Squeezed-limit bispectrum, Non-Bunch-Davies vacuum, Scale-dependent bias, and Multi-field consistency relation

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"Pre-Planckian Inflation," University of Minnesota, Minneapolis

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This talk is based on...

- Squeezed-limit bispectrum
  - *Ganc & Komatsu, JCAP, 12, 009 (2010)*
- Non-Bunch-Davies vacuum
  - *Ganc, PRD 84, 063514 (2011)*
- Scale-dependent bias
  - *Ganc & Komatsu, in preparation*
- Multi-field consistency relation
  - *Sugiyama, Komatsu & Futamase, PRL, 106, 251301 (2011)*

Motivation

- Can we falsify inflation?

Falsifying "inflation"

- We still need inflation to explain the flatness problem!
  - (Homogeneity problem can be explained by a bubble nucleation.)
- However, the observed fluctuations may come from different sources.
- So, what I ask is, "can we rule out inflation as a mechanism for generating the observed fluctuations?"

First Question:

- Can we falsify **single-field** inflation?

An Easy One: Adiabaticity

- Single-field inflation = One degree of freedom.
  - Matter and radiation fluctuations originate from a single source.

$$\mathcal{S}_{c,\gamma} \equiv \frac{\delta\rho_c}{\rho_c} - \frac{3\delta\rho_\gamma}{4\rho_\gamma} = 0$$

Cold Dark Matter | Photon

* A factor of 3/4 comes from the fact that, in thermal equilibrium, $\rho_c \sim (1+z)^3$ and $\rho_\gamma \sim (1+z)^4$.

Komatsu et al. (2011)

Non-adiabatic Fluctuations

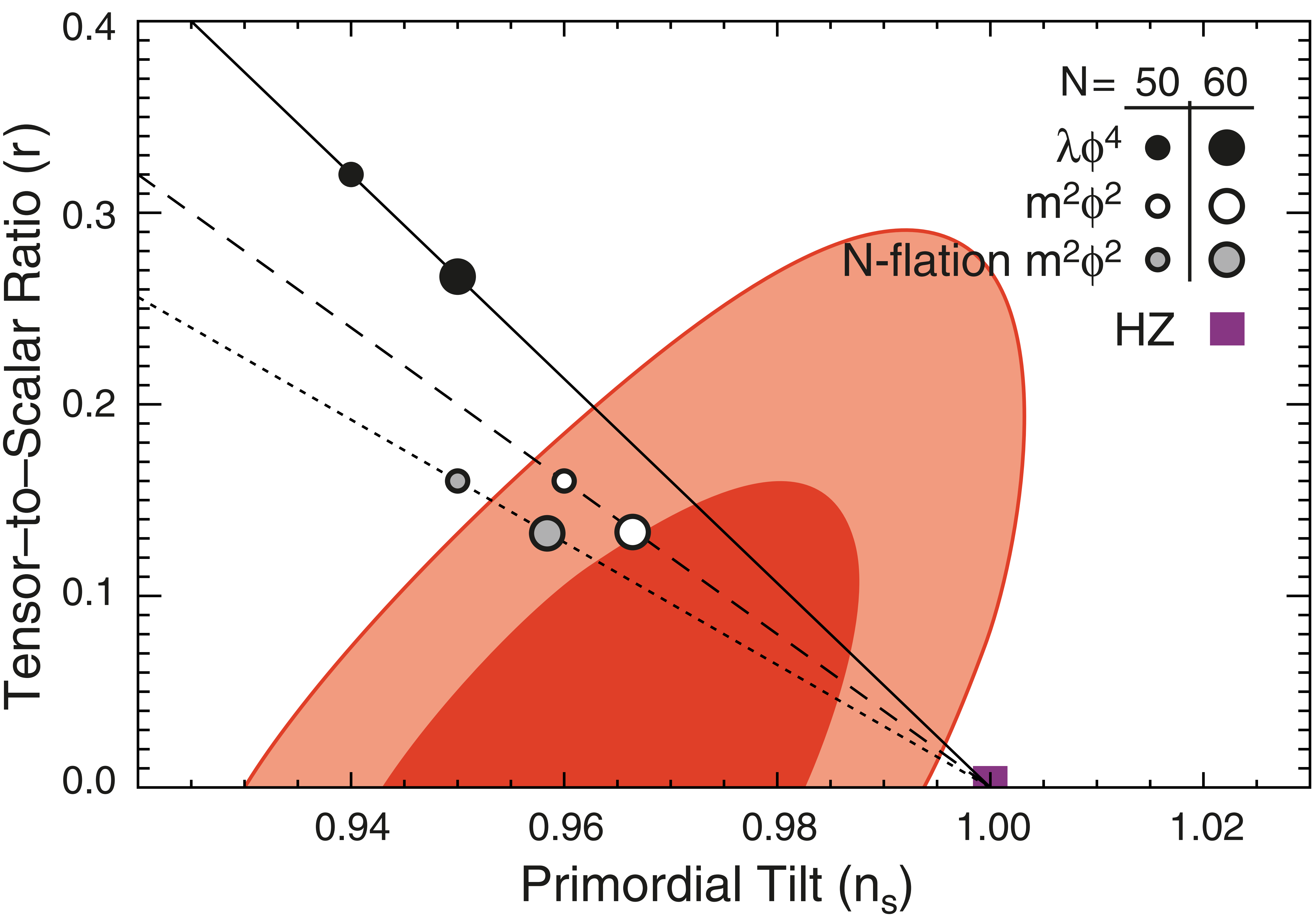
- Detection of non-adiabatic fluctuations immediately rule out single-field inflation models.

The data are consistent with adiabatic fluctuations:

$$\frac{|\delta\rho_c/\rho_c - 3\delta\rho_\gamma/(4\rho_\gamma)|}{\frac{1}{2}[\delta\rho_c/\rho_c + 3\delta\rho_\gamma/(4\rho_\gamma)]} < 0.09 \quad (95\%\ \mathrm{CL})$$

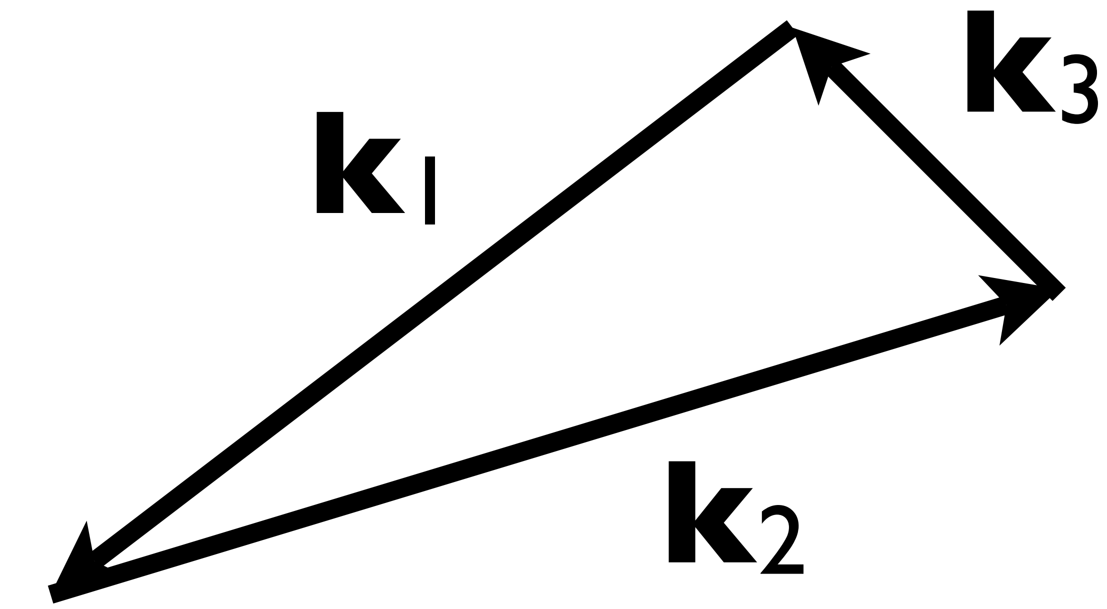
Komatsu et al. (2011)

Single-field inflation looks good (in 2-point function)

