New Probes of Initial State of Quantum Fluctuations during Inflation

Eiichiro Komatsu (Texas Cosmology Center, Univ. of Texas at Austin; Max-Planck-Institut für Astrophysik)
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This talk is based on...

- Squeezed-limit bispectrum
  - Ganc & Komatsu, JCAP, 12, 009 (2010)
- Non-Bunch-Davies vacuum and CMB
  - Ganc, PRD 84, 063514 (2011)
- Scale-dependent bias and $\mu$-distortion
  - Ganc & Komatsu, arXiv:1204.4241
Does this plot prove inflation?

\[
\ell (\ell + 1) C_{\ell}^{TT} / (2\pi) \text{ [\(\mu K^2\)]}
\]

Multipole Moment \((\ell) = 180 \text{ deg/\(\Omega\)}\)
Motivation

• Can we falsify inflation?
Falsifying “inflation”

• We still need inflation to explain the flatness problem!

• (Homogeneity problem can be explained by a bubble nucleation.)

• However, the observed fluctuations may come from different sources.

• So, what I ask is, “can we rule out inflation as a mechanism for generating the observed fluctuations?”
First Question:

- Can we falsify **single-field** inflation?

*I will not be talking about multi-field inflation today: for potentially ruling out multi-field inflation, see Sugiyama, Komatsu & Futamase, PRL, 106, 251301 (2011)*
An Easy One: Adiabaticity

- Single-field inflation = One degree of freedom.
- Matter and radiation fluctuations originate from a single source.

\[ S_{c,\gamma} \equiv \frac{\delta \rho_c}{\rho_c} - \frac{3\delta \rho_\gamma}{4\rho_\gamma} = 0 \]

* A factor of 3/4 comes from the fact that, in thermal equilibrium, \( \rho_c \sim \rho_\gamma^{3/4} \)
Non-adiabatic Fluctuations

- Detection of non-adiabatic fluctuations immediately rule out single-field inflation models.

The current CMB data are consistent with adiabatic fluctuations:

\[
\frac{|\delta \rho_c / \rho_c - 3 \delta \rho_\gamma / (4 \rho_\gamma)|}{\frac{1}{2} [\delta \rho_c / \rho_c + 3 \delta \rho_\gamma / (4 \rho_\gamma)]} < 0.09 \text{ (95% CL)}
\]
Single-field inflation looks good (in 2-point function)

- $P_{\text{scalar}}(k) \sim k^{4-n_s}$
- $n_s = 0.968 \pm 0.012$ (68%CL; WMAP7+BAO+H$_0$)
- $r = 4P_{\text{tensor}}(k)/P_{\text{scalar}}(k)$
- $r < 0.24$ (95%CL; WMAP7+BAO+H$_0$)

Komatsu et al. (2011)
So, let’s use 3-point function

• Three-point function (bispectrum)

• $B_\zeta(k_1, k_2, k_3)$

  $= \langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \rangle = \text{(amplitude)} \times (2\pi)^3 \delta(k_1 + k_2 + k_3) b(k_1, k_2, k_3)$

  model-dependent function
MOST IMPORTANT, for falsifying single-field inflation
Curvature Perturbation

• In the gauge where the energy density is uniform, $\delta \rho = 0$, the metric on super-horizon scales ($k << aH$) is written as

$$ds^2 = -N^2(x,t)dt^2 + a^2(t)e^{2\zeta(x,t)}dx^2$$

• We shall call $\zeta$ the “curvature perturbation.”

• This quantity is independent of time, $\zeta(x)$, on super-horizon scales for single-field models.

• The lapse function, $N(x,t)$, can be found from the Hamiltonian constraint.
Action

• Einstein’s gravity + a canonical scalar field:

\[ S = \frac{1}{2} \int d^4x \sqrt{-g} \left[ R - (\partial \Phi)^2 - 2V(\Phi) \right] \]
Quantum-mechanical Computation of the Bispectrum

\[ \langle \chi^3 (\bar{t}) \rangle = -i \int_{-(1-i\epsilon)\infty}^{\bar{t}} dt' \langle 0 | [\chi^3 (\bar{t}), H_I^{(3)} (t')] | 0 \rangle \]

