The lecture slides are available at
https://wwwmpa.mpa-garching.mpg.de/~komatsu/lectures--reviews.html

Lecture 2: Propagation of Light

Part I: Light propagation in an expanding universe

How does light propagate in space?

Distance between two points

- The first step is to define the distance between two points in space.

y

- In Cartesian coordinates, the distance between two points in Euclidean space is

$$ds^2 = dx^2 + dy^2 + dz^2$$

- To include the isotropic expansion of space,

$$ds^2 = \boxed{a^2(t)}\underbrace{(dx^2 + dy^2 + dz^2)}_{\text{comoving coordinates}}$$

x

Scale Factor

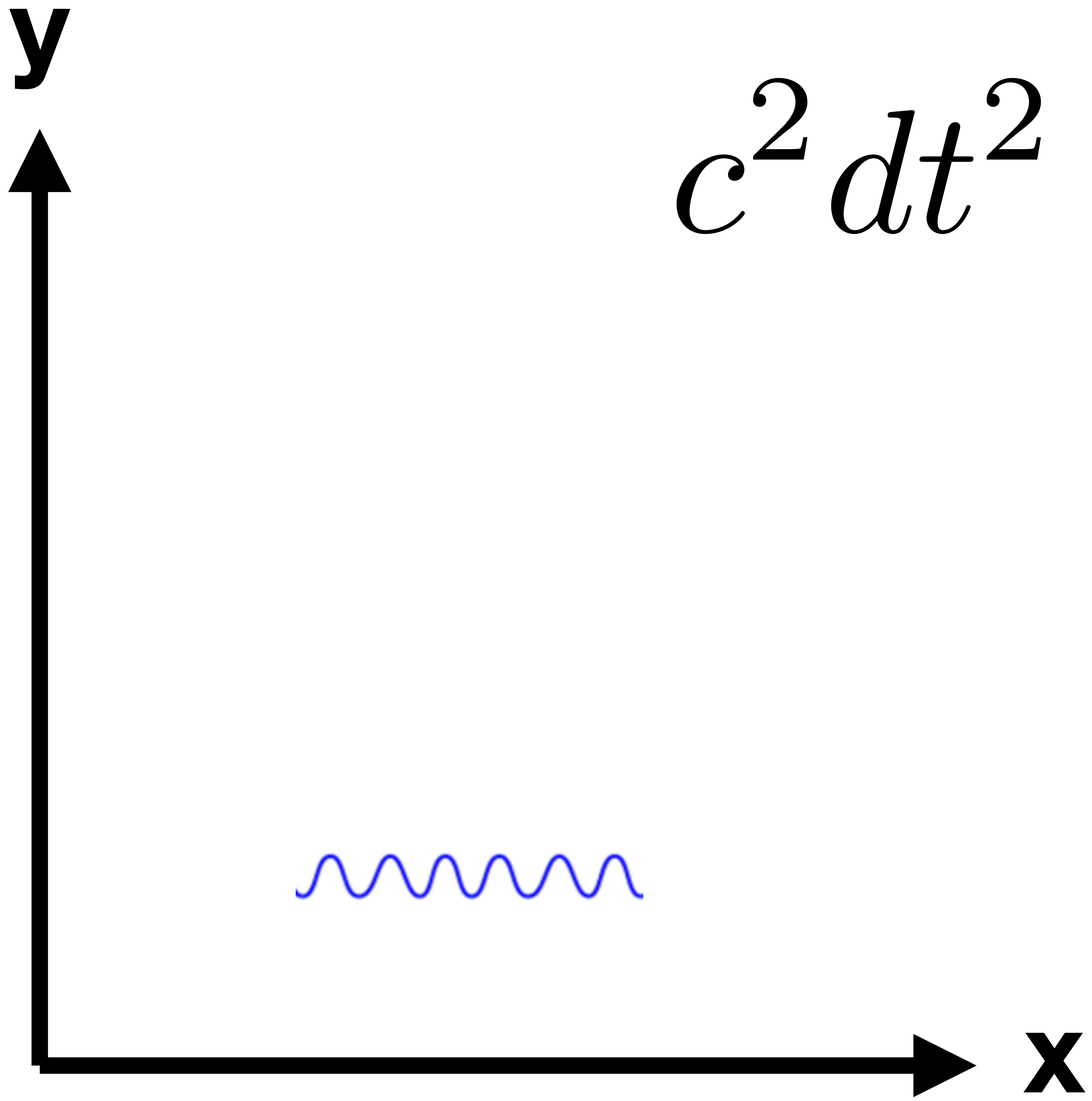
comoving coordinates

How does light propagate in space?