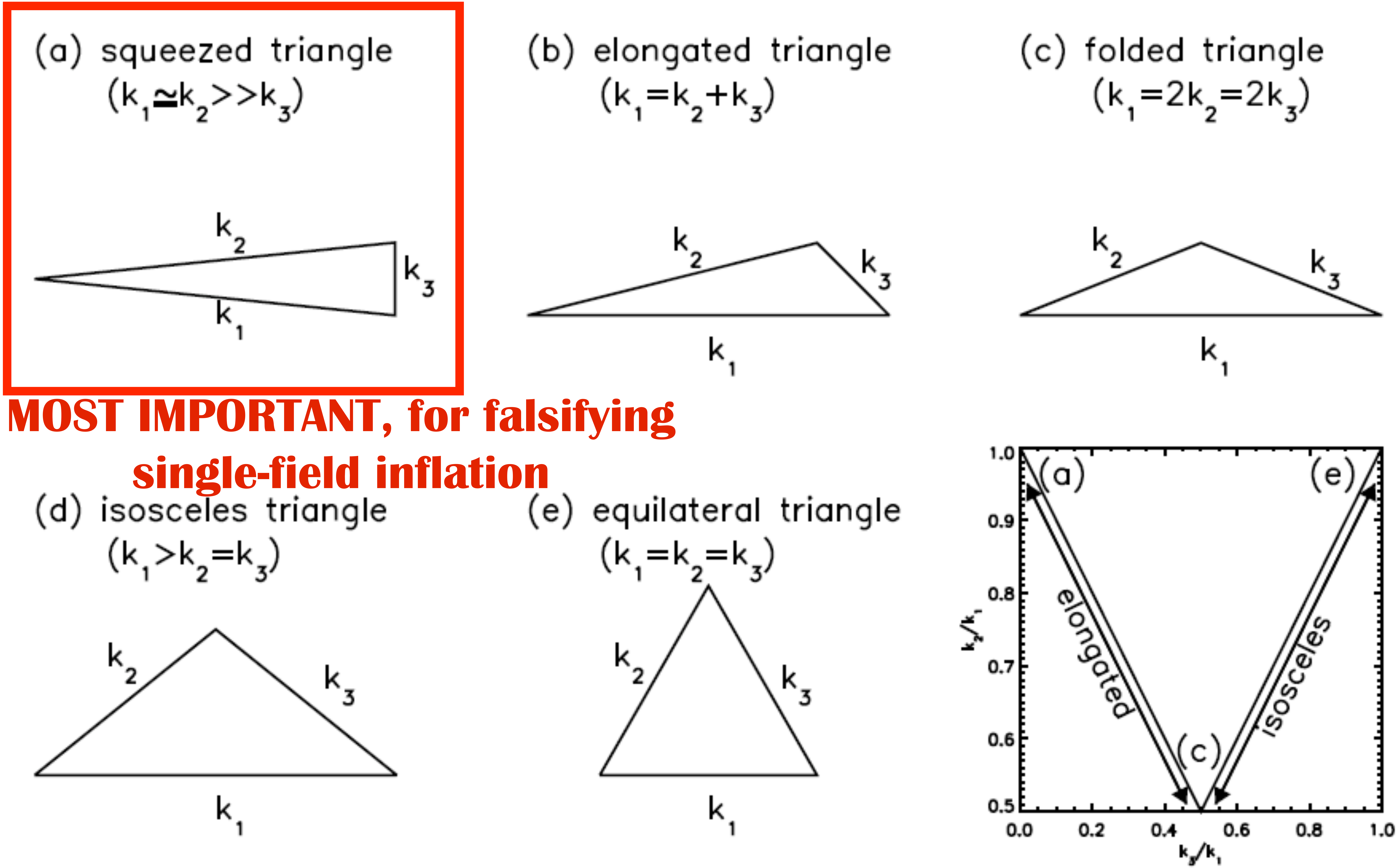


- **n_s=0.968±0.012** (68%CL; WMAP7+BAO+H_0)
- **r < 0.24** (95%CL; WMAP7+BAO+H_0)

So, let's use 3-point function



- Three-point function!
- $B_\zeta(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$
  $= \langle\zeta_{\mathbf{k}1}\zeta_{\mathbf{k}2}\zeta_{\mathbf{k}3}\rangle$ = (amplitude) x $(2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)b(k_1,k_2,k_3)$
  model-dependent function



(a) squeezed triangle
(k1≃k2>>k3)
k2
k3
k1
(b) elongated triangle
(k1=k2+k3)
k2
k3
k1
(c) folded triangle
(k1=2k2=2k3)
k2
k3
k1
MOST IMPORTANT, for falsifying single-field inflation
(d) isosceles triangle
(k1>k2=k3)
k2
k3
k1
(e) equilateral triangle
(k1=k2=k3)
k2
k3
k1
1.0
0.9
0.8
0.7
0.6
0.5
k2/k1
0.0
0.2
0.4
0.6
0.8
1.0
k3/k1
(a)
(e)
(c)
elongated
isosceles

Maldacena (2003); Seery & Lidsey (2005); Creminelli & Zaldarriaga (2004)

Single-field Theorem (Consistency Relation)

- For **ANY** single-field models*, the bispectrum in the squeezed limit is given by
  - $B_\zeta(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) \approx (1-n_s) \times (2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3) \times P_\zeta(k_1)P_\zeta(k_3)$
  - Therefore, all single-field models predict $f_{NL}\approx(5/12)(1-n_s)$.
  - With the current limit $n_s=0.96$, f_{NL} is predicted to be 0.017.

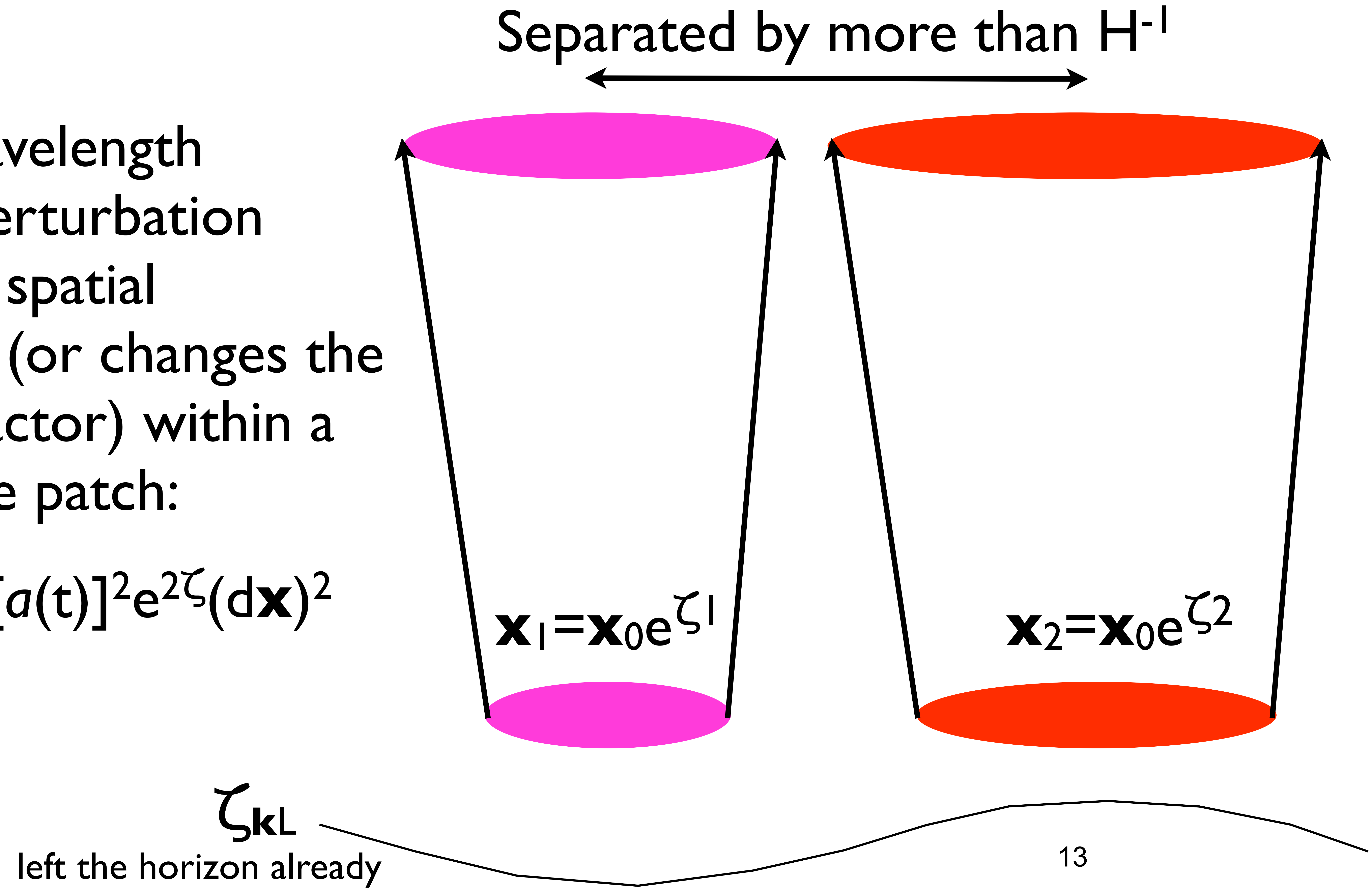
* for which the single field is solely responsible for driving inflation and generating observed fluctuations.

Understanding the Theorem

- First, the squeezed triangle correlates one very long-wavelength mode, k_L (=k_3), to two shorter wavelength modes, k_S (=$k_1 \approx k_2$):
    - $\langle \zeta_{\mathbf{k}1} \zeta_{\mathbf{k}2} \zeta_{\mathbf{k}3} \rangle \approx \langle (\zeta_{\mathbf{k}S})^2 \zeta_{\mathbf{k}L} \rangle$
- Then, the question is: "why should $(\zeta_{\mathbf{k}S})^2$ ever care about $\zeta_{\mathbf{k}L}$?"
    - The theorem says, "it doesn't care, if $\zeta_{\mathbf{k}}$ is exactly scale invariant."