\[ S_{\text{int}}^{(3)} = \int \frac{1}{4} \frac{\dot{\rho}^4}{\rho^4} [e^{3\rho} \dot{\chi}^2 \chi + e^\rho (\partial \chi)^2 \chi] - \frac{\dot{\phi}^2}{\rho^2} e^{3\rho} \dot{\chi} \partial_i \chi \partial_i \chi + \]

\[ - \frac{1}{16} \frac{\dot{\rho}^6}{\rho^6} e^{3\rho} \dot{\chi}^2 \chi + \frac{\dot{\phi}^2}{\rho^2} e^{3\rho} \dot{\chi} \partial_i \chi \partial_i \chi + \frac{1}{4} \frac{\phi^2}{\rho^2} e^{3\rho} \partial_i \partial_j \chi \partial_i \partial_j \chi \]

\[ + f(\chi) \left. \frac{\delta L}{\delta \chi} \right|_1 \]

\[ \partial^2 \chi = \frac{\dot{\phi}}{2} \frac{\ddot{\rho}}{\dot{\rho}^2} \chi \]

\[ H \equiv \dot{\rho} \]
Initial Vacuum State

\[ \zeta_k(t) = u_k(t)a_k + u_k^*(t)a_k^\dagger \]

- Bunch-Davies vacuum, \( a_k|0\rangle = 0 \) with

\[ u_k(\eta) = \frac{H^2}{\phi} \frac{1}{\sqrt{2k^3}} (1 + ik\eta) e^{-ik\eta} \]

[\( \eta \): conformal time]
• $B_\zeta(k_1,k_2,k_3) = \langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \rangle = \text{(amplitude)} \times (2\pi)^3 \delta(k_1+k_2+k_3) b(k_1,k_2,k_3)$

• $b(k_1,k_2,k_3) = \frac{\ddot{\rho}^4_*}{\dot{\rho}^4_*} \frac{H^4_*}{M_{pl}^4} \frac{1}{\prod_i (2k_i^3)}$
  
  $$\left\{ 2 \frac{\dddot{\phi}^2_*}{\dot{\phi}^2_* \ddot{\rho}^2_*} \sum_i k_i^3 + \frac{\dddot{\phi}^2_*}{\dot{\phi}^2_*} \frac{1}{2} \sum_i k_i^3 + \frac{1}{2} \sum_{i \neq j} k_i k_j^2 + 4 \frac{\sum_{i>j} k_i^2 k_j^2}{k_t} \right\}$$

Maldacena (2003)

Complicated? But...
Taking the squeezed limit
\((k_3 \ll k_1 \approx k_2)\)

- \(B_\zeta(k_1, k_2, k_3) = \langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \rangle = (\text{amplitude}) \times (2\pi)^3 \delta(k_1 + k_2 + k_3) b(k_1, k_2, k_3)\)

- \(b(k_1, k_1, k_3 \to 0) = \frac{\dot{\rho}^4_*}{\phi_*} \frac{H^4_*}{M^4_{\text{pl}}} \frac{1}{\prod_i (2k_i^3)}\)

\[
\times \left\{ 2\frac{\ddot{\phi}_*}{\dot{\phi}_* \rho_*} \sum_i k_i^3 + \frac{\dot{\phi}_*^2}{\dot{\rho}_*^2} \right\}
\]

\[
\begin{align*}
&= 2k_1^3 \quad \text{k}_1^3 \quad \text{k}_1^3 \quad \frac{\sum_{i > j} k_i^2 k_j^2}{k_t} \\
&= 2k_1^3
\end{align*}
\]
Taking the squeezed limit \((k_3 << k_1 \approx k_2)\)

1. \(B_\zeta(k_1, k_2, k_3) = \langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \rangle = \text{(amplitude)} \times (2\pi)^3 \delta(k_1 + k_2 + k_3)b(k_1, k_2, k_3)\)

2. \(b(k_1, k_1, k_3 \to 0) = \frac{\dot{\rho}_*^4}{\phi_*^4 M_{pl}^4} \frac{H_*^4}{\dot{\phi}_*^2} \left[ \frac{\ddot{\phi}_*}{\phi_*} + \frac{\dot{\phi}_*^2}{\rho_*^2} \right] \frac{1}{k_1^3 k_3^3}\)

\[= 1 - n_s\]

\[= (1 - n_s) P_\zeta(k_1) P_\zeta(k_3)\]
Single-field Theorem
(Consistency Relation)

- For **ANY** single-field models*, the bispectrum in the squeezed limit \((k_3 \ll k_1 \approx k_2)\) is given by

\[
B_\zeta(k_1,k_1,k_3 \rightarrow 0) = (1-n_s) \times (2\pi)^3 \delta(k_1+k_2+k_3) \times P_\zeta(k_1)P_\zeta(k_3)
\]

* for which the single field is solely responsible for driving inflation and generating observed fluctuations.