Distance traveled by light in an expanding Universe

- When the light propagates in x direction, the time interval of the travel is

$$c^2 dt^2 = ds^2 = a^2(t) dx^2$$



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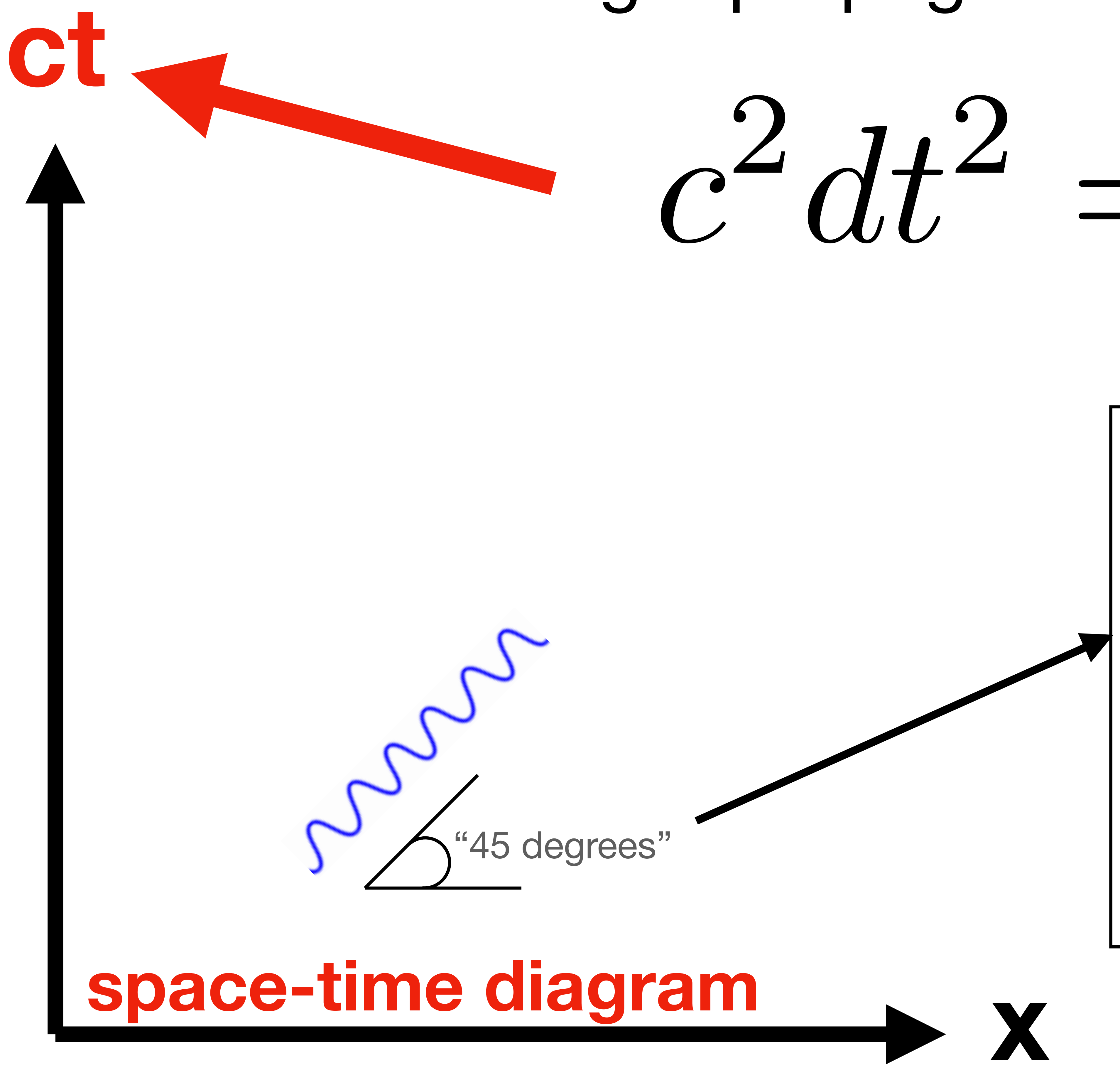
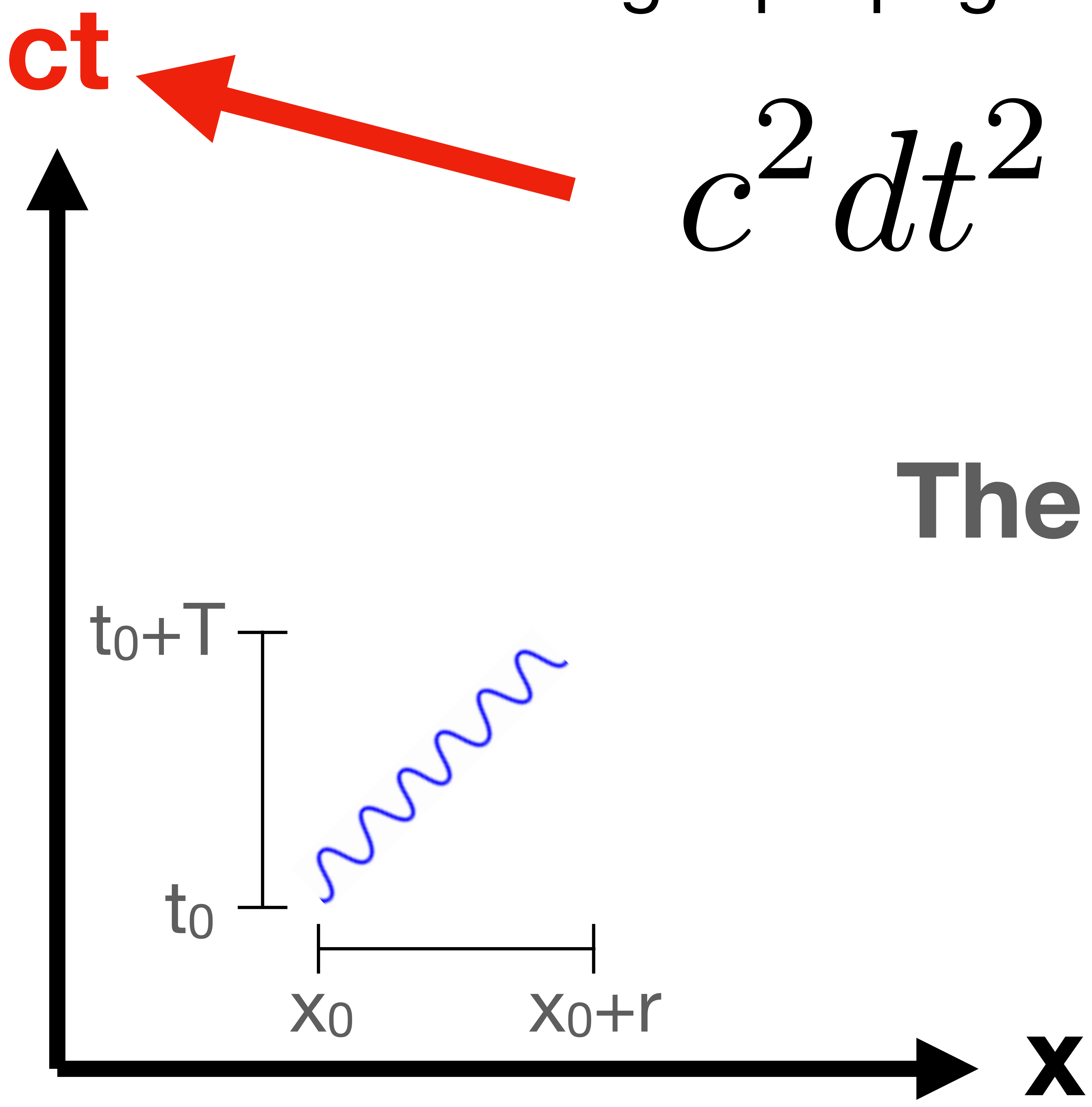


Strictly speaking, the angle is 45 degrees when the y-axis is the so-called "conformal time" instead of the physical time (t). This difference is not essential for a decelerated expansion of the Universe, so we ignore this subtlety in this lecture. For more information on this point, see "further readings" in https://wwwmpa.mpa-garching.mpg.de/~komatsu/lectures--reviews.html

How does light propagate in space?