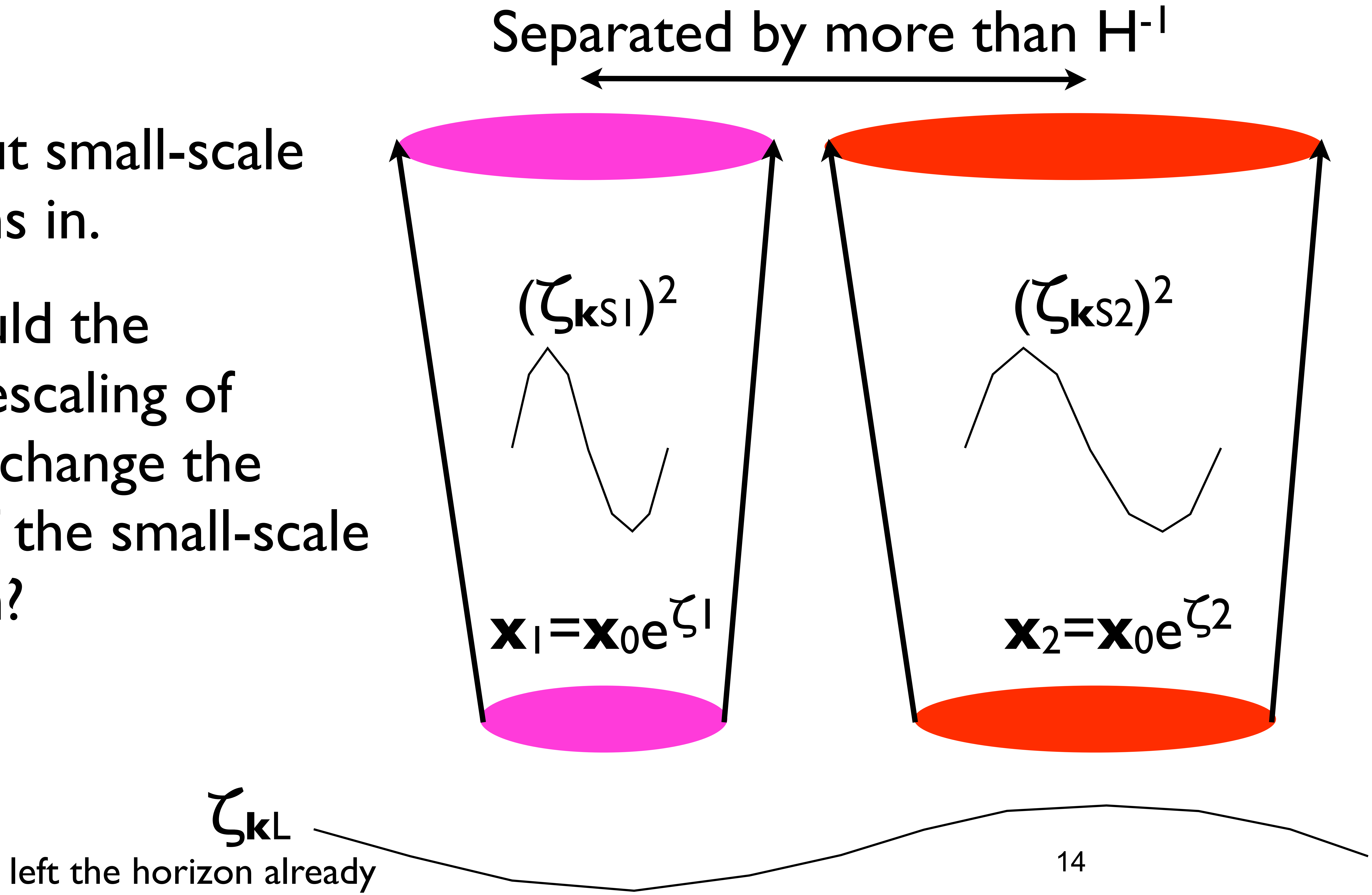
$\zeta_{\mathbf{k}L}$ rescales coordinates

- The long-wavelength curvature perturbation rescales the spatial coordinates (or changes the expansion factor) within a given Hubble patch:
  - $ds^2 = -dt^2 + [a(t)]^2 e^{2\zeta} (d\mathbf{x})^2$



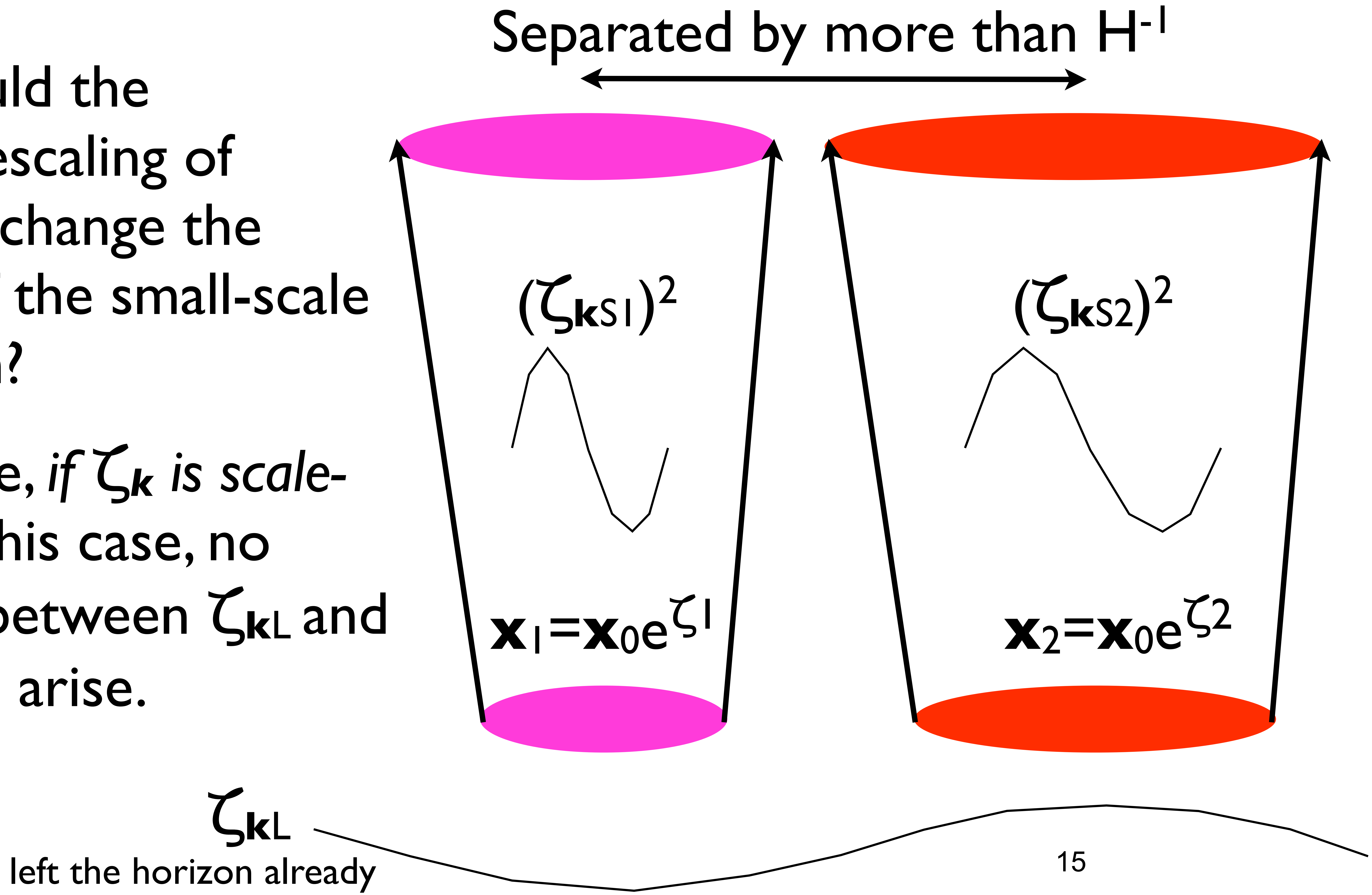
$\zeta_{\mathbf{k}L}$ rescales coordinates

- Now, let's put small-scale perturbations in.
- Q. How would the conformal rescaling of coordinates change the amplitude of the small-scale perturbation?



$\zeta_{\mathbf{k}L}$ rescales coordinates

- Q. How would the conformal rescaling of coordinates change the amplitude of the small-scale perturbation?
- A. No change, *if $\zeta_{\mathbf{k}}$ is scale-invariant*. In this case, no correlation between $\zeta_{\mathbf{k}L}$ and $(\zeta_{\mathbf{k}S})^2$ would arise.

