Reference: EK, Nature Rev. Phys. **4**, 452 (2022)

The lecture slides are available at
https://wwwmpa.mpa-garching.mpg.de/~komatsu/lectures--reviews.html

Parity Violation in Cosmology

In search of new physics for the Universe

Lecture 1

Eiichiro Komatsu (Max Planck Institute for Astrophysics)

Universe+ "Topics in Theoretical Cosmology", MPP

July 22, 2025

MAX-PLANCK-INSTITUT FÜR ASTROPHYSIK

Overarching Theme

Let's find new physics!

- The current cosmological model (*flat ΛCDM*) **requires** new physics beyond the standard model of elementary particles and fields.
    - What is dark matter (*CDM*)?
    - What is dark energy (*Λ*)?
    - Why is the spatial geometry of the Universe Euclidean (*flat*)?
    - What powered the Big Bang?

Overarching Theme

There are many ideas, but how can we make progress?

- The current cosmological model (*flat ΛCDM*) **requires** new physics beyond the standard model of elementary particles and fields.
  - What is dark matter (*CDM*)? => CDM, WDM, FDM, …
  - What is dark energy (*Λ*)? => Dynamical field, modified gravity, quantum gravity, …
  - Why is the spatial geometry of the Universe Euclidean (*flat*)? => Inflation, contracting universe, …
  - What powered the Big Bang? => Scalar field, gauge field, …

Ove

There

New in cosmology!

Violation of parity symmetry may hold the answer to these fundamental questions.

- The current cosmological model (*flat ΛCDM*) **requires** new physics beyond the standard model of elementary particles and fields.
  - What is dark matter (*CDM*)? => CDM, WDM, FDM, …
  - What is dark energy (*Λ*)? => Dynamical field, modified gravity, quantum gravity, …
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  - What powered the Big Bang? => Scalar field, gauge field, …

Reference:

Nature Rev. Phys. **4**, 452 (2022)

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Available also at arXiv:2202.13919

Review Article | Published: 18 May 2022

New physics from the polarized light of the cosmic microwave background

Eiichiro Komatsu

Nature Reviews Physics **4**, 452–469 (2022) | Cite this article

Key Words:

1. Cosmic Microwave Background (CMB)
2. Polarization
3. Parity Symmetry

The Plan [13:15–15:00]

2 x 45 minutes

- <u>Lecture 1</u> "*Known Physics*" [45 min]
- <u>Break</u> and problem solving (voluntary) [15 min]
- <u>Lecture 2</u> "*New Physics*" [45 min]

The lecture slides are available at https://wwwmpa.mpa-garching.mpg.de/~komatsu/lectures--reviews.html

1.1 Parity

Probing Parity Symmetry
Definition

- **Parity transformation = Inversion of all spatial coordinates**
    - (x, y, z) –> (-x, -y, -z)

- Parity symmetry in physics states:
    - *The laws of physics are invariant under inversion of all spatial coordinates.*

- Violation of parity symmetry = The laws of physics are **not** invariant under…

But, who cares about coordinates?

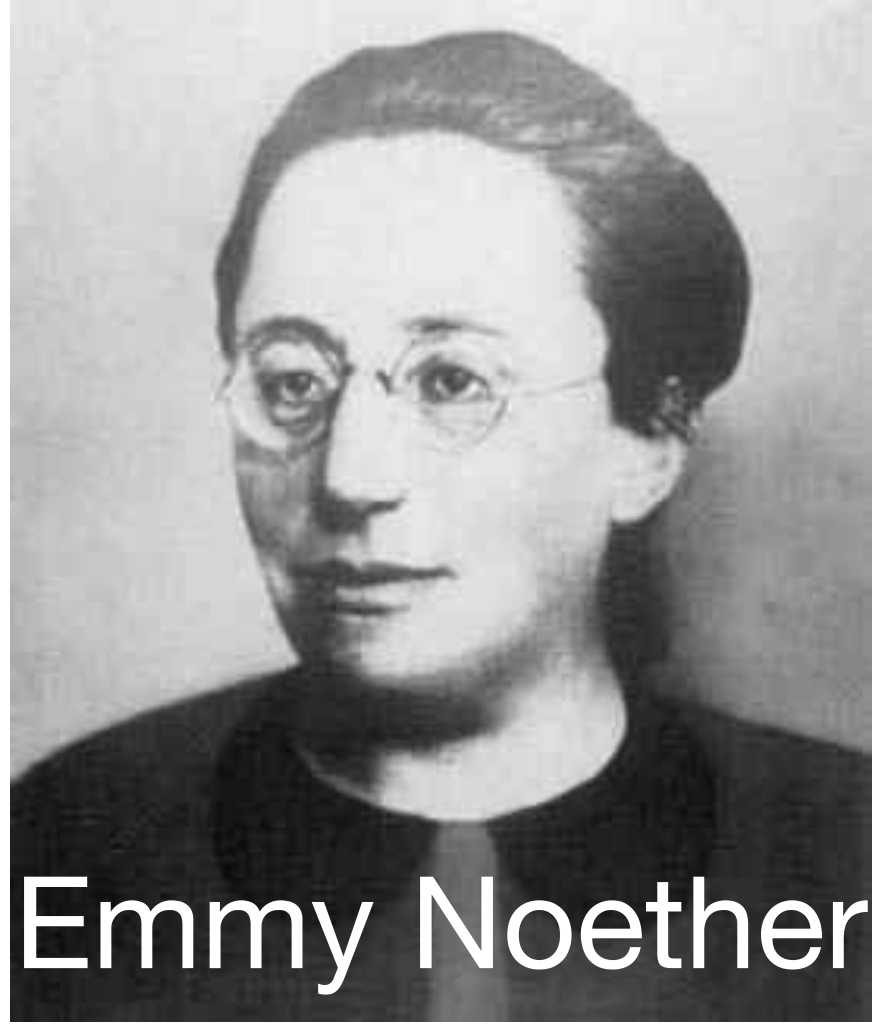
The key is the coordinate *transformation*

- You may say, "*Coordinates are just a convenient mathematical tool. Physics should not depend on how we chart the world with coordinates.*"
    - Yes, that is absolutely correct.
- Coordinate **transformations** are different. The underlying physical principle does not depend on the choice of coordinates. However, "**how a physical system appears to change from one coordinate system to another**" often contains useful information.

Continuous Coordinate Transformation - 1

Spatial translation and homogeneity

- We do an experiment in Munich, and repeat it in Tokyo. We find the same answer (to within the uncertainty).
- This is evidence for **invariance under spatial translation**. We shift spatial coordinates by a constant vector $\boldsymbol{c}$, $\boldsymbol{x} \rightarrow \boldsymbol{x} + \boldsymbol{c}$, and the physics relevant to the experiment does not change.
  - There is no special location in space => **homogeneity**.
  - This even implies that the total momentum is conserved!
    - Noether's theorem



https://mathshistory.st-andrews.ac.uk

Continuous Coordinate Transformation - 2

Spatial rotation and isotropy

- We do an experiment. We repeat it a few times after rotating the experimental apparatus at different angles. We find the same answer (to within the uncertainty).
- This is evidence for **invariance under spatial rotation**. We rotate spatial coordinates by $\boldsymbol{x}$ –> R$\boldsymbol{x}$, where R is a 3-dimensional rotation matrix, and the physics relevant to the experiment does not change.
  - There is no special direction in space => **isotropy**.
  - This even implies that the total angular momentum is conserved!
    - Noether's theorem


Emmy Noether

https://mathshistory.st-andrews.ac.uk

Parity: Discrete Coordinate Transformation

- We ask, “***When we observe a certain phenomenon in nature, do we also observe its mirror image(*) with equal probability?***”

- (*) “Mirror image” is an ambiguous word. A parity transformation is (x, y, z) –> (-x, -y, -z), whereas a “mirror image” often refers to, e.g., (x, y, z) –> (-x, y, z), where only one of (x,y,z) is flipped.

Do we also observe this with equal probability?

Note that this is not full parity transformation, as only one axis is flipped.

Parity and Rotation

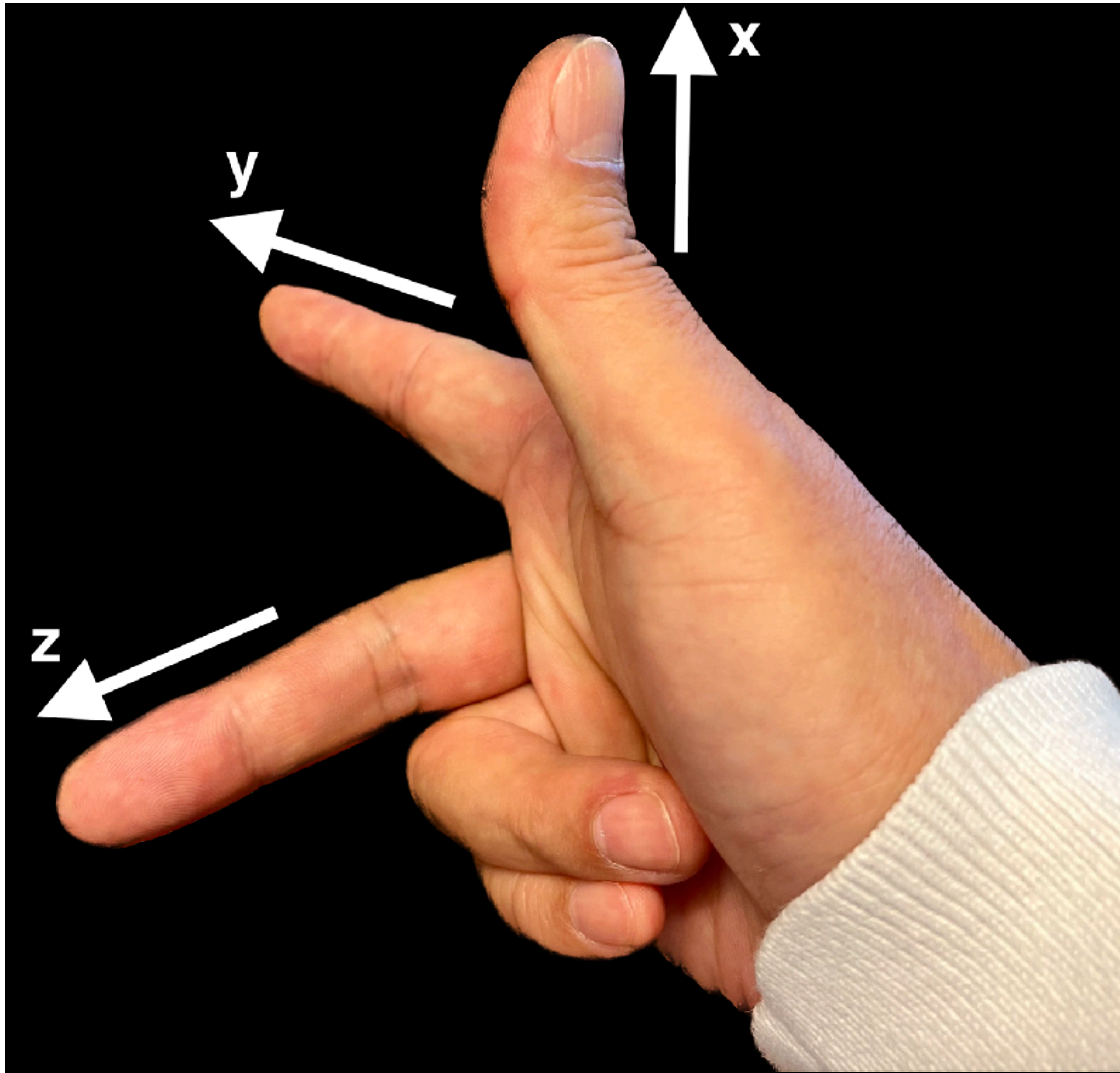
- Parity transformation (***x*** –> -***x***) and 3d rotation (***x*** –> R***x***) are different.
  - R is a continuous transformation and the determinant of R is det(R) = +1.
  - Parity is a discrete transformation and the **determinant is –1**, as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