* Maldacena (2003); Seery & Lidsey (2005); Creminelli & Zaldarriaga (2004)
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\]

\[
\frac{6}{5} f_{NL} = \frac{B_\zeta(k_1, k_2, k_3)}{P_\zeta(k_1) P_\zeta(k_2) + P_\zeta(k_2) P_\zeta(k_3) + P_\zeta(k_3) P_\zeta(k_1)}
\]

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For **ANY** single-field models*, the bispectrum in the squeezed limit ($k_3 \ll k_1 \approx k_2$) is given by

$$B_{\zeta}(k_1, k_1, k_3 \rightarrow 0) = (1-n_s) \times (2\pi)^3 \delta(k_1+k_2+k_3) \times P_{\zeta}(k_1)P_{\zeta}(k_3)$$

Therefore, all single-field models predict $f_{NL} \approx (5/12)(1-n_s)$.

With the current limit $n_s = 0.96$, $f_{NL}$ is predicted to be 0.017.

* for which the single field is solely responsible for driving inflation and generating observed fluctuations.
Limits on $f_{NL}$

$$\frac{6}{5} f_{NL} = \frac{B_3(k_1, k_2, k_3)}{P_g(k_1) P_g(k_2) + P_g(k_2) P_g(k_3) + P_g(k_3) P_g(k_1)}$$

When $f_{NL}$ is independent of wavenumbers, it is called the "local type."
Limits on $f_{\text{NL}}$

- $f_{\text{NL}} = 32 \pm 21$ (68\%C.L.) from WMAP 7-year data
- Planck’s CMB data is expected to yield $\Delta f_{\text{NL}}=5$.
- $f_{\text{NL}} = 27 \pm 16$ (68\%C.L.) from WMAP 7-year data combined with the limit from the large-scale structure (by Slosar et al. 2008)
- Future large-scale structure data are expected to yield $\Delta f_{\text{NL}}=1$. 

\[
\frac{6}{5}f_{\text{NL}} \equiv \frac{B_2(k_1,k_2,k_3)}{P_2(k_1)P_2(k_2)+P_3(k_2)P_2(k_3)+P_2(k_3)P_2(k)}
\] 

Komatsu et al. (2011)
Understanding the Theorem

- First, the squeezed triangle correlates one very long-wavelength mode, $k_L (=k_3)$, to two shorter wavelength modes, $k_S (=k_1 \approx k_2)$:
  - $<\zeta_{k_1} \zeta_{k_2} \zeta_{k_3}> \approx <(\zeta_{k_S})^2 \zeta_{k_L}>$

- Then, the question is: “why should $(\zeta_{k_S})^2$ ever care about $\zeta_{k_L}$?”
  - The theorem says, “it doesn’t care, if $\zeta_k$ is exactly scale invariant.”
The long-wavelength curvature perturbation rescales the spatial coordinates (or changes the expansion factor) within a given Hubble patch:

\[ ds^2 = -dt^2 + [a(t)]^2 e^{2\zeta} (dx)^2 \]

\[ x_1 = x_0 e^{\zeta_1} \]

\[ x_2 = x_0 e^{\zeta_2} \]

\( \zeta_{KL} \) rescales coordinates

Separated by more than \( H^{-1} \)

left the horizon already
\( \zeta_{kL} \) rescales coordinates

- Now, let’s put small-scale perturbations in.
- Q. How would the conformal rescaling of coordinates change the amplitude of the small-scale perturbation?

\[ x_1 = x_0 e^{\zeta_1} \]

\[ x_2 = x_0 e^{\zeta_2} \]

\( x_1 \) and \( x_2 \) are separated by more than \( H^{-1} \).
\( \zeta_{kL} \) rescales coordinates

- Q. How would the conformal rescaling of coordinates change the amplitude of the small-scale perturbation?
- A. No change, if \( \zeta_k \) is scale-invariant. In this case, no correlation between \( \zeta_{kL} \) and \( (\zeta_{kS})^2 \) would arise.