Comoving distance traveled by light in an expanding Universe

- When the light propagates in x direction, the time interval of the travel is



$$c^2 dt^2 = ds^2 = a^2(t) dx^2$$

The *comoving* distance traveled by light:

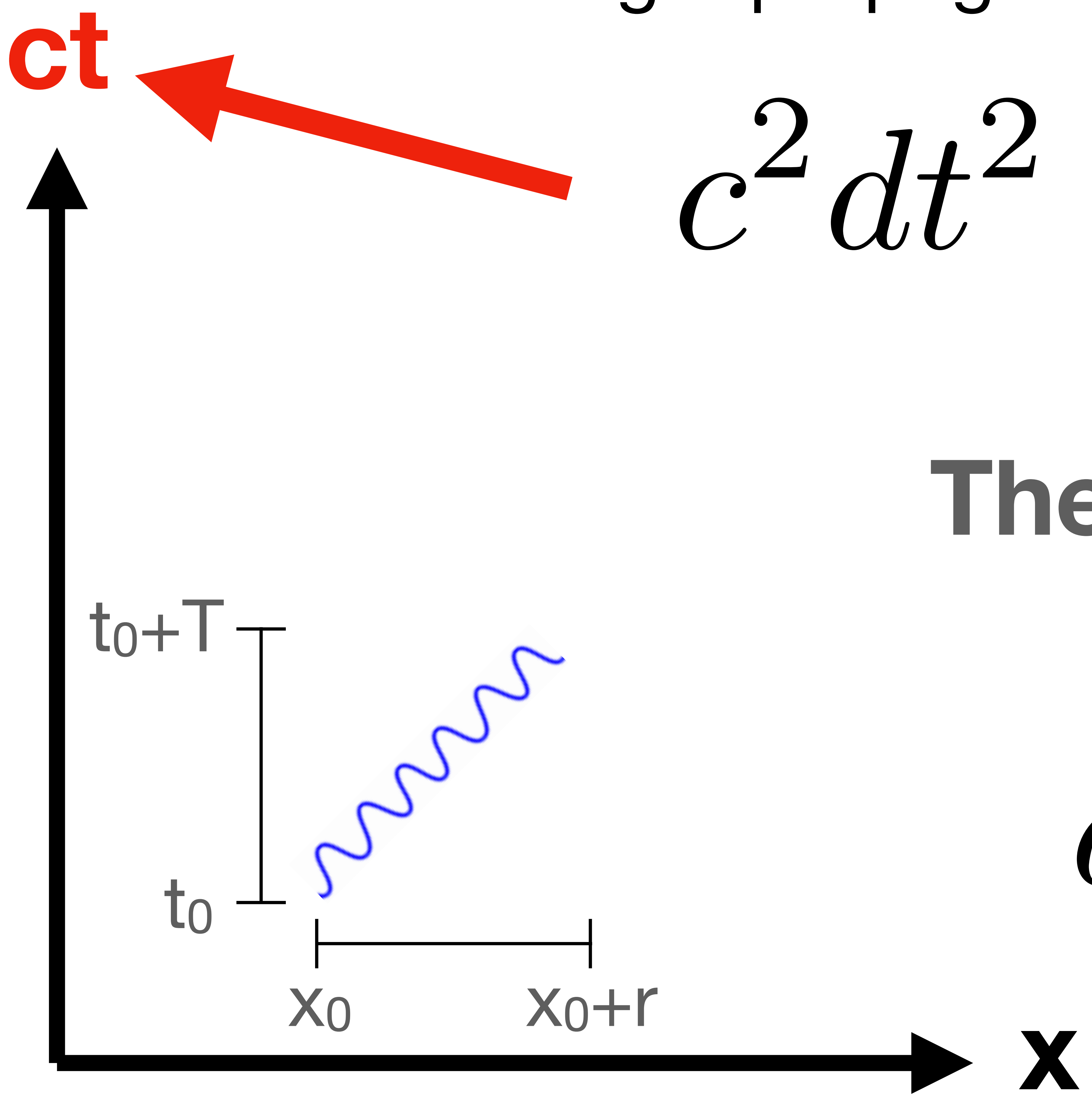
$$r = c \int_{t_0}^{t_0+T} \frac{dt}{a(t)}$$

How does light propagate in space?