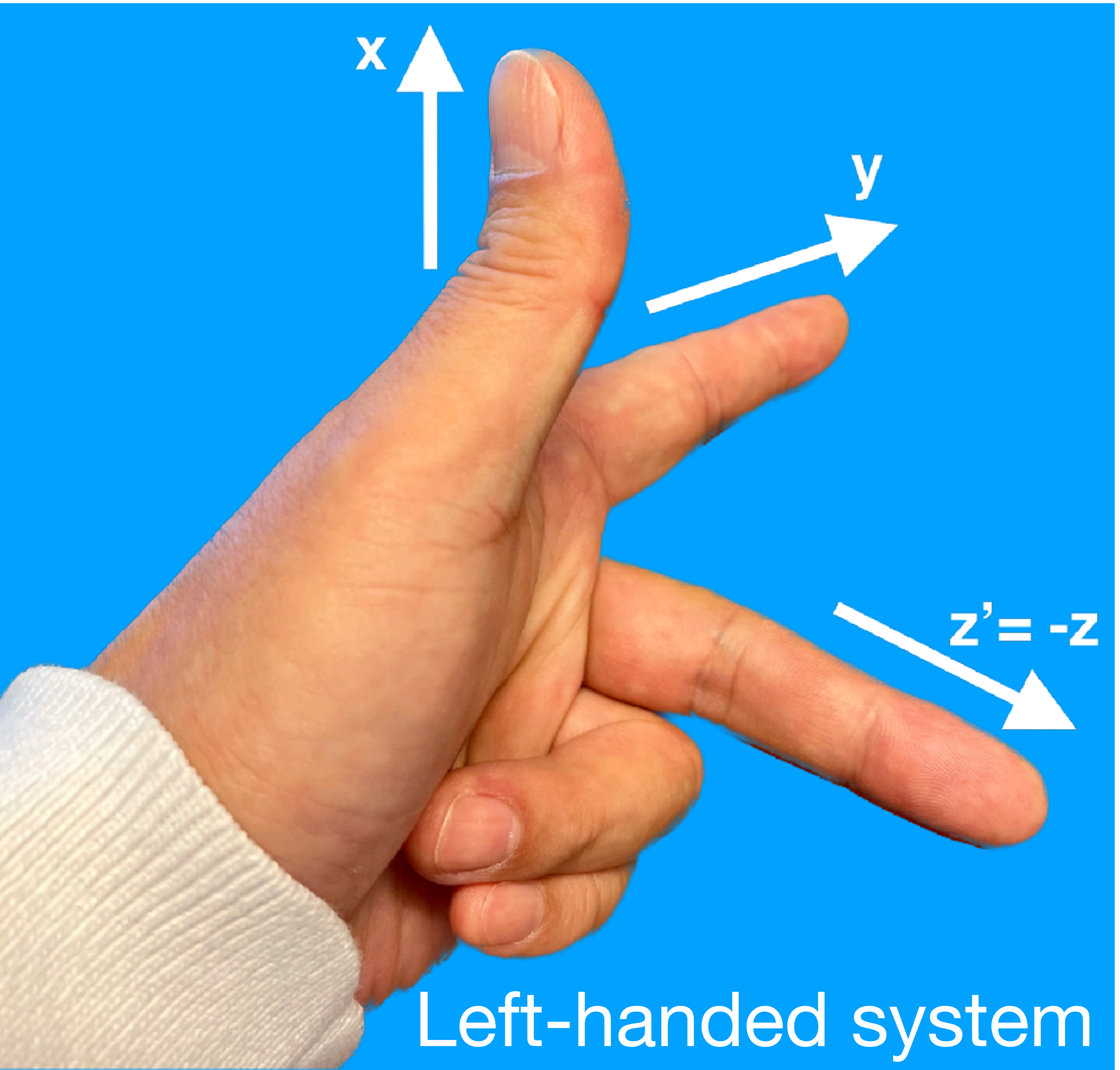
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Parity = Mirror + 2d Rotation

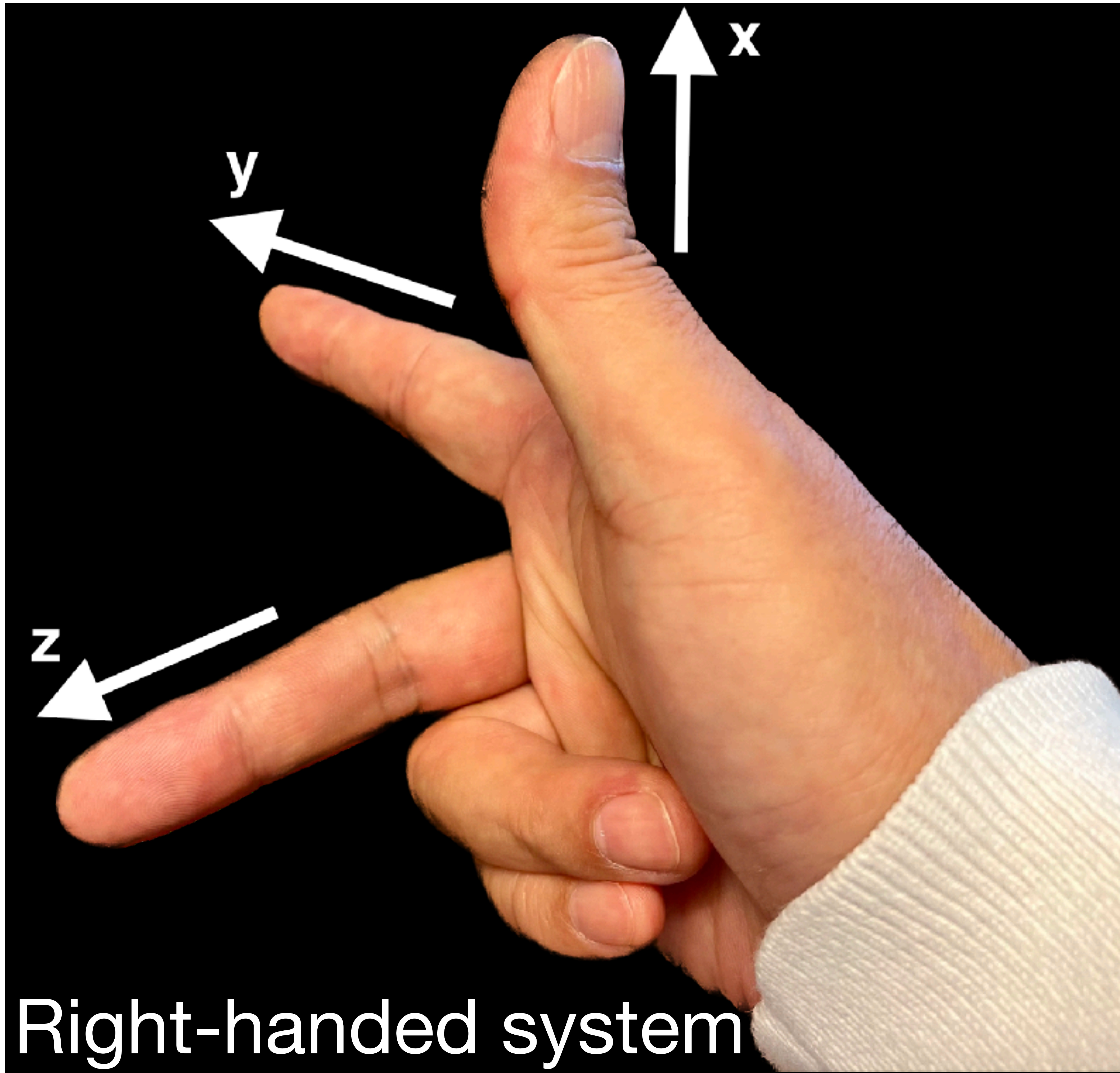
- One may think of parity transformation as a mirror in one of the coordinates (e.g., z –> -z) and **2d** rotation by π in the others.
- Let's demonstrate it!