\[ \begin{align*}
\text{Separated by more than } H^{-1} & \\
\mathbf{x}_1 &= \mathbf{x}_0 e^{\zeta_1} \\
(\zeta_{kS1})^2 & \\
\mathbf{x}_2 &= \mathbf{x}_0 e^{\zeta_2} \\
(\zeta_{kS2})^2 &
\end{align*} \]

\( \zeta_{kL} \leftarrow \) left the horizon already
Real-space Proof

- The 2-point correlation function of short-wavelength modes, $\xi = \langle \zeta_S(x) \zeta_S(y) \rangle$, within a given Hubble patch can be written in terms of its vacuum expectation value (in the absence of $\zeta_L$), $\xi_0$, as:

  - $\xi_{\zeta L} \approx \xi_0(|x-y|) + \zeta_L \left[ d\xi_0(|x-y|)/d\zeta_L \right]$
  - $\xi_{\zeta L} \approx \xi_0(|x-y|) + \zeta_L \left[ d\xi_0(|x-y|)/d\ln|x-y| \right]$
  - $\xi_{\zeta L} \approx \xi_0(|x-y|) + \zeta_L \left( 1-n_s \right) \xi_0(|x-y|)$

$3$-pt func. $= \langle (\zeta_S)^2 \zeta_L \rangle = \langle \xi_{\zeta L} \zeta_L \rangle$
$= \left( 1-n_s \right) \xi_0(|x-y|) \langle \zeta_L^2 \rangle$
This is great, but...

• The proof relies on the following Taylor expansion:
  
  $\langle \zeta_S(x)\zeta_S(y) \rangle_{\zeta_L} = \langle \zeta_S(x)\zeta_S(y) \rangle_0 + \zeta_L \left[ d\langle \zeta_S(x)\zeta_S(y) \rangle_0 / d\zeta_L \right]$

• Perhaps it is interesting to show this explicitly using the in-in formalism.

  • Such a calculation would shed light on the limitation of the above Taylor expansion.

  • Indeed it did - we found a non-trivial “counter-example” (more later)
An Idea

• How can we use the in-in formalism to compute the two-point function of short modes, given that there is a long mode, $\langle \zeta_S(x) \zeta_S(y) \rangle_{\zeta_L}$?

• Here it is!

$$\langle \zeta_S^2(t) \rangle_{\zeta_L} = -i \int_{-(1-i\epsilon)\infty}^{t} dt' \langle 0 \left| [\zeta_S^2(t), H_I^{(3)}(t')] \right| 0 \rangle$$

Ganc & Komatsu, JCAP, 12, 009 (2010)
Inserting $\zeta = \zeta_L + \zeta_S$ into the cubic action of a scalar field, and retain terms that have one $\zeta_L$ and two $\zeta_S$'s.

\[
S_{\text{int}}^{(3)} = \int d^4x \left[ \left( \frac{1}{4} \frac{\dot{\phi}_0^4}{H^4} - \frac{1}{16} \frac{\dot{\phi}_0^6}{H^6} \right) a^3 \zeta_L \dot{\zeta}_S^2 + \frac{1}{4} \frac{\dot{\phi}_0^4}{H^4} a \zeta_L (\partial \zeta_S)^2 - \frac{\dot{\phi}_0^4}{2H^4} a^3 \zeta_S \partial_i \zeta_S \partial_i \partial^{-2} \dot{\zeta}_L + \right.
\]
\[
+ \frac{1}{16} \frac{\dot{\phi}_0^6}{H^6} a^3 \partial_i \partial_j \partial^{-2} \dot{\zeta}_S \partial_i \partial_j \partial^{-2} \dot{\zeta}_S \zeta_L + 2 \frac{\dot{\phi}_0^2}{H^2} a^3 \zeta_L \frac{d}{dt} \left[ \frac{1}{2} \frac{\dot{\phi}_0}{\phi_0 H} + \frac{1}{4} \frac{\dot{\phi}_0^2}{H^2} \right] \dot{\zeta}_S \zeta_S
\]
\[
- f(\zeta) \frac{\delta L_0}{\delta \zeta_S} \right] ,
\]
\[ \langle \zeta S, k_1 \zeta S, k_2 \rangle \zeta_{k_3} = \zeta_{L, k_1+k_2} \left[ K + \left( \frac{\dot{\phi}_0}{\phi_0 H} + \frac{1}{2} \frac{\phi_0^2}{H^2} \right) P(k_1) \right] \]