Physical **distance traveled by light in an expanding Universe**

- When the light propagates in x direction, the time interval of the travel is

$$c^2 dt^2 = ds^2 = a^2(t) dx^2$$



The *physical* distance traveled by light:

$$d = ca(t_0 + T) \int_{t_0}^{t_0+T} \frac{dt}{a(t)}$$

The physical distance does not depend on the arbitrary, unphysical normalisation of a(t)!

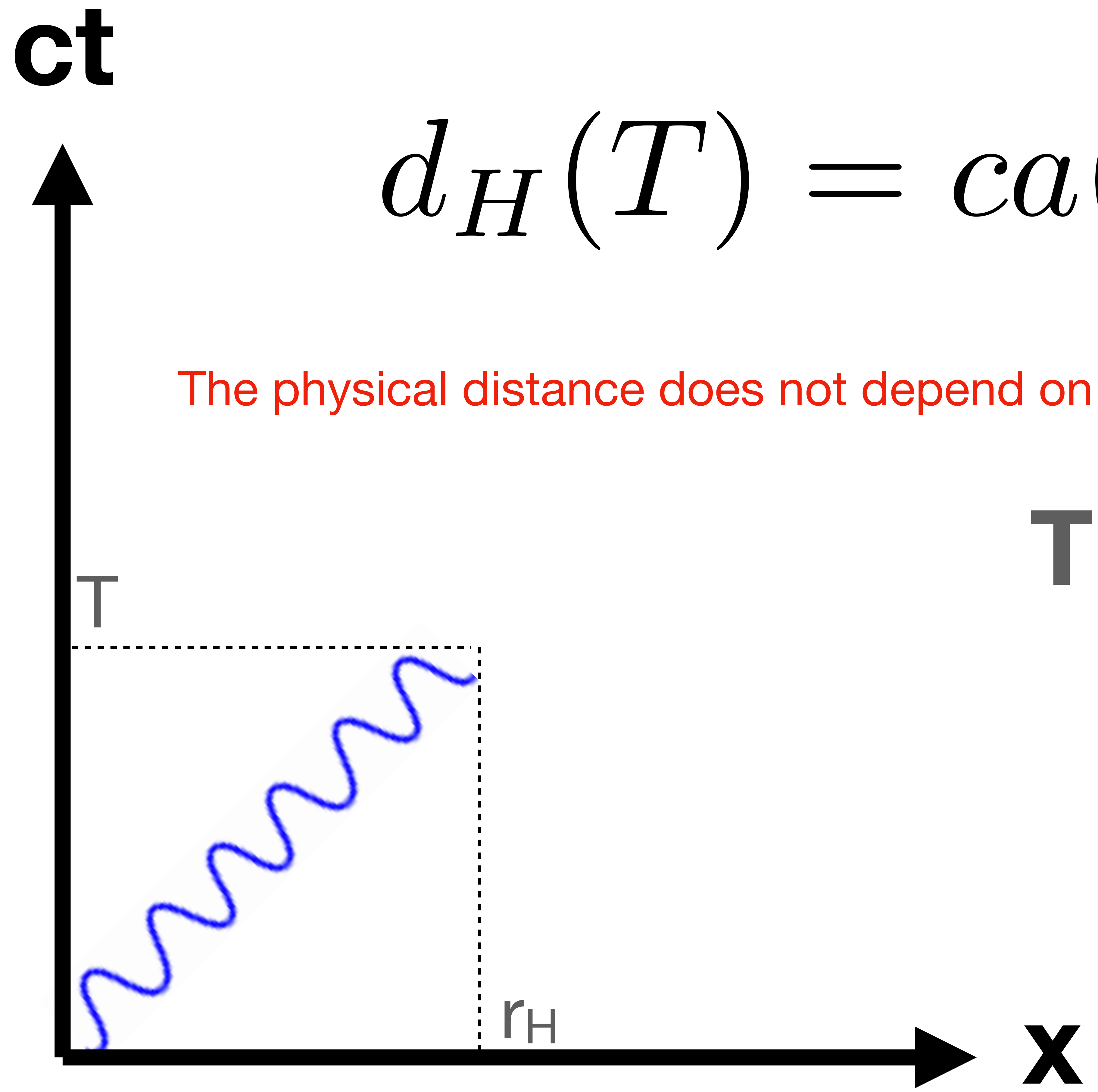
Horizon

How far has the light traveled since t=0?