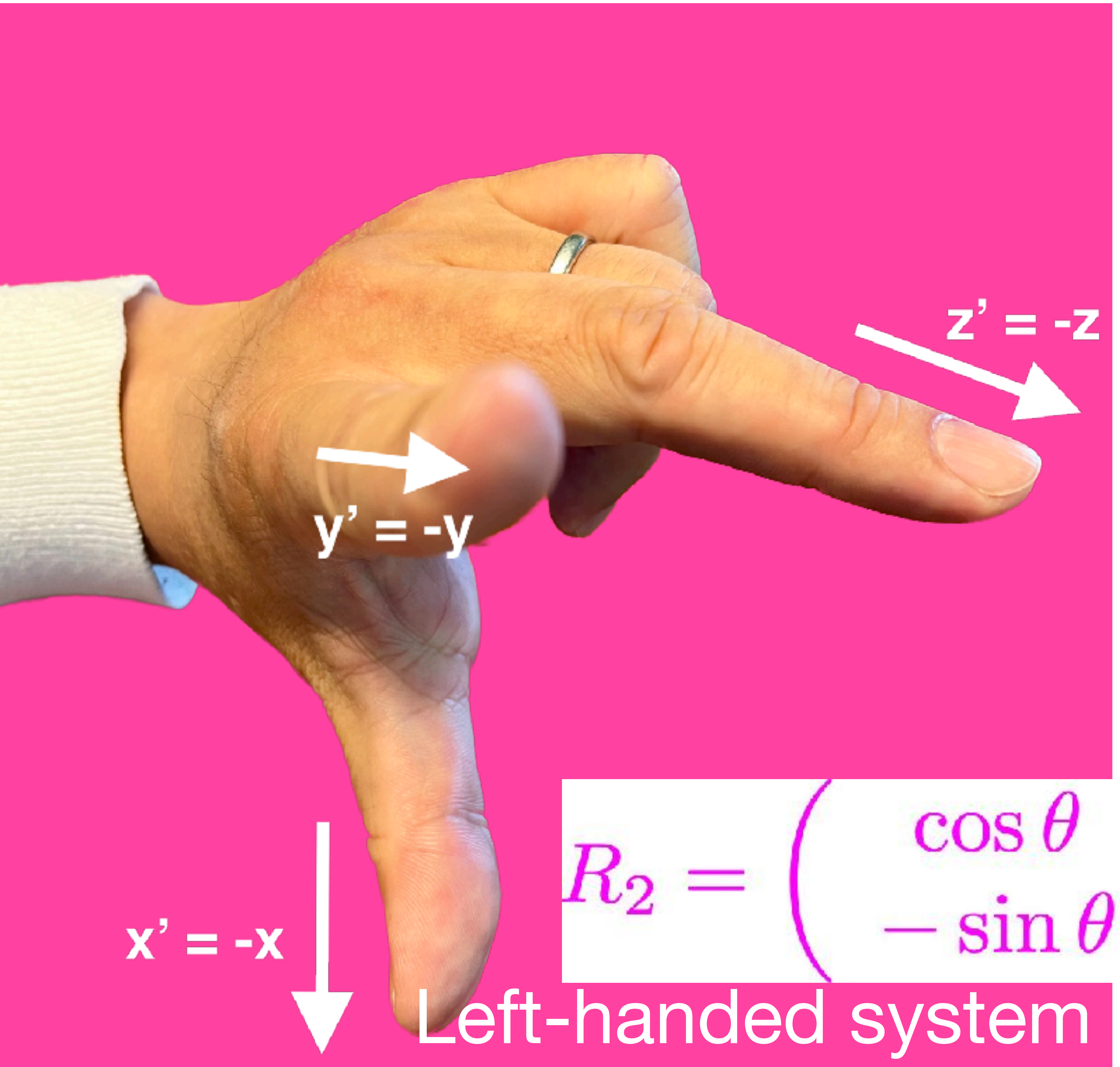
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \boxed{-1} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$



z -> z' = -z



$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \boxed{-1} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

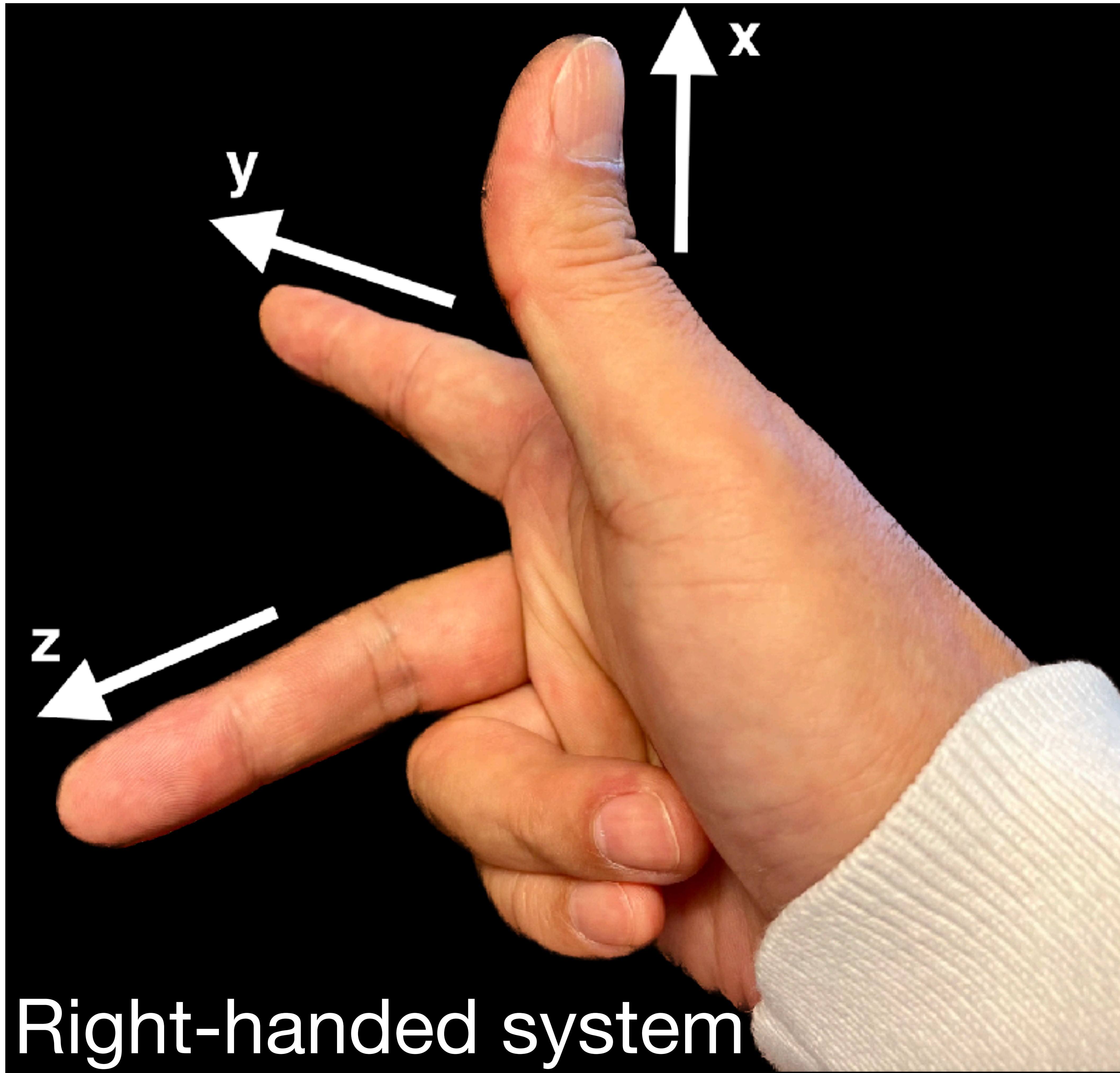


Left-handed system

Rotation In x-y

$$R_2 = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

with θ = π

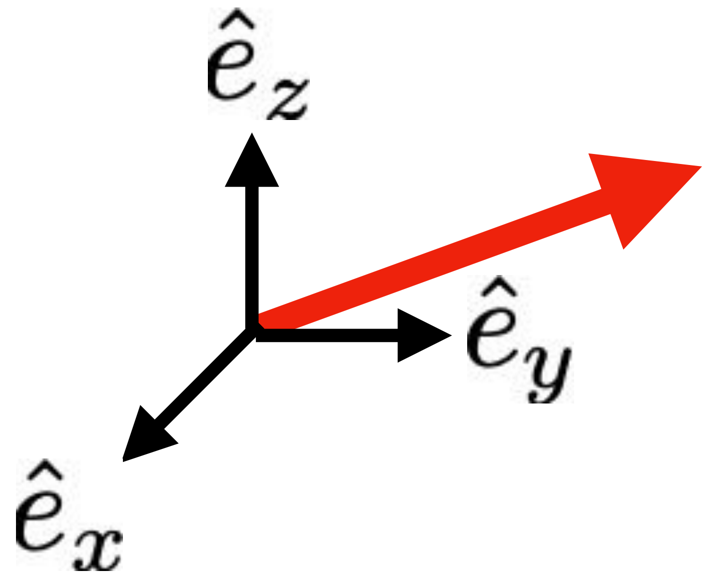


Right-handed system

1.2 Pseudovector, Pseudoscalar

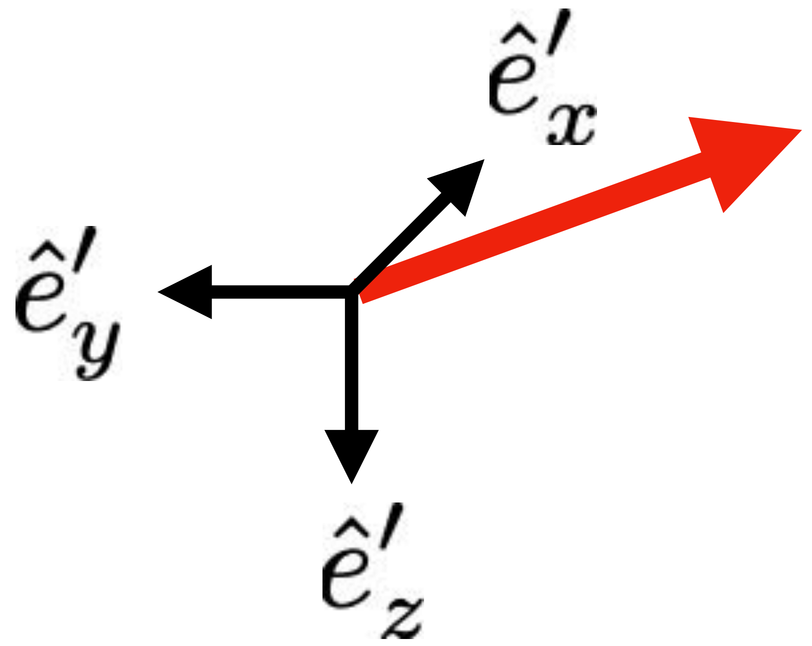
Parity Transformation: Vector

E.g., momentum, electric field



$$\mathbf{p} = p_x\hat{e}_x + p_y\hat{e}_y + p_z\hat{e}_z$$

$\hat{e}_i$ is a unit vector.