- where

\[
K \equiv i u_{k_1}^2(\bar{\eta}) \int_{-\infty}^{\bar{\eta}} d\eta \left[ \frac{1}{2} \frac{\dot{\phi}_0^4}{H^4} a^2 u_{k_1}^{*2}(\eta) + \frac{1}{2} \frac{\dot{\phi}_0^4}{H^4} a^2 k_1^2 u_{k_1}^{*2}(\eta) + \frac{\phi_0^2}{H^2} a^3 \frac{d}{dt} \left( \frac{\ddot{\phi}_0}{\phi_0 H} + \frac{1}{2} \frac{\phi_0^2}{H^2} \right) u_{k_1}^{*}(\eta) u_{k_1}^{*}(\eta) \right] + \text{c.c.}
\]
• Although this expression looks nothing like $(1-n_S)P(k_1)\zeta_{kL}$, we have verified that it leads to the known consistency relation for (i) slow-roll inflation, and (ii) power-law inflation.

• But, there was a curious case – Alexei Starobinsky’s exact $n_S=1$ model.
  • If the theorem holds, we should get a vanishing bispectrum in the squeezed limit.
Starobinsky’s Model

- The famous Mukhanov-Sasaki equation for the mode function is

\[ \frac{d^2 u_k}{d\eta^2} + \left( k^2 - \frac{1}{z} \frac{d^2 z}{d\eta^2} \right) u_k = 0 \]

where

\[ z = \frac{a \dot{\phi}}{H} \]

- The scale-invariance results when

\[ \frac{1}{z} \frac{d^2 z}{d\eta^2} = \frac{2}{\eta^2} \]

So, let’s write

\[ z = \frac{B}{\eta} \]
Starobinsky’s Potential

- This potential is a one-parameter family; this particular example shows the case where inflation lasts very long: \( \phi_{\text{end}} \rightarrow \infty \)
\[
\langle \zeta_{S,k_1} \zeta_{S,k_2} \rangle \zeta_{k_3} = \zeta_{L,k_1+k_2} 4P(k_1)(k_1 \eta_{\text{start}})^2 e^{-\frac{1}{2} \phi_{\text{end}}^2}
\]

- It does not vanish!

- But, it approaches zero when \( \Phi_{\text{end}} \) is large, meaning the duration of inflation is very long.

- In other words, this is a condition that the longest wavelength that we observe, \( k_3 \), is far outside the horizon.

- In this limit, the bispectrum approaches zero.
Initial Vacuum State?

• What we learned so far:
  • The squeezed-limit bispectrum is proportional to $(1-n_s)P(k_1)P(k_3)$, provided that $\zeta_{k_3}$ is far outside the horizon when $k_1$ crosses the horizon.
  
• What if the state that $\zeta_{k_3}$ sees is not a Bunch-Davies vacuum, but something else?
  
• The exact squeezed limit ($k_3->0$) should still obey the consistency relation, but perhaps something happens when $k_3/k_1$ is small but finite.
• With CMB, we can measure primordial modes in $l=2$–3000. Therefore, $k_3/k_1$ can be as small as $1/1500$. 

Keisler et al. (2011)
With large-scale structure, we can measure primordial modes in $k = 10^{-3} - 1$ Mpc$^{-1}$. Therefore, $k_3/k_1$ can be as small as $1/1000$.

Hlozek et al. (2011)
Using the distortion of the thermal spectrum of CMB, we can reach $k_3/k_1$ as small as $10^{-8}$! (Pajer & Zaldarriaga 2012)

(plot from Samtleben et al. 2007)
Back to in-in

\[ \langle \zeta^3(t^*) \rangle = -i \int_{t_0}^{t^*} dt' \langle 0 | [\zeta^3(t^*), H_I(t')] | 0 \rangle \]

\[ B_\zeta(k_1, k_2, k_3) = 2i \frac{\phi^4}{H^6} \sum_i \left( \frac{1}{k_i^2} \right) \tilde{u}_{k_1}(\bar{\eta}) \tilde{u}_{k_2}(\bar{\eta}) \tilde{u}_{k_3}(\bar{\eta}) \int_{\eta_0}^{\bar{\eta}} d\eta \frac{1}{\eta^3} u'_{k_1} u'_{k_2} u'_{k_3} + \text{c.c.} \]