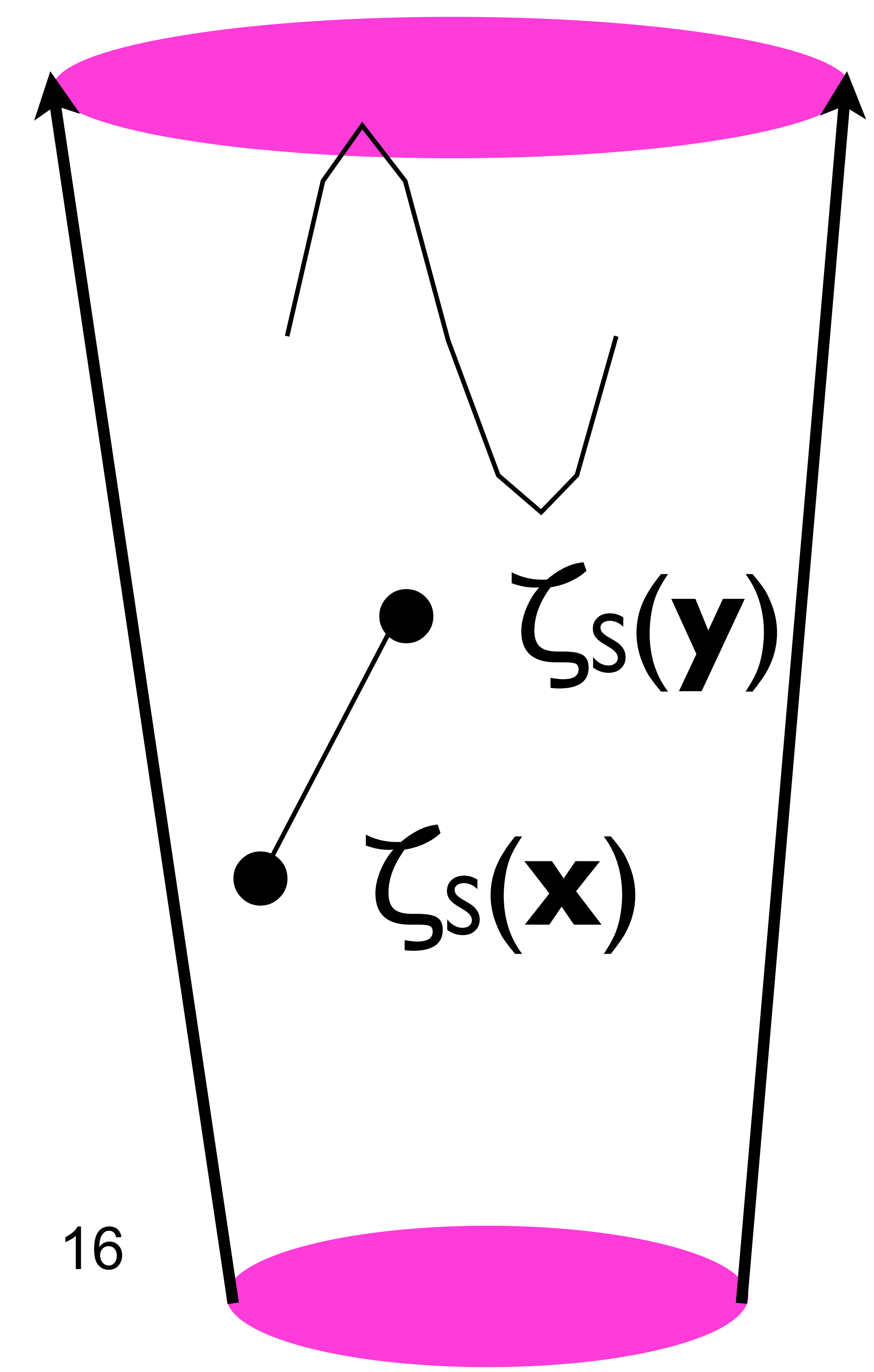


Creminelli & Zaldarriaga (2004); Cheung et al. (2008)

Real-space Proof

- The 2-point correlation function of short-wavelength modes, $\xi = \langle \zeta_S(\mathbf{x}) \zeta_S(\mathbf{y}) \rangle$, within a given Hubble patch can be written in terms of its vacuum expectation value (in the absence of ζ_L), ξ_0, as:

  - $\xi_{\zeta L} \approx \xi_0(|\mathbf{x}-\mathbf{y}|) + \zeta_L \, [d\xi_0(|\mathbf{x}-\mathbf{y}|)/d\zeta_L]$
  - $\xi_{\zeta L} \approx \xi_0(|\mathbf{x}-\mathbf{y}|) + \zeta_L \, [d\xi_0(|\mathbf{x}-\mathbf{y}|)/d\ln|\mathbf{x}-\mathbf{y}|]$
  - $\xi_{\zeta L} \approx \xi_0(|\mathbf{x}-\mathbf{y}|) + \zeta_L \, (1-n_s)\xi_0(|\mathbf{x}-\mathbf{y}|)$



$$\text{3-pt func.} = \langle (\zeta_S)^2 \zeta_L \rangle = \langle \xi_{\zeta L} \zeta_L \rangle$$
$$= (1-n_s)\xi_0(|\mathbf{x}-\mathbf{y}|)\langle \zeta_L^2 \rangle$$

This is great, but...

- The proof relies on the following Taylor expansion:
  - $<\zeta_S(\mathbf{x})\zeta_S(\mathbf{y})>_{\zeta_L} = <\zeta_S(\mathbf{x})\zeta_S(\mathbf{y})>_0 + \zeta_L\,[d<\zeta_S(\mathbf{x})\zeta_S(\mathbf{y})>_0/d\zeta_L]$
- Perhaps it is interesting to show this explicitly using the in-in formalism.
  - Such a calculation would shed light on the limitation of the above Taylor expansion.
  - Indeed it did - we found a non-trivial "counter-example" (more later)

An Idea

- How can we use the in-in formalism to compute the two-point function of short modes, given that there is a long mode, $\langle\zeta_S(\mathbf{x})\zeta_S(\mathbf{y})\rangle_{\zeta_L}$?
- Here it is!

$$\left\langle \zeta_S^2(\bar{t}) \right\rangle_{\zeta_L} = -i \int_{-(1-i\epsilon)\infty}^{\bar{t}} dt' \left\langle 0 \right| [\zeta_S^2(\bar{t}), H_I^{(3)}(t')] \left| 0 \right\rangle$$

Ganc & Komatsu, JCAP, 12, 009 (2010)

Long-short Split of H_I

$$\langle \zeta_S^2(\bar{t}) \rangle_{\zeta_L} = -i \int_{-(1-i\epsilon)\infty}^{\bar{t}} dt' \langle 0 | [\zeta_S^2(\bar{t}), H_I^{(3)}(t')] | 0 \rangle$$

- Inserting $\zeta=\zeta_L+\zeta_S$ into the cubic action of a scalar field, and retain terms that have one ζ_L and two ζ_S's.

$$S_{\rm int}^{(3)} = \int d^4x \left[\left(\frac{1}{4}\frac{\dot\phi_0^4}{H^4} - \frac{1}{16}\frac{\dot\phi_0^6}{H^6} \right) a^3 \zeta_L \dot\zeta_S^2 + \frac{1}{4}\frac{\dot\phi_0^4}{H^4} a \zeta_L (\partial\zeta_S)^2 - \frac{\dot\phi_0^4}{2H^4} a^3 \dot\zeta_S \partial_i \zeta_S \partial_i \partial^{-2} \dot\zeta_L + \right.$$

$$+ \frac{1}{16}\frac{\dot\phi_0^6}{H^6} a^3 \partial_i\partial_j\partial^{-2}\dot\zeta_S\, \partial_i\partial_j\partial^{-2}\dot\zeta_S\, \zeta_L + 2\frac{\dot\phi_0^2}{H^2} a^3 \zeta_L \frac{d}{dt}\left[\frac{1}{2}\frac{\ddot\phi_0}{\dot\phi_0 H} + \frac{1}{4}\frac{\dot\phi_0^2}{H^2} \right] \dot\zeta_S \zeta_S$$

$$\left. - f(\zeta)\frac{\delta L_0}{\delta \zeta_S} \right],$$

Result

$$\langle \zeta_{S,\mathbf{k}_1} \zeta_{S,\mathbf{k}_2} \rangle_{\zeta_{\mathbf{k}_3}} = \zeta_{L,\mathbf{k}_1+\mathbf{k}_2} \left[K + \left(\frac{\ddot{\phi}_0}{\dot{\phi}_0 H} + \frac{1}{2}\frac{\dot{\phi}_0^2}{H^2} \right) P(k_1) \right]$$

- where

$$K \equiv i u_{k_1}^2(\bar{\eta}) \int_{-\infty(1-i\epsilon)}^{\bar{\eta}} d\eta \left[\frac{1}{2}\frac{\dot{\phi}_0^4}{H^4} a^2 u_{k_1}^{\prime *2}(\eta) + \frac{1}{2}\frac{\dot{\phi}_0^4}{H^4} a^2 k_1^2 u_{k_1}^{*2}(\eta) + \right.$$
$$\left. + 2\frac{\dot{\phi}_0^2}{H^2} a^3 \frac{d}{dt} \left(\frac{\ddot{\phi}_0}{\dot{\phi}_0 H} + \frac{1}{2}\frac{\dot{\phi}_0^2}{H^2} \right) u_{k_1}^{\prime *}(\eta) u_{k_1}^{*}(\eta) \right] + \text{c.c.}$$

Result

- Although this expression looks nothing like $(1-n_S)P(k_1)\zeta_{kL}$, we have verified that it leads to the known consistency relation for (i) slow-roll inflation, and (ii) power-law inflation.
- But, there was a curious case – Alexei Starobinsky's exact $n_S=1$ model.
  - If the theorem holds, we should get a vanishing bispectrum in the squeezed limit.

