_1' & -m_2' & m_3 \end{array} \right). \end{aligned} \tag{B.21}$$