The ***physical*** horizon distance:

$$d_H(T) = ca(T) \int_0^T \frac{dt}{a(t)}$$

The physical distance does not depend on the arbitrary, unphysical normalisation of a(t)!

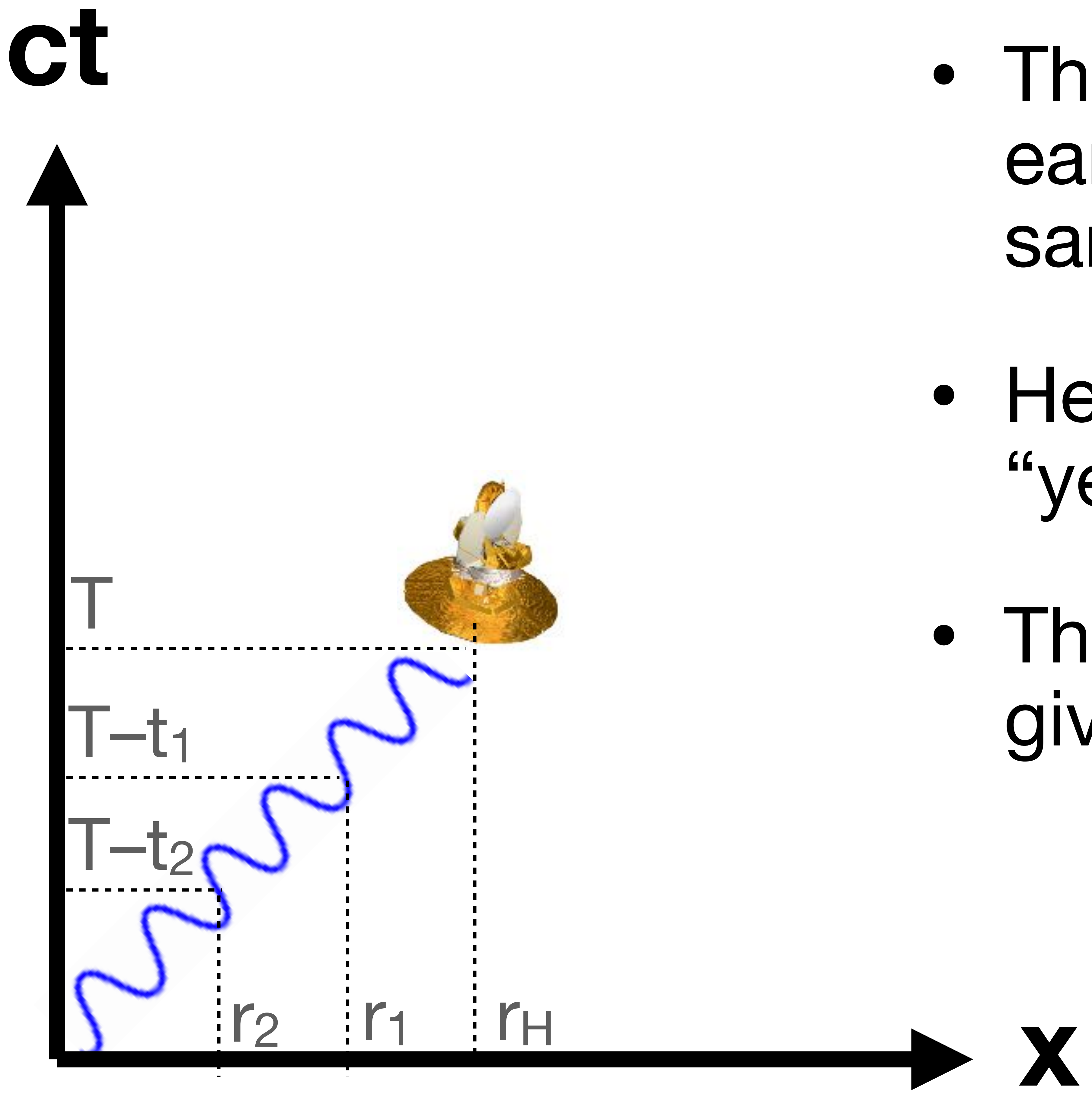


The ***comoving*** horizon distance:

$$r_H(T) = c \int_0^T \frac{dt}{a(t)}$$

Concept of the "light cone"