- The Bunch-Davies vacuum: \( u_k' \sim \eta e^{-ik\eta} \) (positive frequency mode)
- The integral yields \( 1/(k_1+k_2+k_3) \rightarrow 1/(2k_1) \) in the squeezed limit
Back to in-in

\[ \langle \zeta^3(t^*) \rangle = -i \int_{t_0}^{t^*} dt' \langle 0 | [\zeta^3(t^*), H_I(t')] | 0 \rangle \]

\[ B_\zeta(k_1, k_2, k_3) = 2i \frac{\phi^4}{H^6} \sum_i \left( \frac{1}{k_i^2} \right) \bar{u}_{k_1}(\vec{\eta}) \bar{u}_{k_2}(\vec{\eta}) \bar{u}_{k_3}(\vec{\eta}) \int_{\vec{\eta}_0}^{\vec{\eta}} d\eta \frac{1}{\eta_3} u'_{k_1} u'_{k_2} u'_{k_3} + \text{c.c.} \]

- Non-Bunch-Davies vacuum: \( u'_k \sim \eta(A_k e^{-i\eta} + B_k e^{i\eta}) \)
- The integral yields \( 1/(k_1-k_2+k_3) \), peaking in the folded limit \( \text{Chen et al. (2007); Holman & Tolley (2008)} \)
- The integral yields \( 1/(k_1-k_2+k_3) \to 1/(2k_3) \) in the squeezed limit \( \text{Enhanced by } k_1/k_3: \text{this can be a big factor!} \) \( \text{Agullo & Parker (2011)} \)
How about the consistency relation?

\[ B_\zeta(k_1, k_2, k_3) \xrightarrow{k_3/k_1 \ll 1} P_\zeta(k_1) P_\zeta(k_3) \left\{ (1 - n_s) + 4 \frac{\dot{\phi}^2 k_1}{H^2 k_3} [1 - \cos(k_3 \eta_0)] \right\} \]

- When \( k_3 \) is far outside the horizon at the onset of inflation, \( \eta_0 \) (whatever that means), \( k_3 \eta_0 \rightarrow 0 \), and thus the above additional term vanishes.

- The consistency relation is restored.
An interesting possibility:

- What if $k_3 \eta_0 = O(1)$?
- The squeezed bispectrum receives an enhancement of order $\varepsilon k_1/k_3$, which can be sizable.
- Most importantly, the bispectrum grows faster than the local-form toward $k_3/k_1 \to 0$!
  - $B_\zeta(k_1,k_2,k_3) \sim 1/k_3^3$ [Local Form]
  - $B_\zeta(k_1,k_2,k_3) \sim 1/k_3^4$ [non-Bunch-Davies]
- This has an observational consequence – particularly a scale-dependent bias and distortion of CMB spectrum.
Power Spectrum of Galaxies

• Galaxies do not trace the underlying matter density fluctuations perfectly. They are biased tracers.

• “Bias” is operationally defined as

\[ b_{\text{galaxy}}^2(k) = \frac{\langle |\delta_{\text{galaxy},k}|^2 \rangle}{\langle |\delta_{\text{matter},k}|^2 \rangle} \]
Density-$\zeta$ Relation

- It is given by the Poisson equation:

$$\delta_{m,k}(z) = \frac{2k^2}{5H_0^2\Omega_m} \zeta_k T(k) D(k, z)$$

- $T(k) \rightarrow 1$ for $k << 10^{-2}$ Mpc$^{-1}$
- $T(k) \rightarrow (\ln k)^2/k^4$ for $k >> 10^{-2}$ Mpc$^{-1}$

- $D(k,z) = 1/(1+z)$ during the matter-dominated era

Positive $\zeta_k \rightarrow$ positive $\delta_{m,k}$!
Galaxy clustering modified by the squeezed limit

- The existence of long-wavelength $\zeta$ changes the small-scale power of $\delta_m$.

- **A positive long-wavelength $\zeta$ -> more power on small scales.**

- More power on small scales -> more galaxies formed.
Scale-dependent Bias

\[ \Delta b(k,R) = 2 \frac{\mathcal{F}_R(k)}{M_R(k)} \left[ (b_1 - 1) \delta_c \right], \]

\[ \mathcal{F}_R(k) \approx \frac{1}{4\sigma^2_R P_\zeta(k)} \int \frac{d^3k_1}{(2\pi)^3} M^2_R(k_1) B_\zeta(k_1, k_1, k) \]

- For a local-form (p=3), it goes like \( b(k) \sim 1/k^2 \)

- For a non-Bunch-Davies vacuum (p=4), would it go like \( b(k) \sim 1/k^3 \)?
It does!