Starobinsky (2005)

Starobinsky's Model

- The famous Mukhanov-Sasaki equation for the mode function is

$$\frac{d^2u_k}{d\eta^2} + \left(k^2 - \frac{1}{z}\frac{d^2z}{d\eta^2}\right)u_k = 0$$

where

$$z = \frac{a\dot{\phi}}{H}$$

- The scale-invariance results when $\frac{1}{z}\frac{d^2z}{d\eta^2} = \frac{2}{\eta^2}$

So, let's write **$z=B/\eta$**

Result

$$\langle \zeta_{S,\mathbf{k}_1} \zeta_{S,\mathbf{k}_2} \rangle_{\zeta_{\mathbf{k}_3}} = \zeta_{L,\mathbf{k}_1+\mathbf{k}_2} 4P(k_1)(k_1\eta_{\rm start})^2 e^{-\frac{1}{2}\phi_{\rm end}^2}$$

- **It does not vanish!**
- But, it approaches zero when $\Phi_{\rm end}$ is large, meaning the duration of inflation is very long.
  - In other words, this is a condition that **the longest wavelength that we observe, k_3, is far outside the horizon.**
  - In this limit, the bispectrum approaches zero.

Vacuum State

- What we learned so far:
  - The squeezed-limit bispectrum is proportional to $(1-n_S)P(k_1)P(k_3)$, provided that ζ_{k3} is far outside the horizon when k_1 crosses the horizon.
- What if the state that ζ_{k3} sees is not a Bunch-Davies vacuum, but something else?
  - The exact squeezed limit (k_3->0) should still obey the consistency relation, but perhaps something happens when **k_3/k_1 is small but finite**.

Back to in-in

$$\langle \zeta^3(t^*) \rangle = -i \int_{t_0}^{t^*} dt' \langle 0 | [\zeta^3(t^*), H_I(t')] | 0 \rangle$$

$$B_\zeta(k_1, k_2, k_3) = 2i \frac{\dot{\phi}^4}{H^6} \sum_i \left(\frac{1}{k_i^2} \right) \tilde{u}_{k_1}(\bar{\eta}) \tilde{u}_{k_2}(\bar{\eta}) \tilde{u}_{k_3}(\bar{\eta}) \int_{\eta_0}^{\bar{\eta}} d\eta \frac{1}{\eta^3} u'^*_{k_1} u'^*_{k_2} u'^*_{k_3} + \text{c.c.}$$

- The Bunch-Davies vacuum: $u_k' \sim \eta e^{-ik\eta}$ (positive frequency mode)
  - The integral yields $1/(k_1+k_2+k_3)$ -> $1/(2k_1)$ in the squeezed limit

Back to in-in

$$\langle \zeta^3(t^*) \rangle = -i \int_{t_0}^{t^*} dt' \langle 0 | [\zeta^3(t^*), H_I(t')] | 0 \rangle$$

$$B_\zeta(k_1, k_2, k_3) = 2i \frac{\dot\phi^4}{H^6} \sum_i \left(\frac{1}{k_i^2} \right) \tilde u_{k_1}(\bar\eta) \tilde u_{k_2}(\bar\eta) \tilde u_{k_3}(\bar\eta) \int_{\eta_0}^{\bar\eta} d\eta \frac{1}{\eta^3} u'^*_{k_1} u'^*_{k_2} u'^*_{k_3} + \text{c.c.}$$

- Non-Bunch-Davies vacuum: $u_k' \sim \eta(A_k e^{-ik\eta} + B_k e^{+ik\eta})$ **negative frequency mode**
- The integral yields $1/(k_1-k_2+k_3)$, peaking in the folded limit *Chen et al. (2007); Holman & Tolley (2008)*
- The integral yields $1/(k_1-k_2+k_3)$ -> $1/(2k_3)$ in the squeezed limit

Enhanced by k_1/k_3: this can be a big factor!

Agullo & Parker (2011)

Agullo & Parker (2011)

How about the consistency relation?

$$B(k_1,k_2,k_3) \underset{k_3/k_1<<1}{\longrightarrow} P(k_1)P(k_3)\left\{(1-n_s) + 4\frac{\dot{\phi}^2}{H^2}\frac{k_1}{k_3}\left[1-\cos(k_3\eta_0)\right]\right\}$$

- When k_3 is far outside the horizon at the onset of inflation, η_0 (whatever that means), $k_3\eta_0$->0, and thus the above additional term vanishes.
    - The consistency relation is restored. *Sounds familiar!*

Checking "Not-so-squeezed Limit"

- Creminelli, D'Amico, Musso & Norena, arXiv:1106.1462 showed that all single-field models have the next-to-leading behavior of the squeezed bispectrum given by

$$\langle\zeta(\vec{k}_1)\zeta(\vec{k}_2)\zeta(\vec{k}_3)\rangle \simeq -(2\pi)^3\delta(\vec{k}_1+\vec{k}_2+\vec{k}_3)P(k_1)P(k_S)\left[\frac{d\ln(k_S^3 P(k_S))}{d\ln k_S}+\mathcal{O}\left(\frac{k_1^2}{k_S^2}\right)\right], \quad k_1 \ll k_S$$

The non-Bunch-Davies vacuum case seems to violate this: the solution is that, in order for their result to hold, k_3 must be small enough so that k_3 is already **far** outside the horizon.

➡ We already saw that, in this limit, the non-Bunch-Davies vacuum result reproduces the standard result. But...

Checking "Not-so-squeezed Limit"

$$B(k_1, k_2, k_3) \underset{k_3/k_1 \ll 1}{\longrightarrow} P(k_1)P(k_3)\left\{(1-n_s) + 4\frac{\dot{\phi}^2}{H^2}\frac{k_1}{k_3}\left[1-\cos(k_3\eta_0)\right]\right\}$$

- The Taylor expansion of the second term yields O ($k_1 k_3 \eta_0^2$), which is not the same as $(k_3/k_1)^2$. Hmm...

Anyway, an interesting possibility:

- What if $k_3\eta_0 = O(1)$?
- The squeezed bispectrum receives an enhancement of order $\varepsilon k_1/k_3$, which can be sizable.
- Most importantly, **the bispectrum grows faster than the local-form toward k_1/k_3 -> 0!**
  - $B(k_1,k_2,k_3) \sim 1/k_3^3$ [Local Form]
  - $B(k_1,k_2,k_3) \sim 1/k_3^{\mathbf{4}}$ [non-Bunch-Davies]

- This has an observational consequence – particularly a scale-dependent bias.

Dalal et al. (2008); Matarrese & Verde (2008); Desjacques et al. (2011)