$$\begin{aligned}\mathbf{p} &= p'_x\hat{e}'_x + p'_y\hat{e}'_y + p'_z\hat{e}'_z \\ &= -p'_x\hat{e}_x - p'_y\hat{e}_y - p'_z\hat{e}_z\end{aligned}$$

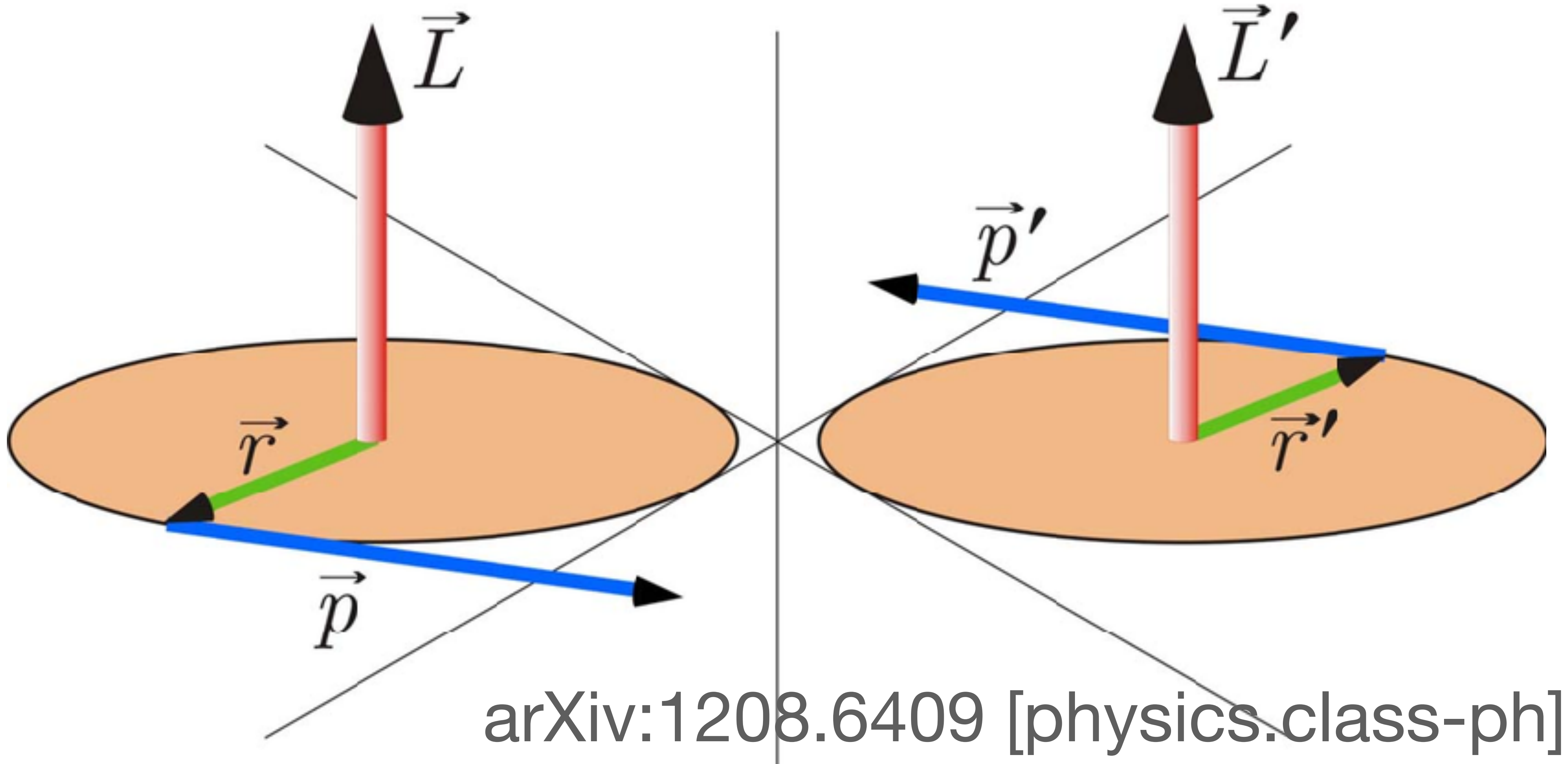
- **p** is the same vector, written using two different basis vectors.
- Therefore, **p**’s components are transformed as $(p'_x, p'_y, p'_z) = (-p_x, -p_y, -p_z)$

Parity Transformation: Pseudovector

E.g., angular momentum, magnetic field

- Orbital angular momentum, $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, is a *pseudovector*. Its *components* do **<u>not</u>** change under parity transformation: $(L'_x, L'_y, L'_z) = (L_x, L_y, L_z)$
  - Both $\mathbf{r} = (X, Y, Z)$ and $\mathbf{p} = (p_x, p_y, p_z)$ are vectors whose components change sign. Thus, their products do not change, e.g.,

$$
\begin{aligned}
L'_x &= Y' p'_z - Z' p'_y \\
&= (-Y)(-p_z) - (-Z)(-p_y) \\
&= L_x
\end{aligned}
$$



arXiv:1208.6409 [physics.class-ph]

Key Point

Parity Transformation: Pseudoscalar

How to test parity symmetry?

- A dot product of a vector and a pseudovector is a **pseudoscalar**.
    - Like a scalar, a pseudoscalar is invariant under rotation.
    - But, a pseudoscalar changes sign under parity transformation.
- **Experimental test of parity symmetry**: Construct a pseudoscalar and see if the average value is zero. If not, the system violates parity symmetry!
    - *Example*: a dot product of particle A's momentum and particle B's angular momentum: $\mathbf{p}_{\mathrm{A}} \cdot \mathbf{L}_{\mathrm{B}}$. Measure this and average over many trials. Does the average vanish, $\langle \mathbf{p}_{\mathrm{A}} \cdot \mathbf{L}_{\mathrm{B}} \rangle = 0$?

1.3 Discovery of Parity Violation in β-decay (weak interaction)

Letters to the Editor,
Physical Review, 105, 1413 (1957)

Experimental Test of Parity Conservation in Beta Decay*

C. S. Wu, *Columbia University, New York, New York*

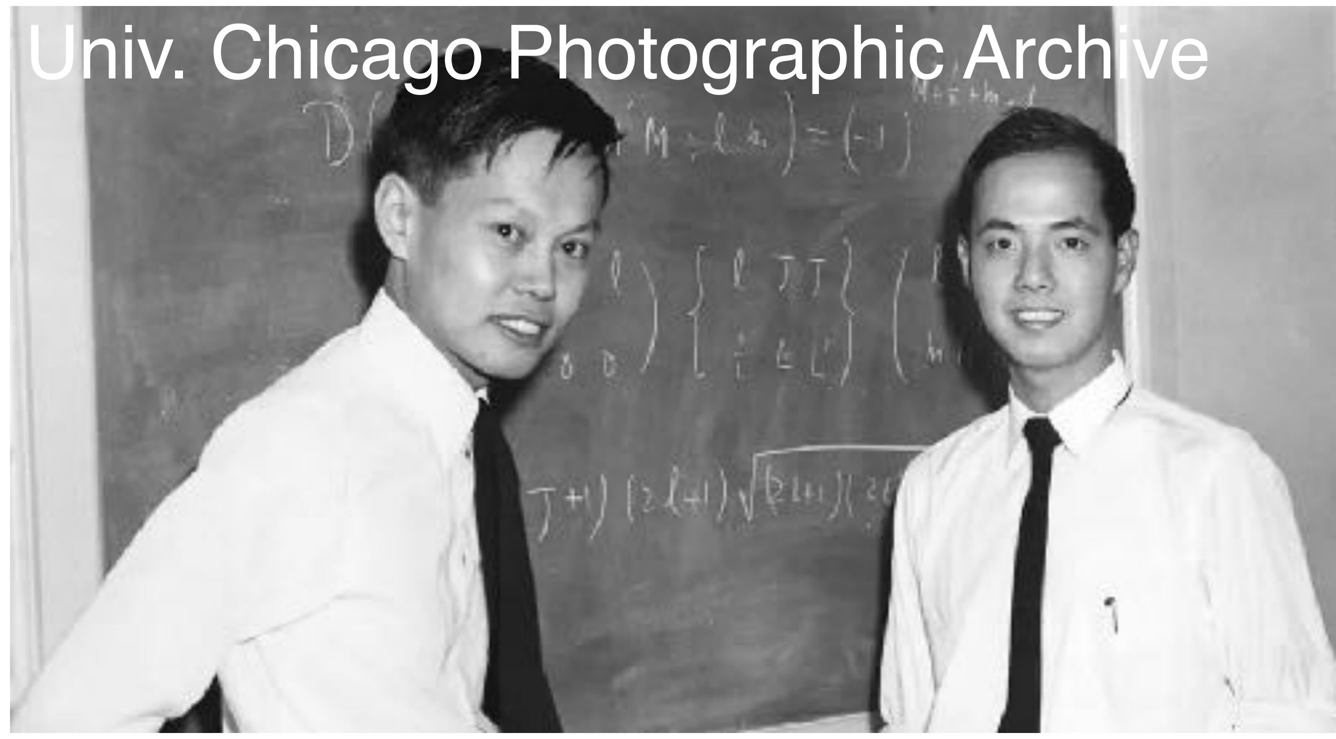
AND

E. Ambler, R. W. Hayward, D. D. Hoppes, and R. P. Hudson,
National Bureau of Standards, Washington, D. C.

(Received January 15, 1957)

In a recent paper[1] on the question of parity in weak interactions, Lee and Yang critically surveyed the experimental information concerning this question and reached the conclusion that there is no existing evidence either to support or to refute parity conservation in weak interactions. They proposed a number of experiments on beta decays and hyperon and meson decays which would provide the necessary evidence for parity conservation or nonconservation. In beta decay, one could measure the angular distribution of the electrons coming from beta decays of polarized nuclei. If an asymmetry in the

Chien-Shiung Wu



Chen-Ning Yang Tsung-Dao Lee

The Wu Experiment of β-decay

$^{60}Co \rightarrow {}^{60}Ni + e^- + \bar{\nu}_e + 2\gamma$

- Electrons must be emitted with equal probability in all directions relative to **J**, if parity symmetry is respected in β-decay.
  - This was not observed: $\langle \mathbf{p}_\mathrm{e} \cdot \mathbf{J} \rangle \neq 0$. **Parity symmetry is violated in β-decay!**

Initial reaction


Bildarchiv der ETH-Bibliothek

Many physicists did not believe it initially.

