Appendix A

Slow-roll Approximation

In this appendix, we describe the *slow-roll approximation*, which has played a central role in the analysis of inflation dynamics. We then apply the approximation to the effective mass of scalar-field fluctuations, $m_\chi^2(\tau)$, to show explicitly the approximation that we have used in equation (2.22).

A slowly-rolling scalar field on a potential, $V(\phi)$, is a key ingredient of a successful inflation model, for to achieve the accelerated expansion of the universe the kinetic energy of ϕ needs to be smaller than the potential energy (see Eq.(2.2)),

$$\left(\frac{d\phi}{dt}\right)^2 < V(\phi). \tag{A.1}$$

It then follows from this condition that the second-order time derivative, $d^2\phi/dt^2$, needs to be smaller than the potential slope, $dV/d\phi$,

$$2\left|\frac{d^2\phi}{dt^2}\right| < \left|\frac{dV}{d\phi}\right|. \tag{A.2}$$

These conditions provide us an useful scheme of approximation, the slow-roll approximation. The approximation demands that the l.h.s's of the conditions be much smaller than the r.h.s's.

The trace-part Einstein equation and the Friedmann equation give the exact relation between the time derivative of H and $d\phi/dt$,

$$\frac{dH}{dt} = -4\pi G\left(\frac{d\phi}{dt}\right)^2. \tag{A.3}$$

Hence, we obtain another form of the slow-roll condition, $|dH/dt| < 4\pi G V(\phi)$. In the slow-roll approximation, the dimensionless variables such as $V^{-1}(d\phi/dt)^2$, $(H d\phi/dt)^{-1}(d^2\phi/dt^2)$, $H^{-2}(dH/dt)$, and so on, are small order parameters which control the approximation.

By the physical requirement, the accelerated expansion of the universe, the slow-roll conditions are defined by the time derivative with respect to the physical time, t. If we use the conformal time, τ ($d\tau = a^{-1}dt$), then we have $\dot{\phi} = a(d\phi/dt)$, and

$$\ddot{\phi} = aH\dot{\phi} + a^2\frac{d^2\phi}{dt^2}, \tag{A.4}$$

where the dots denote the conformal time derivative: $\dot{x} \equiv d\phi/d\tau$. In contrast to $d^2\phi/dt^2$, $\ddot{\phi}$ is not so small compared with $dV/d\phi$ because of the first term. From this, we obtain a rule for the slow-roll analysis: *evaluate the second- or the higher-order time derivative with respect to the physical time, not the conformal time.* Using this rule, we can use the slow-roll approximation consistently in the conformal time coordinate.

Using the slow-roll approximation, we derive the approximation to the effective mass of scalar-field fluctuations, m_χ^2, which we have done in chapter 2 (Eq. (2.22)). To derive

$$m_\chi^2 = -\frac{H}{\dot{\phi}}\frac{d^2(\dot{\phi}/H)}{d\tau^2} \approx \left(\frac{d^2V}{d\phi^2} + 9\frac{dH}{dt}\right)a^2 - \frac{2}{\tau^2}, \tag{A.5}$$

we begin with

$$-\frac{H}{\dot{\phi}}\frac{d^2(\dot{\phi}/H)}{d\tau^2} = \frac{\ddot{H}}{H} - \frac{\dddot{\phi}}{\dot{\phi}} + 2\frac{\dot{H}}{H}\left(\frac{\ddot{\phi}}{\dot{\phi}} - \frac{\dot{H}}{H}\right), \tag{A.6}$$

where

$$\begin{aligned}
\frac{\ddot{H}}{H} &= -a\dot{H} - 8\pi G\frac{\dot{\phi}\ddot{\phi}}{aH} = a\dot{H} - 8\pi G\frac{a\dot{\phi}}{H}\left(\frac{d^2\phi}{dt^2}\right), &\text{(A.7)}\\
\frac{\dddot{\phi}}{\dot{\phi}} &= 2a^2H^2 - 2a\dot{H} - a^2\frac{d^2V}{d\phi^2}, &\text{(A.8)}\\
\frac{\ddot{\phi}}{\dot{\phi}} &= aH + a^2\frac{d^2\phi}{dt^2}. &\text{(A.9)}
\end{aligned}$$

Here, to calculate $\dddot{\phi}$, we have used the Klein–Gordon equation for a homogeneous scalar field, $\ddot{\phi} + 2aH\dot{\phi} + a^2(dV/d\phi) = 0$. These equations are exact.

We then neglect the higher-order slow-roll terms such as $\dot{H}^2$ and $(d^2\phi/dt^2)\dot{H}$, and obtain

$$-\frac{H}{\dot{\phi}}\frac{d^2(\dot{\phi}/H)}{d\tau^2} \approx \left(\frac{d^2V}{d\phi^2} - 2H^2 + 5\frac{dH}{dt}\right)a^2. \tag{A.10}$$

Next step is to relate H to the conformal time, τ, through

$$\tau \equiv \int \frac{dt}{a} = \int \frac{da}{a^2H} = -\frac{1}{aH} - \int da\frac{\dot{H}}{(aH)^3}. \tag{A.11}$$

By rewriting the conformal-time derivative in the second term with the physical-time derivative, $\int da(dH/dt)/(a^2H^3)$, and neglecting the higher-order slow-roll terms, we obtain the first-order slow-roll correction to the conformal time,

$$\tau \approx -\frac{1}{aH} + \frac{dH/dt}{aH^3}, \tag{A.12}$$

and hence $H^2 \approx (a\tau)^{-1} - 2(dH/dt)$. Finally, substituting this for H^2 in equation (A.10), we obtain

$$-\frac{H}{\dot{\phi}}\frac{d^2(\dot{\phi}/H)}{d\tau^2} \approx \left(\frac{d^2V}{d\phi^2} + 9\frac{dH}{dt}\right)a^2 - \frac{2}{\tau^2}. \tag{A.13}$$