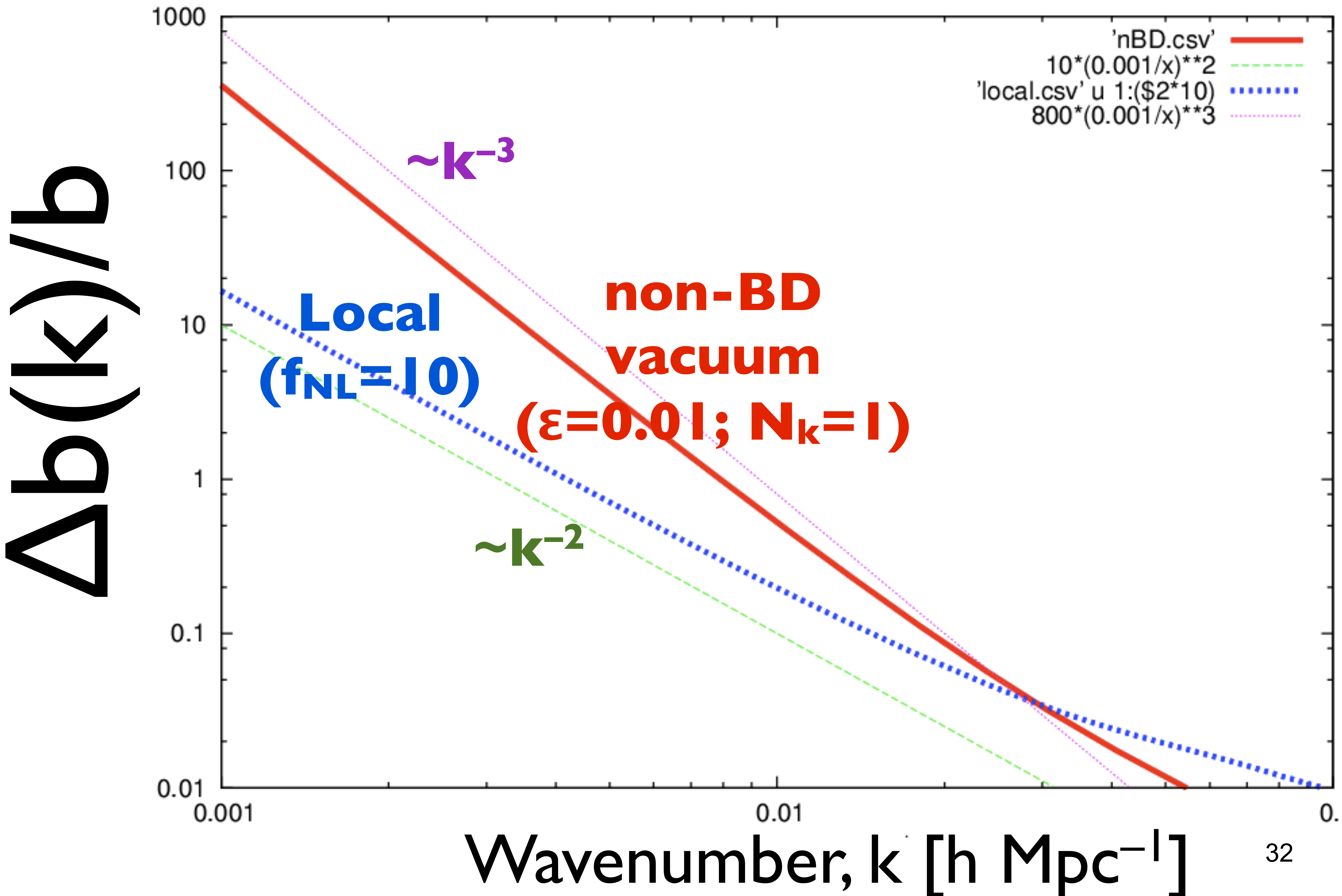
Scale-dependent Bias

$$\frac{\Delta b_h(k,R)}{b_h} = \frac{\delta_c}{D(z)\mathcal{M}_R(k)} \frac{1}{8\pi^2\sigma_R^2} \int_0^\infty dk_1\, k_1^2 \mathcal{M}_R(k_1) \times \int_{-1}^{1} d\mu\, \mathcal{M}_R\left(\sqrt{k^2+k_1^2+2kk_1\mu}\right) \frac{\mathrm{B}\left(k_1, \sqrt{k^2+k_1^2+2kk_1\mu}, k\right)}{P_\zeta(k)}$$

- A rule-of-thumb:
  - For $B(k_1,k_2,k_3) \sim 1/k_3^{p}$, the scale-dependence of the halo bias is given by $b(k) \sim 1/k^{p-1}$
  - For a local-form (p=3), it goes like $b(k) \sim 1/k^2$
  - For a non-Bunch-Davies vacuum (p=4), would it go like $b(k) \sim 1/k^3$?

It does!

Ganc & Komatsu (in prep)



CMB?

- The expected contribution to f_{NL}^{local} as measured by CMB is typically $f_{NL}^{local} < 2(\varepsilon/0.01)$.
  - A lot bigger than $(5/12)(1-n_S)$, but still small enough.

How about...

- Falsifying multi-field inflation?

Sugiyama, Komatsu & Futamase, PRL, 106, 251301 (2011)

Strategy

- We look at the local-form four-point function (trispectrum).
- Specifically, we look for a consistency relation between the local-form bispectrum and trispectrum that is respected by (almost) all models of multi-field inflation.
- We found one: $\tau_{\rm NL} > \frac{1}{2}\left(\frac{6}{5}f_{\rm NL}\right)^2$

provided that 2-loop and higher-order terms are ignored.

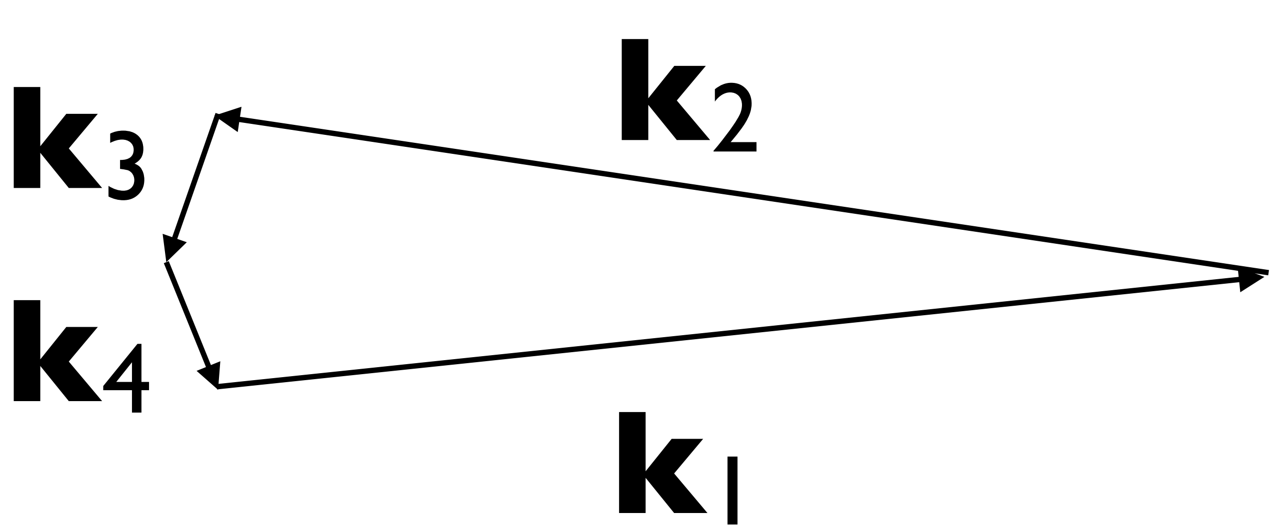
Which Local-form Trispectrum?

- The local-form bispectrum:
  - $B_\zeta(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)=(2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)f_{NL}[(6/5)P_\zeta(k_1)P_\zeta(k_2)+\text{cyc.}]$
- can be produced by a curvature perturbation in position space in the form of:
  - $\zeta(\mathbf{x})=\zeta_g(\mathbf{x}) + (3/5)f_{NL}[\zeta_g(\mathbf{x})]^2$
- This can be extended to higher-order:
  - $\zeta(\mathbf{x})=\zeta_g(\mathbf{x}) + (3/5)f_{NL}[\zeta_g(\mathbf{x})]^2 + (9/25)g_{NL}[\zeta_g(\mathbf{x})]^3$

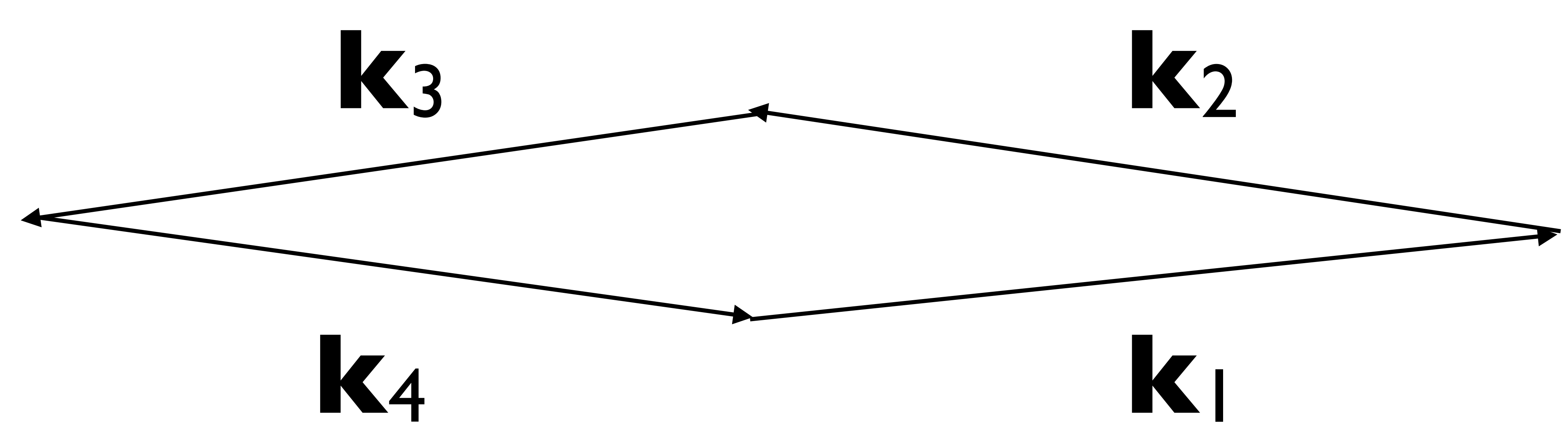
This term (ζ^3) is too small to see, so I will ignore this in this talk.