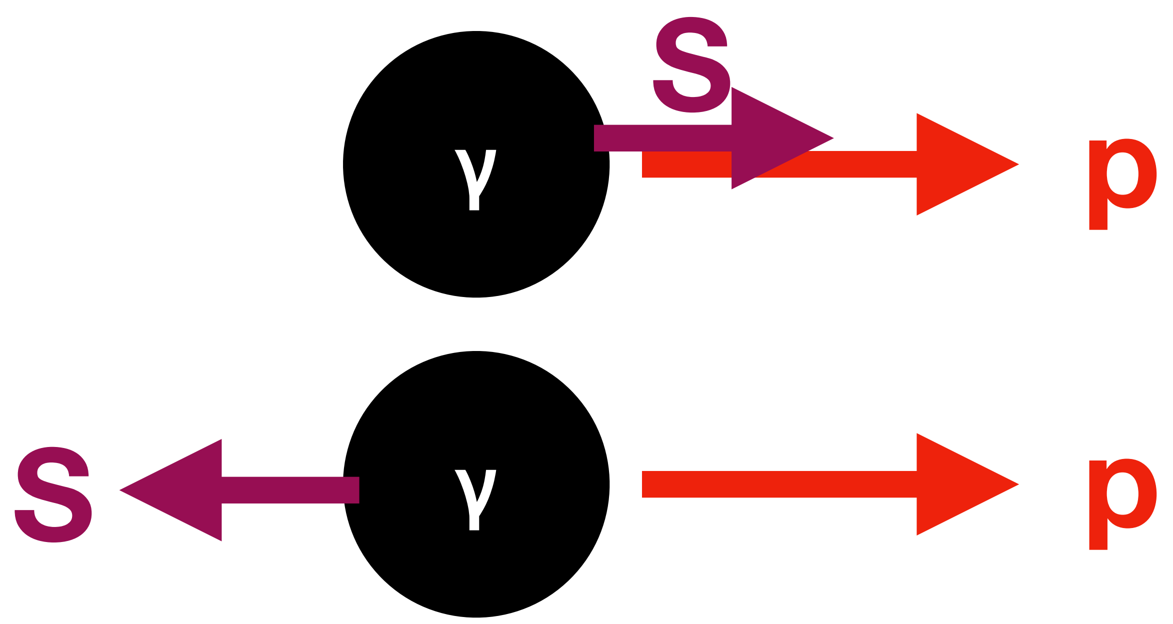
- To Lee and Yang’s theoretical paper on parity violation in β-decay:
    - Wolfgang Pauli said, “*Ich glaube aber nicht, daß der Herrgott ein schwacher Linkshänder ist*” (I do not believe that the Lord is a weak left-hander).
- To Wu’s discovery paper:
    - Wolfgang Pauli said, “*Sehr aufregend. Wie sicher ist die Nachricht?*” (Very exciting. How sure is this news?)
- **This was shocking news. The weak interaction distinguishes between left and right!**
- In this lecture we ask, “*Does the Universe distinguish between left and right?*”

1.4 Helicity

Momentum and spin of a massless particle

- Imagine a photon (γ) traveling at the speed of light with a momentum vector **p**.

- The photon is a spin-1 particle. The spin angular momentum vector **S** is a pseudovector. We then define “parallel” (right-handed) and “antiparallel” (left-handed) states as



“Parallel” (right-handed): $\mathbf{S} \cdot \mathbf{p} > 0$

“Antiparallel” (left-handed): $\mathbf{S} \cdot \mathbf{p} < 0$

Momentum and spin of a massless particle

- Imagine a photon (γ) traveling at the speed of light with a momentum vector **p**.

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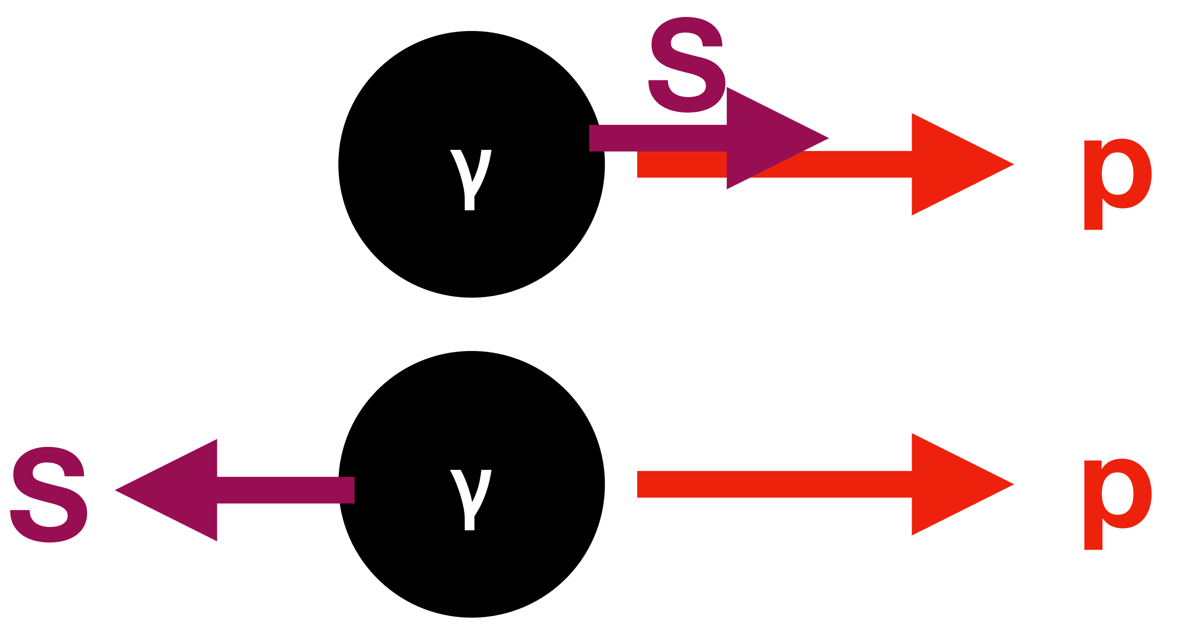
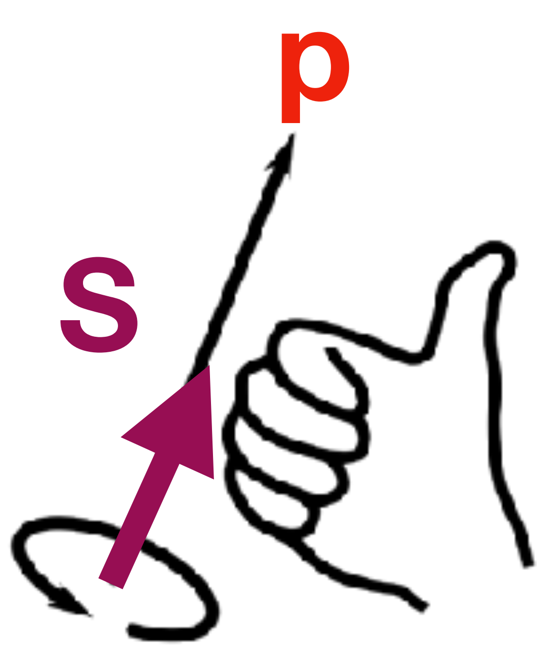