Ganc & Komatsu (2012)

\[ \Delta b_{\text{galaxy}}(k)/b_{\text{galaxy}} \sim k^{-3} \]

\[ \sim k^{-2} \]

non-BD vacuum

\( (\varepsilon=0.01; N_k=1) \)

Local

(\( f_{NL}=10 \))

- local form, \( f_{NL}=1 \)

- nBD model, \( \theta_k \approx k \eta_0 \)

Wavenumber, \( k [h \text{ Mpc}^{-1}] \)

\( \Delta b_{\text{galaxy}}(k)/b_{\text{galaxy}} \)
The expected contribution to $f_{NL}$ as measured by the CMB bispectrum is typically $f_{NL} \approx 8(\varepsilon/0.01)$.

- A lot bigger than $(5/12)(1-n_S)$, and could be detectable with Planck.

- Note that this does not mean a violation of the single-field consistency condition, which is valid in the exact squeezed limit, $k_3 \rightarrow 0$.

- We have an enhanced bispectrum in the squeezed configuration where $k_3/k_1$ is small but finite.
Using the distortion of the thermal spectrum of CMB, we can reach $k_3/k_1$ as small as $10^{-8}$! (Pajer & Zaldarriaga 2012)

(plot from Samtleben et al. 2007)
Damping of Acoustic Waves

- Energy stored in the acoustic waves must go somewhere -> heating of CMB photons -> distortion of the thermal spectrum
Chemical potential from energy injection

• Suppose that some energy, \( \Delta E \), is injected into the cosmic plasma during the radiation dominated era.

• What happens? The thermal spectrum of CMB should be distorted!
Chemical potential from energy injection

- For $z > z_i = 2 \times 10^6$, double Compton scattering, $e^- + \gamma \rightarrow e^- + 2\gamma$, is effective, erasing the distortion of the thermal spectrum of CMB.

- Black-body spectrum is restored.
Chemical potential from energy injection

• For $z < z_i = 2 \times 10^6$, double Compton scattering, $e^- + \gamma \rightarrow e^- + 2\gamma$, freezes out.

• However, the elastic scattering, $e^- + \gamma \rightarrow e^- + \gamma$, remains effective [until $z_f = 5 \times 10^4$]

• Black-body spectrum is not restored, but the spectrum relaxes to a Bose-Einstein spectrum with a non-zero chemical potential, $\mu$, for $z_f < z < z_i$:

$$n(\nu) = \frac{1}{e^{h\nu/(k_BT)} - 1} \rightarrow \frac{1}{e^{h\nu/(k_BT) + \mu} - 1}$$
Chemical potential from energy injection

\[
n(\nu) = \frac{1}{\frac{e^{h\nu/(k_B T)}}{1}} \rightarrow \frac{1}{e^{h\nu/(k_B T)+\mu}}
\]

• Energy density is added to the plasma ($\mu << 1$):
  • $aT^4 + \Delta E/V = a(T')^4(1 - 1.11\mu)$

• Number density is conserved ($\mu << 1$):
  • $bT^3 = b(T')^3(1 - 1.37\mu)$

• Solving for $\mu$ gives
  • $\mu = 1.4[\Delta E/(aT^4V)] = 1.4(\Delta E/E)$
How much energy?

• Only $1/3$ of the total energy stored in the acoustic wave during radiation era is used to heat CMB (thus distorting the CMB spectrum) (papers by Jens Chluba):

  • $Q = (1/3)(9/4)c_s^2 \rho_Y (\delta_Y)^2 = (1/4)\rho_Y (\delta_Y)^2$

  • $\mu \approx 1.4 \int dz \left[\frac{dQ}{dz}/\rho_Y\right]$

    $= (1.4/4)[(\delta_Y)^2(z_i) – (\delta_Y)^2(z_f)]$

  • where $z_i = 2 \times 10^6$ and $z_f = 5 \times 10^4$
Bottom Line

• Therefore, the chemical potential is generated by the photon density perturbation squared.