The farther away we look in distance, the further back we look in time



- The light arrives at us at the time T.
- The light emitted at a distance r_2 had to be emitted earlier than that at r_1, for us to see them both at the same time T.
- Here, "t_i" is called the "look-back time", in units of "years **ago**".
- The corresponding comoving distances **from us** are given by $r_H - r_i$.

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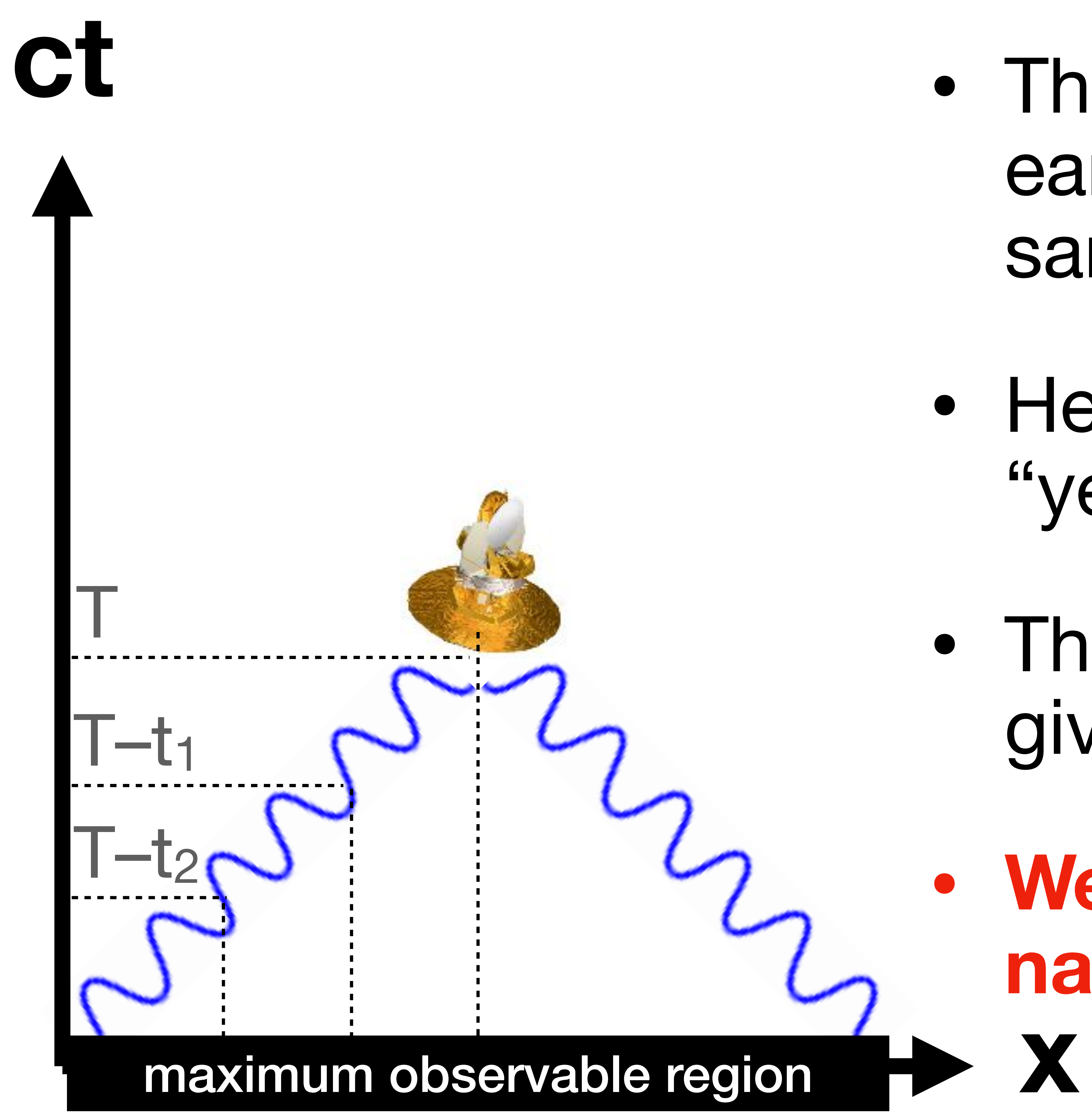