“Parallel” (right-handed): $\mathbf{S} \cdot \mathbf{p} > 0$

“Antiparallel” (left-handed): $\mathbf{S} \cdot \mathbf{p} < 0$

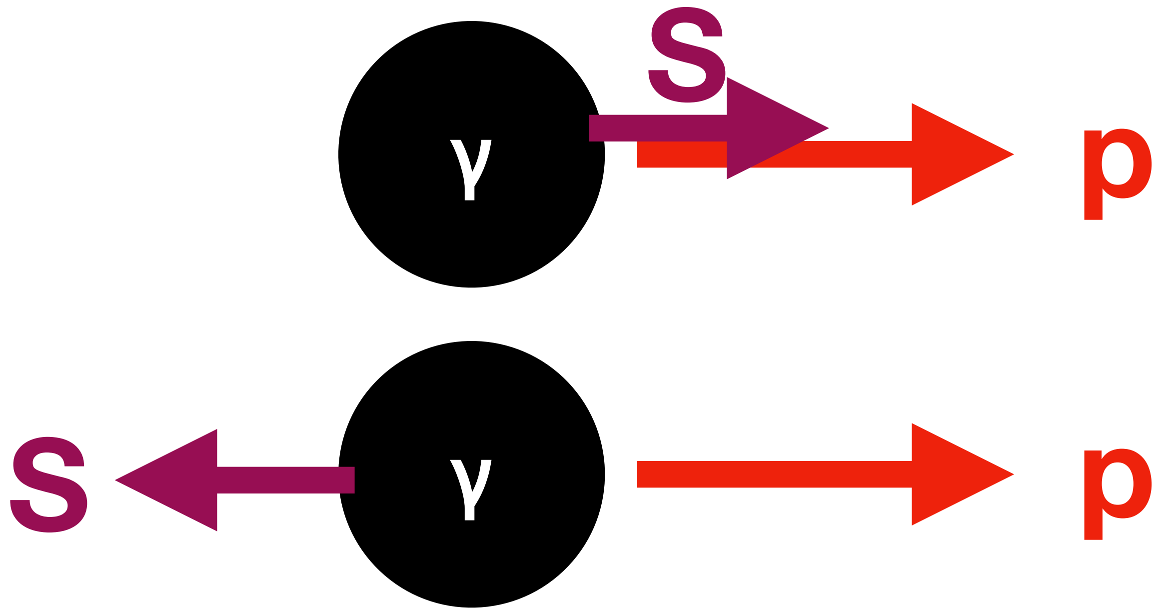


Right-handed

Momentum and spin of a massless particle

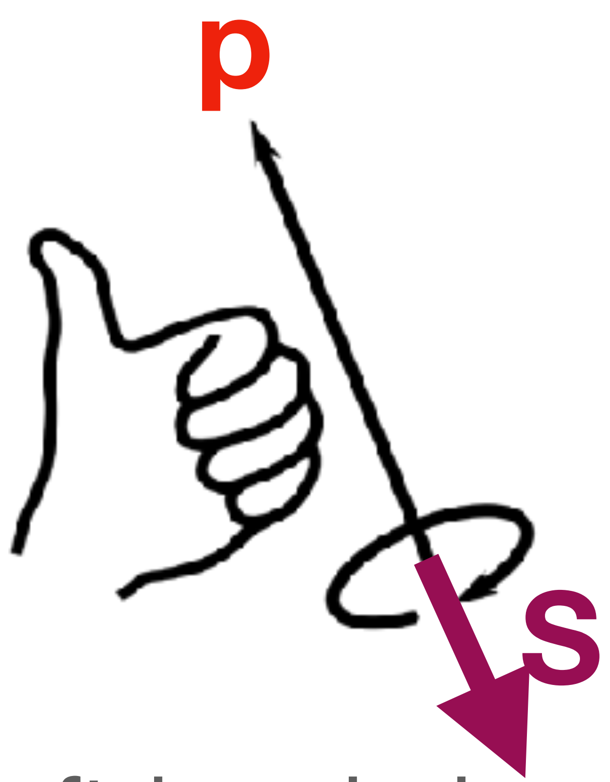
- Imagine a photon (γ) traveling at the speed of light with a momentum vector **p**.

- The photon is a spin-1 particle. The spin angular momentum vector **S** is a pseudovector. We then define “parallel” (right-handed) and “antiparallel” (left-handed) states as



“Parallel” (right-handed): $\mathbf{S} \cdot \mathbf{p} > 0$

“Antiparallel” (left-handed): $\mathbf{S} \cdot \mathbf{p} < 0$



Left-handed

Key Point

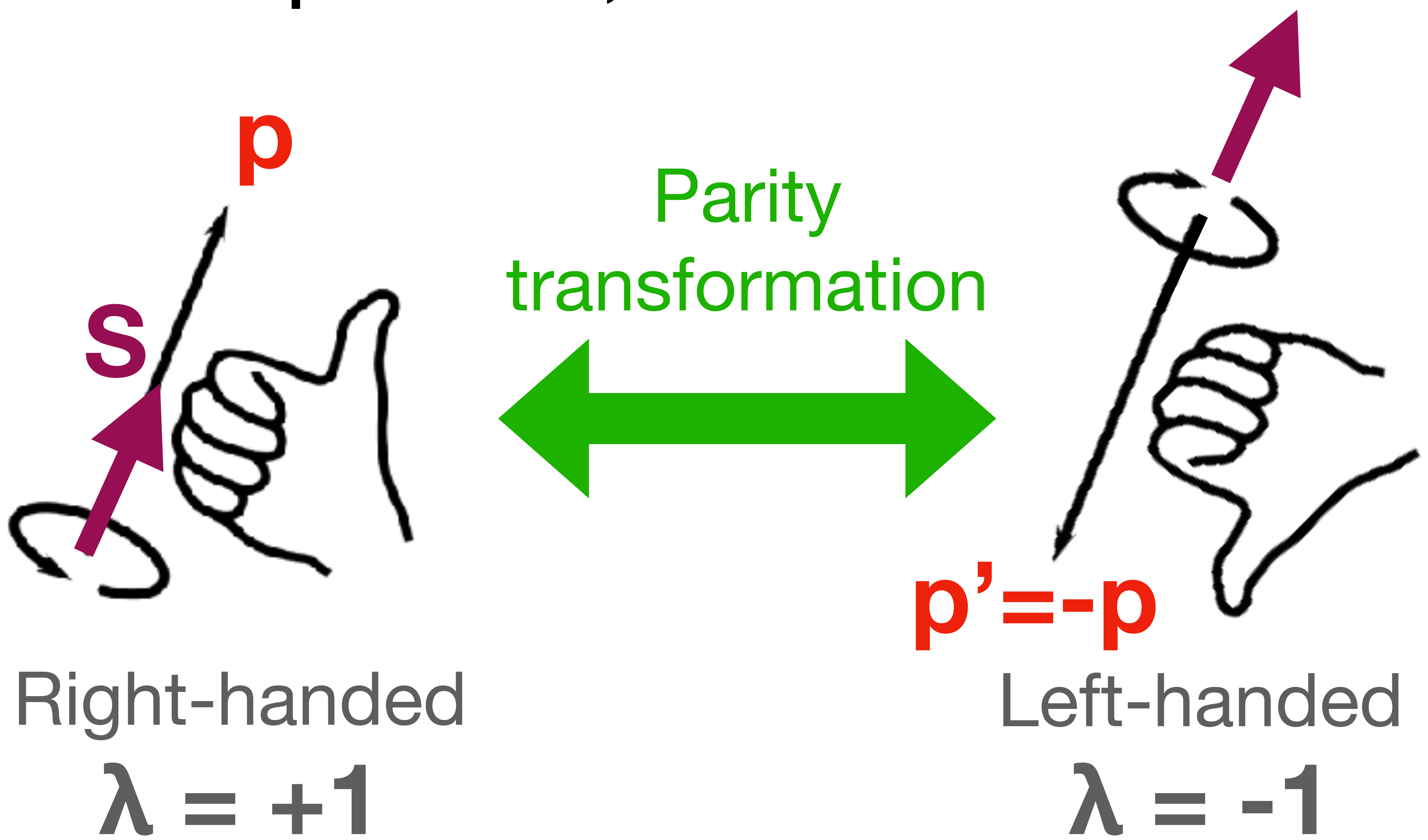
Helicity is a pseudoscalar

Party transformation changes “right-handed” to “left-handed” and vice versa

- For massless particles, we define the “helicity”, λ, as

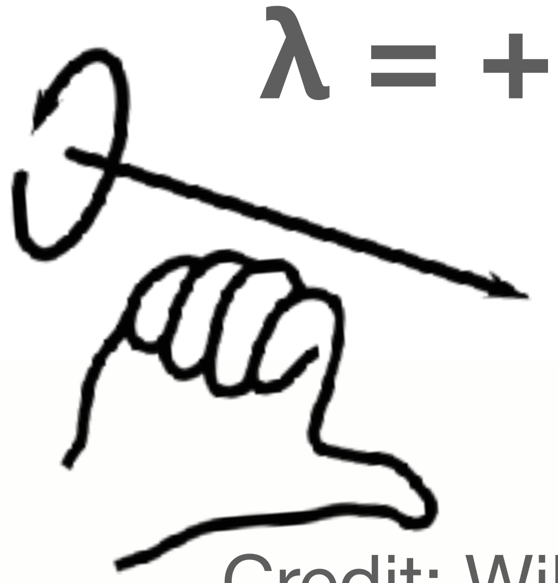
$$\mathbf{S}\cdot\frac{\mathbf{p}}{|\mathbf{p}|}=\lambda\hbar$$

- For a photon, λ=±1.



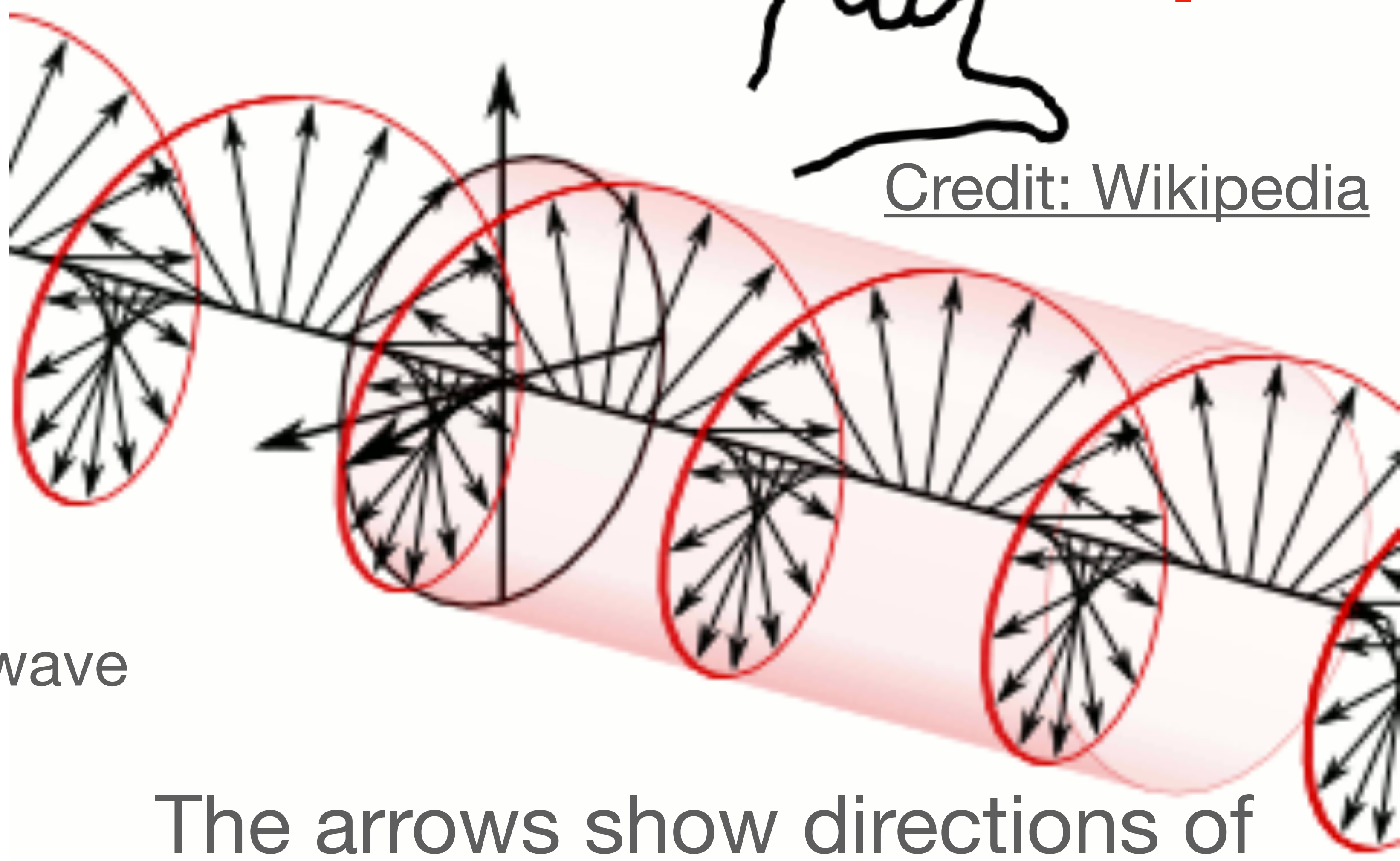
- **λ is a pseudoscalar** because it is a product of a momentum vector (**p**) and a spin pseudovector (**S**).
  - On the other hand, “scalar”, such as $\mathbf{p}^2$ and $\mathbf{S}^2$, does not change sign.
- For a graviton, λ=±2.
- **Asymmetry between λ=±1 and ±2 is the sign of parity violation!**