• At what scale? The diffusion damping occurs at the mean free path of photons. In terms of the wavenumber, it is given by:

\[ k_D \approx 130 \left[ \frac{(1 + z)}{10^5} \right]^{3/2} \text{ Mpc}^{-1} \]

\[ k_D(z_i) \approx 12000 \text{ Mpc}^{-1}; \quad k_D(z_f) \approx 46 \text{ Mpc}^{-1} \]

It’s a very small scale! (compared to the large-scale structure, k~1 Mpc\(^{-1}\))
μ-distortion modified by the squeezed limit

• The existence of long-wavelength $\zeta$ changes the small-scale power of $\delta_Y$.

• A positive long-wavelength $\zeta$ \(\rightarrow\) more power on small scales.

• More power on small scales \(\rightarrow\) more $\mu$-distortion.

• $\mu$-distortion becomes anisotropic on the sky! (Pajer & Zaldarriaga 2012)
\( \mu - T \) cross-correlation

- In real space:
  - \( \mu = (1.4/4)[(\delta_{\gamma})^2(z_i) - (\delta_{\gamma})^2(z_f)] \) at \( k_1 \sim O(10^2) - O(10^4) \)
  - \( \Delta T/T = -(1/5)\zeta \) at \( k_3 \sim O(10^{-4}) \) [in the Sachs-Wolfe limit]
- Correlating these will probe the bispectrum in the squeezed configuration with \( k_3/k_1 = O(10^{-6}) - O(10^{-8}) \)
More exact treatment

• Going to harmonic space:
  
  \[ \Delta T/T(n) = \sum a_{lm} \Upsilon_{lm}(n); \mu(n) = \sum a_{lm} \mu \Upsilon_{lm}(n) \]

• \[ a_{lm}^T = \frac{12\pi}{5} (-i)^l \int \frac{d^3k}{(2\pi)^3} \zeta(k) g_{Tl}(k) Y_{lm}^*(\hat{k}) \]

• \[ a_{lm}^\mu = 18\pi (-i)^l \int \frac{d^3k_1d^3k_2}{(2\pi)^6} Y_{lm}(\hat{k}) \zeta(k_1) \zeta(k_2) W \left( \frac{k}{k_s} \right) \times \]

\[ \times j_l(k r_L) \langle \cos(k_1 r) \cos(k_2 r) \rangle_p \left[ e^{-\left( k_1^2 + k_2^2 \right)/k_D^2(z)} \right]_{z_f} \]

[\[ g_{Tl}(k) \text{ contains info about the acoustic oscillation} \]
The integral is dominated by \( k_1 \approx k_2 \approx k_D \) (which is big) and \( k \approx l/r_L \) (which is small because \( r_L = 14000 \) Mpc).

- Very squeezed limit bispectrum
Local-form Result

$\mu-T$ cross-correlation

$C^\mu_T \times 10^{16} \ell^2 f_{\text{NL}}$

- Full calculation (our result)
  [sign changes]

- Sachs-Wolfe approximation (Pajer & Zaldarriaga)
  [always negative]
Can we detect the local-form bispectrum?

- No, unless $f_{NL} \gg 2300$
But, a modified initial state enhances the signal

\[ \frac{S}{N} \frac{\epsilon}{\eta_0} \]

maximum signal

more realistic estimate

\[ \theta_k \approx k \eta_0 \]

\[ \theta_k = \text{const., max} \]
Future Work

• All we did was to impose the following mode function at a finite past:

\[ u_k = \frac{H^2}{\phi} \frac{1}{\sqrt{2k^3}} [\alpha_k (1+ik\eta)e^{-ik\eta} + \beta_k (1-ik\eta)e^{ik\eta}] \]

• with the condition: \( \beta_k \rightarrow 0 \) for \( k \rightarrow \infty \)

• However, it is desirable to construct an explicit model which will give explicit forms of \( \alpha_k \) and \( \beta_k \), so that we do not need to put an arbitrary model function at an arbitrary time by hand.
Summary

• A more insight into the single-field consistency relation for the squeezed-limit bispectrum using in-in formalism.

• Non-Bunch-Davies vacuum can give an enhanced bispectrum in the $k_3/k_1 << 1$ limit, yielding a distinct form of the scale-dependent bias.

• The $\mu$-type distortion of the CMB spectrum becomes anisotropic, and it can be detected by correlating $\mu$ on the sky with the temperature anisotropy.

Squeezed-limit bispectrum = Test of single-field inflation & initial state of quantum fluctuations