Two Local-form Shapes

- For $\zeta(\mathbf{x})=\zeta_g(\mathbf{x}) + (3/5)f_{NL}[\zeta_g(\mathbf{x})]^2 + (9/25)g_{NL}[\zeta_g(\mathbf{x})]^3$, we obtain the trispectrum:
  - $T_\zeta(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)=(2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\ \{\mathbf{g_{NL}}[(54/25)P_\zeta(k_1)P_\zeta(k_2)P_\zeta(k_3)+\text{cyc.}] +\mathbf{(f_{NL})^2}[(18/25)P_\zeta(k_1)P_\zeta(k_2)(P_\zeta(|\mathbf{k}_1+\mathbf{k}_3|)+P_\zeta(|\mathbf{k}_1+\mathbf{k}_4|))+\text{cyc.}]\}$



g_{NL}

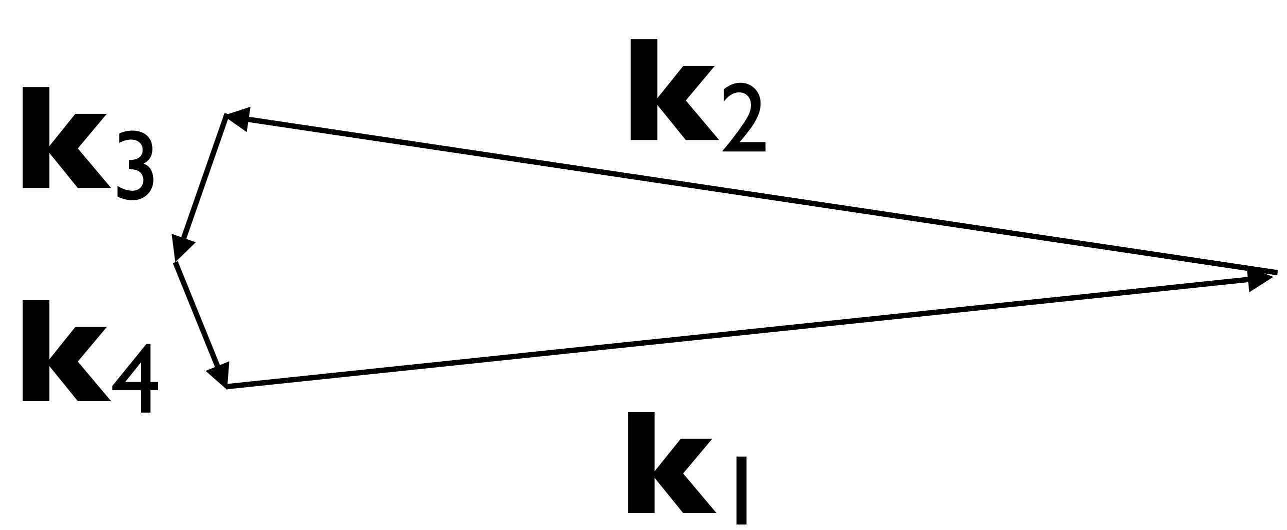


$f_{NL}{}^2$

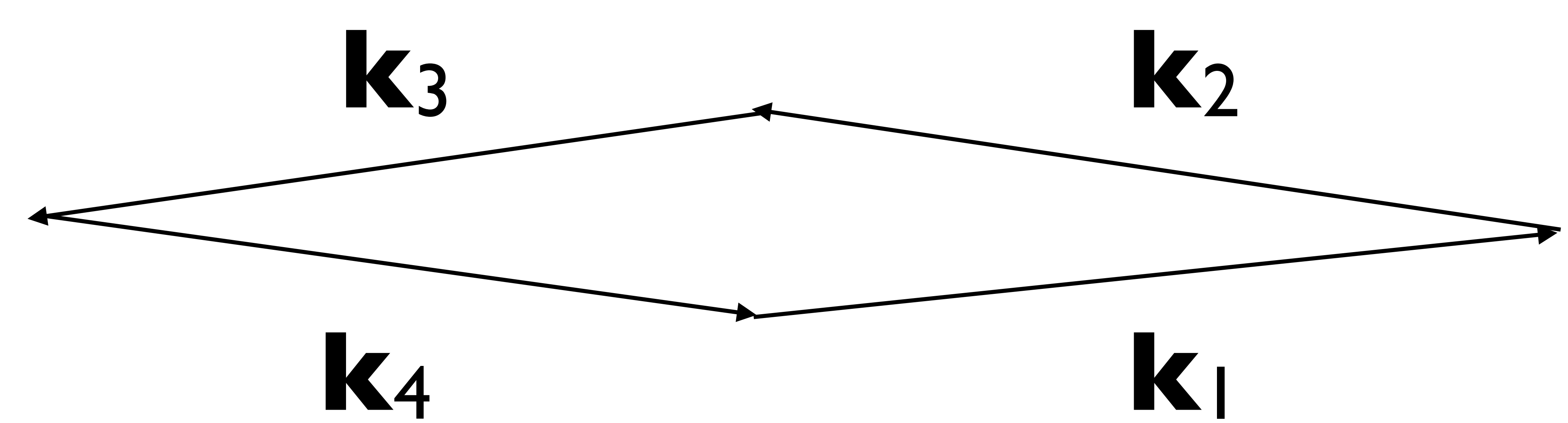
Generalized Trispectrum

- $T_\zeta(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)=(2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\ \{g_{NL}[(54/25)P_\zeta(k_1)P_\zeta(k_2)P_\zeta(k_3)+\text{cyc.}]+\tau_{NL}[P_\zeta(k_1)P_\zeta(k_2)(P_\zeta(|\mathbf{k}_1+\mathbf{k}_3|)+P_\zeta(|\mathbf{k}_1+\mathbf{k}_4|))+\text{cyc.}]\}$

The single-source local form consistency relation,
$\tau_{NL}=(6/5)(f_{NL})^2$, may not be respected –
additional test of multi-field inflation!