Helicity and circular polarization



- λ=±1 states of a photon describe circular polarization.
  - λ=+1: Right-handed circular polarization
  - λ=–1: Left-handed circular polarization

Many photons
= Electromagnetic wave



Credit: Wikipedia

The arrows show directions of the electric field vector **E**.

- Linear polarization is given by a super position of λ=±1 states with the equal number of photons. **Linear polarization carries no angular momentum!**
  - This means that circular polarization is a more fundamental state for photons. Circularly polarized light carries angular momentum, which is equal to $(N_+ - N_-)\hbar$ where $N_\pm$ is the number of λ=±1 photons.

1.5 Parity Symmetry in Electromagnetism (EM)

Throughout this talk, I will assume homogeneity and isotropy of space (invariance under 3d translation and rotation).

Maxwell's Equations

In Minkowski space, Heaviside units and *c*=1

$$\nabla \cdot \mathbf{E} = \rho\,, \qquad -\dot{\mathbf{E}} + \nabla \times \mathbf{B} = \mathbf{j}$$

$$\nabla \cdot \mathbf{B} = 0\,, \qquad \dot{\mathbf{B}} + \nabla \times \mathbf{E} = 0$$

- These equations are invariant under Poincaré transformation (spatial translation and rotation and Lorentz boost).

Throughout this talk, I will assume homogeneity and isotropy of space (invariance under 3d translation and rotation).

Parity-flipping Maxwell's Equations

In Minkowski space, Heaviside units and *c*=1

$$(-\nabla)\cdot(-\mathbf{E}) = \rho\,, \qquad -(-\dot{\mathbf{E}}) + (-\nabla)\times\mathbf{B} = (-\mathbf{j})$$

$$(-\nabla)\cdot\mathbf{B} = 0\,, \qquad \dot{\mathbf{B}} + (-\nabla)\times(-\mathbf{E}) = 0$$

- These equations are invariant under Poincaré transformation (spatial translation and rotation and Lorentz boost).
- They are also invariant under parity transformation, if **E** and **j** are vectors, ρ is a scalar, and **B** is a pseudovector.

Throughout this talk, I will assume homogeneity and isotropy of space (invariance under 3d translation and rotation).

Maxwell's Equations in a covariant form

$$\boxed{\nabla \cdot \mathbf{E} = \rho\,, \qquad -\dot{\mathbf{E}} + \nabla \times \mathbf{B} = \mathbf{j}}$$

$$\boxed{\nabla \cdot \mathbf{B} = 0\,, \qquad \dot{\mathbf{B}} + \nabla \times \mathbf{E} = 0}$$

- These equations can be written in a covariant form as

Dual tensor

$$\boxed{\partial_\nu F^{\mu\nu} = j^\mu} \qquad \boxed{\partial_\nu \tilde{F}^{\mu\nu} = 0}$$

$$\mu = 0, 1, 2, 3\,, \quad j^\mu = (\rho, \mathbf{j})\,, \quad \partial_\mu = \partial/\partial x^\mu\,, \quad x^\mu = (t, \mathbf{x})$$

Antisymmetric Field Strength Tensor, $F^{\mu\nu}$

$F^{\mu\nu} = -F^{\nu\mu}$

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}$$

- Equivalently,

$$\boxed{\begin{aligned} F^{0i} &= E_i \\ F^{ij} &= \epsilon^{ijk} B_k \end{aligned}}$$

$$F^{12} = B_z\,, F^{23} = B_x\,, F^{31} = B_y$$

$$F^{21} = -B_z\,, F^{32} = -B_x\,, F^{13} = -B_y$$

$$\epsilon^{ijk} = \begin{cases} +1 & \text{if (i,j,k) is even permutation of (1,2,3)} \\ -1 & \text{if (i,j,k) is odd permutation of (1,2,3)} \\ 0 & \text{otherwise} \end{cases}$$

Levi-Civita symbol

$$\epsilon^{123} = 1\,, \epsilon^{132} = -1\,, \epsilon^{312} = 1\,, \ldots$$

Antisymmetric Field Strength Tensor, $F_{\mu\nu}$

$F_{\mu\nu} = -F_{\nu\mu}$

$$F_{\mu\nu} = \eta_{\mu\alpha}\eta_{\nu\beta}F^{\alpha\beta}$$ where $$\eta_{\mu\alpha} = \mathrm{diag}(-1, 1, 1, 1)$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

- Therefore,

$$\boxed{F^2 \equiv F_{\mu\nu}F^{\mu\nu} = 2(\mathbf{B}\cdot\mathbf{B} - \mathbf{E}\cdot\mathbf{E})}$$

This is a *scalar* and is invariant under parity transformation.

Dual Field Strength Tensor, $\tilde{F}^{\mu\nu}$

$\tilde{F}^{\mu\nu} = -\tilde{F}^{\nu\mu}$

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$$ where $$\epsilon^{\mu\nu\alpha\beta} = \begin{cases} +1 & \text{if } (\mu,\nu,\alpha,\beta) \text{ is even perm. of } (0,1,2,3) \\ -1 & \text{if } (\mu,\nu,\alpha,\beta) \text{ is odd perm. of } (0,1,2,3) \\ 0 & \text{otherwise} \end{cases}$$

Levi-Civita symbol

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{pmatrix}$$

- Equivalently,

$$\boxed{\begin{aligned} \tilde{F}^{0i} &= B_i \\ \tilde{F}^{ij} &= -\epsilon^{ijk}E_k \end{aligned}}$$

- Therefore,

$$\boxed{F\tilde{F} \equiv F_{\mu\nu}\tilde{F}^{\mu\nu} = -4\mathbf{B}\cdot\mathbf{E}}$$

This is a *pseudoscalar* and changes sign under parity transformation!

1.6 Action Principle for EM

$$\partial_\nu F^{\mu\nu} = j^\mu, \quad \partial_\nu \tilde{F}^{\mu\nu} = 0$$

What is the action that gives Maxwell's equations?

In Heaviside units and *c*=1

- The answer is

$$I = -\frac{1}{4}\int d^4x \; F^2 + \int d^4x \; A_\mu j^\mu \qquad \boxed{d^4x = dt d^3\mathbf{x}}$$

with

Vector potential

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad \text{where } A_\mu = (-\phi, \mathbf{A})$$

Therefore,

$$\begin{cases} F_{i0} = \partial_i A_0 - \dot{A}_i = E_i & \Rightarrow \mathbf{E} = -\nabla\phi - \dot{\mathbf{A}} \\ F_{ij} = \partial_i A_j - \partial_j A_i = \epsilon_{ijk} B_k & \Rightarrow \mathbf{B} = \nabla \times \mathbf{A} \end{cases}$$

$$\partial_\nu F^{\mu\nu} = j^\mu, \checkmark \partial_\nu \tilde{F}^{\mu\nu} = 0$$

What is the action that gives Maxwell's equations?

In Heaviside units and *c*=1

- The answer is

$$I = -\frac{1}{4}\int d^4x\ F^2 + \int d^4x\ A_\mu j^\mu \qquad \boxed{d^4x = dt d^3\mathbf{x}}$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad \text{where } A_\mu = (-\phi, \mathbf{A})$$

One set of Maxwell's equations is simply given by the definition of $F_{\mu\nu}$:

$$\boxed{\partial_\nu \tilde{F}^{\mu\nu}} = \partial_\nu \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}(\partial_\alpha A_\beta - \partial_\beta A_\alpha) = \epsilon^{\mu\nu\alpha\beta}\partial_\nu\partial_\alpha A_\beta \boxed{= 0}$$

$$✓\partial_\nu F^{\mu\nu} = j^\mu, \quad ✓\partial_\nu \tilde{F}^{\mu\nu} = 0$$

What is the action that gives Maxwell's equations?

In Heaviside units and *c*=1

$$I = -\frac{1}{4}\int d^4x\ F^2 + \int d^4x\ A_\mu j^\mu$$

- **The idea**: The equation of motion for A_μ is the path that gives a stationary point. For a small change in $A_\mu \to A_\mu + \delta A_\mu$, the corresponding change in $I \to I + \delta I$ is also small.

$$\delta I = \int d^4x\ F^{\mu\nu}\partial_\nu(\delta A_\mu) + \int d^4x\ (\delta A_\mu) j^\mu$$

Hint:

$$\delta(F^2) = 2F^{\mu\nu}\delta F_{\mu\nu} = -4F^{\mu\nu}\partial_\nu(\delta A_\mu)$$

Integration by parts

$$= \int d^4x\ (-\partial_\nu F^{\mu\nu} + j^\mu)\delta A_\mu = 0 \Rightarrow \boxed{\partial_\nu F^{\mu\nu} = j^\mu}$$

1.7 Propagation of EM Waves

Finding symmetries in the action

It is like a treasure hunt!

$$I = -\frac{1}{4}\int d^4x\ F^2 + \int d^4x\ A_\mu j^\mu$$

- This action is invariant under spatial translation, rotation, and parity transformation.
- **It is also invariant under the following "gauge transformation",**

$$A_\mu \to A_\mu + \partial_\mu f$$

Hint:

$$\int d^4x\ (\partial_\mu f) j^\mu \overset{\text{Integration by parts}}{=} -\int d^4x\ f\partial_\mu j^\mu = 0$$

due to the charge conservation:

$$\partial_\mu j^\mu = 0 \Rightarrow \dot{\rho} + \nabla \cdot \mathbf{j} = 0$$

- Here, f is an arbitrary scalar function.