g_{NL}

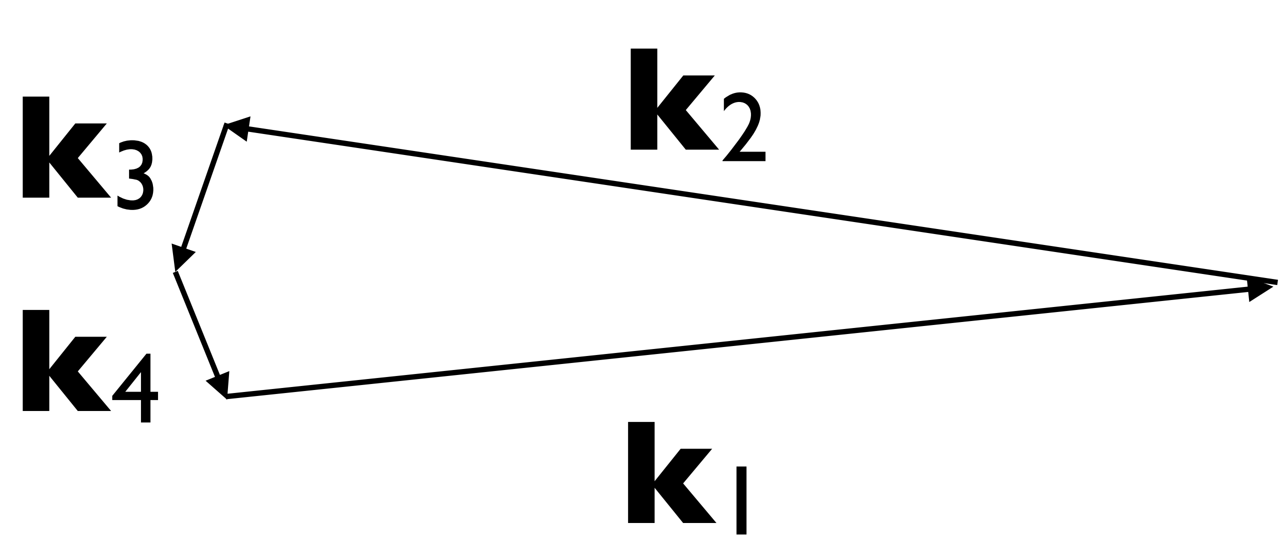


τ_{NL}

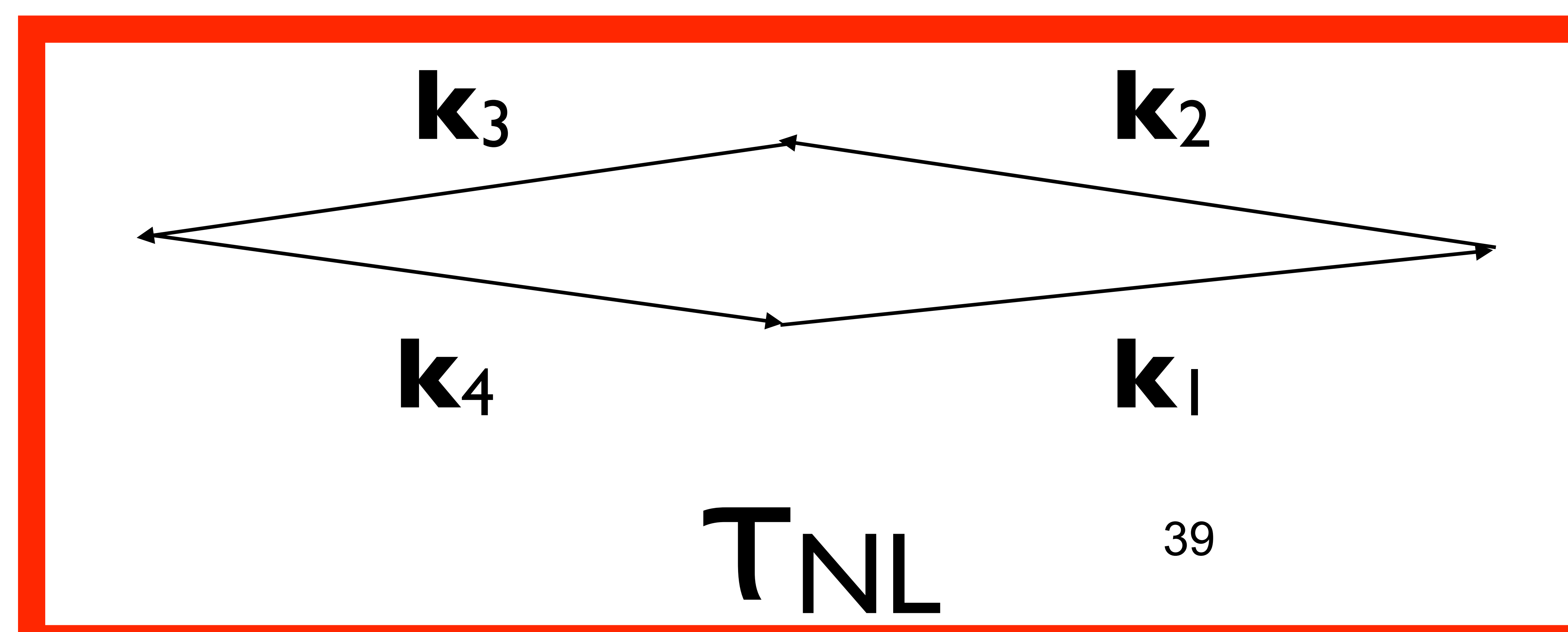
(Slightly) Generalized Trispectrum

- $T_\zeta(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)=(2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\ \{g_{NL}[(54/25)P_\zeta(k_1)P_\zeta(k_2)P_\zeta(k_3)+\text{cyc.}]\ +\tau_{NL}[P_\zeta(k_1)P_\zeta(k_2)(P_\zeta(|\mathbf{k}_1+\mathbf{k}_3|)+P_\zeta(|\mathbf{k}_1+\mathbf{k}_4|))+\text{cyc.}]\}$

The single-source local form consistency relation, $\tau_{NL}=(6/5)(f_{NL})^2$, may not be respected – additional test of multi-field inflation!



g_{NL}



Tree-level Result (Suyama & Yamaguchi)

- Usual δN expansion to the second order

$$\zeta = \sum_I \frac{\partial N}{\partial \phi_I}\delta\phi_I + \frac{1}{2}\sum_{IJ}\frac{\partial^2 N}{\partial\phi_I\partial\phi_J}\delta\phi_I\delta\phi_J + \ldots$$

gives:

$$\frac{6}{5}f_{\rm NL}^{\rm local} = \frac{\sum_{IJ} N_{,IJ}N_{,I}N_{,J}}{[\sum_I (N_{,I})^2]^2},$$

$$\tau_{\rm NL} = \frac{\sum_{IJK} N_{,IJ}N_{,J}N_{,IK}N_{,K}}{[\sum_I (N_{,I})^2]^3} = \frac{\sum_I(\sum_J N_{,IJ}N_{,J})^2}{[\sum_I (N_{,I})^2]^3}$$

Now, stare at these.

$$\frac{6}{5} f_{\rm NL}^{\rm local} = \frac{\sum_{IJ} N_{,IJ} N_{,I} N_{,J}}{[\sum_I (N_{,I})^2]^2},$$

$$\tau_{\rm NL} = \frac{\sum_{IJK} N_{,IJ} N_{,J} N_{,IK} N_{,K}}{[\sum_I (N_{,I})^2]^3} = \frac{\sum_I (\sum_J N_{,IJ} N_{,J})^2}{[\sum_I (N_{,I})^2]^3}$$

Change the variable...

$$\frac{6}{5} f_{\rm NL}^{\rm local} = \frac{\sum_{IJ} N_{,IJ} N_{,I} N_{,J}}{[\sum_I (N_{,I})^2]^2},$$

$$\tau_{\rm NL} = \frac{\sum_{IJK} N_{,IJ} N_{,J} N_{,IK} N_{,K}}{[\sum_I (N_{,I})^2]^3} = \frac{\sum_I (\sum_J N_{,IJ} N_{,J})^2}{[\sum_I (N_{,I})^2]^3}$$

$$a_I = \frac{\sum_J N_{,IJ} N_{,J}}{[\sum_J (N_{,J})^2]^{3/2}}$$

$$b_I = \frac{N_{,I}}{[\sum_J (N_{,J})^2]^{1/2}}$$

$(6/5)f_{NL}=\sum_I a_I b_I$

$\tau_{NL}=(\sum_I a_I^2)(\sum_I b_I^2)$

Then apply the Cauchy-Schwarz Inequality

$$\left(\sum_I a_I^2\right)\left(\sum_J b_J^2\right) \geq \left(\sum_I a_I b_I\right)^2$$

- Implies (Suyama & Yamaguchi 2008)

$$\tau_{\rm NL} \geq \left(\frac{6 f_{\rm NL}^{\rm local}}{5}\right)^2$$

But, this is valid only at the tree level!

Harmless models can violate the tree-level result

- The Suyama-Yamaguchi inequality does not always hold because the Cauchy-Schwarz inequality can be 0=0. For example:

$$\zeta = \frac{\partial N}{\partial \phi_1}\delta\phi_1 + \frac{1}{2}\frac{\partial^2 N}{\partial \phi_2^2}\delta\phi_2^2$$

In this harmless two-field case, the Cauchy-Schwarz inequality becomes 0=0 (both f_{NL} and τ_{NL} result from the second term). In this case,

$$\tau_{\rm NL} \sim 10^3 (f_{\rm NL}^{\rm local})^{4/3}$$

(Suyama & Takahashi 2008)

"1 Loop"

$$\zeta = \frac{\partial N}{\partial \phi_1}\delta\phi_1 + \frac{1}{2}\frac{\partial^2 N}{\partial \phi_2^2}\delta\phi_2^2$$

Fourier transform this, and multiply 3 times

$$\int \frac{d^3p}{(2\pi)^3}\int \frac{d^3q}{(2\pi)^3}\int \frac{d^3s}{(2\pi)^3}\langle \delta\tilde{\phi}_2(\mathbf{k}_1-\mathbf{p})\delta\tilde{\phi}_2(\mathbf{p})\delta\tilde{\phi}_2(\mathbf{k}_2-\mathbf{q})\delta\tilde{\phi}_2(\mathbf{q})\delta\tilde{\phi}_2(\mathbf{k}_3-\mathbf{s})\delta\tilde{\phi}_2(\mathbf{s})\rangle$$

$$= \left(\frac{H^2}{2}\right)^3 (2\pi)^3\delta_D(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\int \frac{d^3p}{(2\pi)^3}\frac{1}{p^3|\mathbf{k}_1-\mathbf{p}|^3|\mathbf{k}_3+\mathbf{p}|^3} + (\text{permutations})$$

$p_{min}=1/L$

$$\approx \left(\frac{H^2}{2}\right)^3 (2\pi)^3\delta_D(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\frac{8\ln(k_bL)}{2\pi^2}\left[\frac{1}{k_1^3k_3^2}+\frac{1}{k_2^3k_3^2}+\frac{1}{k_1^3k_2^2}\right]$$

- $k_b=\min(k_1,k_2,k_3)$

Sugiyama, Komatsu & Futamase, PRL, 106, 251301 (2011)

Ignoring details...

- I don't have time to show you the derivation (you can look it up in the paper), but the result is somewhat weaker than the Suyama-Yamaguchi inequality:

$$\tau_{\rm NL} > \frac{1}{2}\left(\frac{6}{5} f_{\rm NL}\right)^2$$

Detection of a violation of this relation can potentially falsify inflation as a mechanism for generating cosmological fluctuations.

A Comment

- Even without using the physics argument, the statistics argument can give a bound (*Smith, LoVerde & Zaldarriaga, arXiv:1108.1805*):

$$\tau_{NL} \geq \left(\frac{6}{5} f_{NL}\right)^2 - A$$

where

$$A = \frac{1}{2P(k_L)V_S} = \frac{2\pi^2}{k_L^3 P(k_L)} \frac{1}{4\pi^2} \frac{3}{4\pi} \left(\frac{k_L}{k_S}\right)^3 \approx 2 \times 10^6 \left(\frac{k_L}{k_S}\right)^3 \quad \mathbf{\neq 0}$$

The statistics argument does **not** preclude a physical violation of the Suyama-Yamaguchi inequality

Summary

- A more insight into the single-field consistency relation for the squeezed-limit bispectrum using in-in formalism.
- Non-Bunch-Davies vacuum can give an enhanced bispectrum in the $k_3/k_1 \ll 1$ limit, yielding a distinct form of the scale-dependent bias.
- Multi-field consistency relation between the 3-point and 4-point function can be used to rule out multi-field inflation, as well as single-field.