Finding symmetries in the action

It is like a treasure hunt!

$$I = -\frac{1}{4}\int d^4x\ F^2 + \int d^4x\ A_\mu j^\mu$$

- This action is invariant under spatial translation, rotation, and parity transformation.
- **It is also invariant under the following "gauge transformation",**

$$\phi \to \phi - \dot{f}$$
$$\mathbf{A} \to \mathbf{A} + \nabla f$$

Hint:

Integration by parts

$$\int d^4x\ (\partial_\mu f) j^\mu \overset{\Downarrow}{=} -\int d^4x\ f \partial_\mu j^\mu = 0$$

- Here, *f* is an arbitrary scalar function.

due to the charge conservation:

$$\partial_\mu j^\mu = 0 \Rightarrow \dot{\rho} + \nabla \cdot \mathbf{j} = 0$$

Warm-up: The wave equation for *A*$^\mu$

Maxwell's equations in vacuum

- Maxwell's equations in vacuum $\partial_\nu F^{\mu\nu} = 0$ gives

$$\boxed{-\Box A^\mu + \eta^{\mu\alpha}\partial_\alpha(\partial_\nu A^\nu) = 0}$$

  where

$$\Box = \eta^{\alpha\beta}\partial_\alpha\partial_\beta = -\frac{\partial^2}{\partial t^2} + \nabla^2$$

$$\boxed{\eta^{\alpha\beta} = \mathrm{diag}(-1, \mathbf{1})}$$

$$A^\mu = \eta^{\mu\alpha}A_\alpha = (\phi, \mathbf{A})$$

"Lorenz gauge condition"

- Now, using invariance under the gauge transformation, we can set $\partial_\nu A^\nu = 0$ by choosing $\Box f = -\partial_\nu A^\nu$ in $A_\mu \to A_\mu + \partial_\mu f$. Then…

Warm-up: The wave equation for *A*$^\mu$

The use case of the gauge invariance

"Lorenz gauge condition"

- Maxwell's equations in vacuum $\partial_\nu F^{\mu\nu} = 0$ and $\partial_\nu A^\nu = 0$ gives

$$\boxed{\Box A^\mu = 0}$$

➡ The equation for a wave traveling at the speed of light!

where

$$\Box = \eta^{\alpha\beta}\partial_\alpha\partial_\beta = -\frac{\partial^2}{\partial t^2} + \nabla^2$$

$$A^\mu = \eta^{\mu\alpha}A_\alpha = (\phi, \mathbf{A}) \text{ with } \dot{\phi} + \nabla \cdot \mathbf{A} = 0$$

- The number of degrees of freedom for *A*$^\mu$ is **3** due to the Lorenz gauge condition.

Physical degrees of freedom of EM waves

3? 2?

- We know that photons must have only 2 helicity states, $\lambda=\pm1$ (two circular polarization states).
  - Shouldn't the number of physical degrees of freedom be **2**, instead of 3? The answer is yes.
- The Lorenz gauge does not fully specify A^μ. We can still add

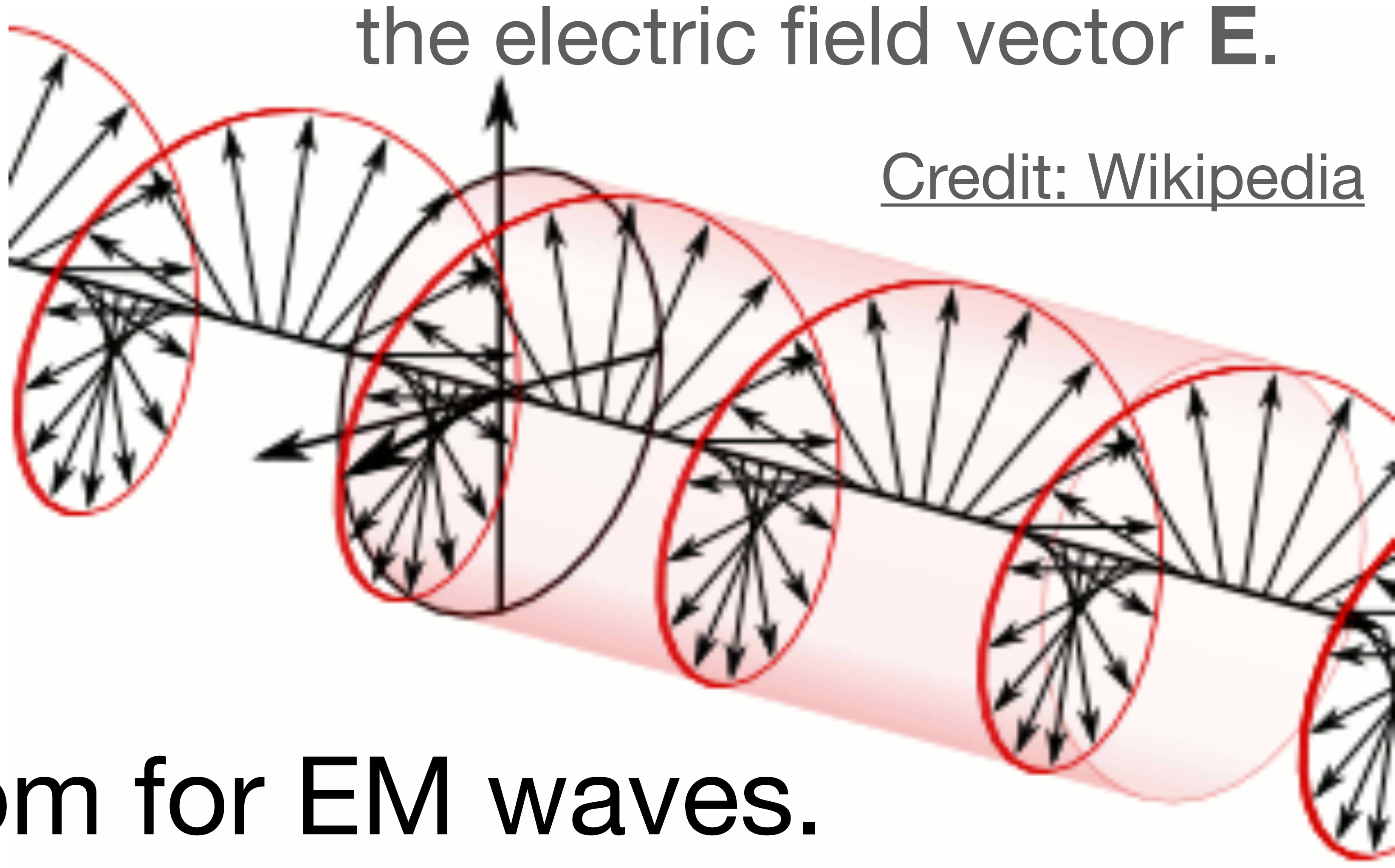
  $$A_\mu \to A_\mu + \partial_\mu f_2$$

  which satisfies $\Box f_2 = 0$
- Choosing f_2 will fully specify A^μ. This leaves **2** degrees of freedom.

EM waves must be transverse

The arrows show directions of the electric field vector **E**.

Credit: Wikipedia



- It is common to choose *f* and f_2 such that

$$\phi = 0, \qquad \nabla \cdot \mathbf{A} = 0$$

in vacuum "Coulomb gauge condition"

- We have 2 conditions. That leaves **2** degrees of freedom for EM waves.
- This choice is consistent with the Lorenz gauge condition $\dot{\phi} + \nabla \cdot \mathbf{A} = 0$
- $\nabla \cdot \mathbf{A} = 0$ requires that the EM wave be *transverse*, i.e., the change in **A** is perpendicular to the direction of propagation of the EM wave.
- **We will use this condition throughout the lecture.**

Maxwell's equations in vacuum

Expanding space, in Heaviside units and *c*=1

- In expanding space, one obtains

$$\boxed{\mathbf{A}'' - \nabla^2 \mathbf{A} = 0} \quad \text{where} \quad ' = \frac{\partial}{\partial \tau} = a \frac{\partial}{\partial t}$$

- Remarkably, it takes the same form as in non-expanding space, except for **the change of variables** from the physical time, *t*, to the **conformal time, τ.**

- The distance between two points in 4d spacetime:

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -dt^2 + d\mathbf{x}^2 \qquad \text{[Non-expanding space]}$$

$$\to -dt^2 + a^2(t) d\mathbf{x}^2 = a^2(\tau)[-d\tau^2 + d\mathbf{x}^2] \qquad \text{[Expanding space]}$$

$$= g_{\mu\nu} dx^\mu dx^\nu \quad \text{with } x^\mu = (\tau, \mathbf{x}) \qquad \boxed{g_{\mu\nu} = a^2 \mathrm{diag}(-1, \mathbf{1}) = a^2 \eta_{\mu\nu}}$$

Recap: Lecture 1

Known Physics

- Parity transformation changes “right-handed” to “left-handed” and vice versa.
  - Violation of parity symmetry: Nature distinguishes right- and left-handed states, which has been observed in β-decay. ***How about the Universe?***
- Transformation properties:
  - **Vector** (such as momentum) changes sign under parity transformation.
  - **Pseudovector** (such as angular momentum and spin) does not.
  - **Pseudoscalar** (such as helicity) does => Key to testing for parity violation!

Problem Set

Prelude to the next lecture

Show that $F\tilde{F}$ is a total derivative and can be written as

$$F_{\mu\nu}\tilde{F}^{\mu\nu} = 2\partial_\mu(A_\nu\tilde{F}^{\mu\nu})$$