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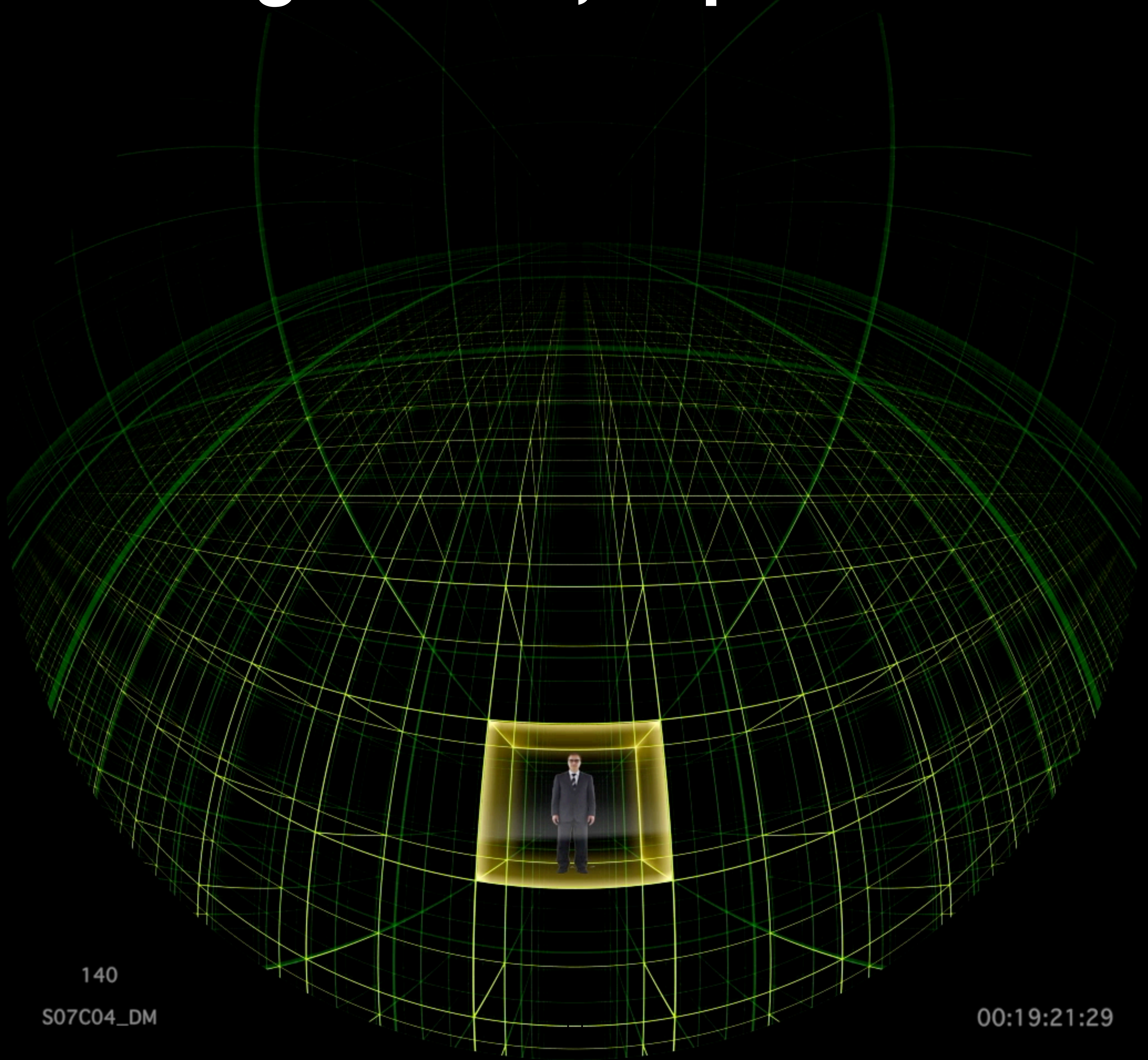
- The corresponding comoving distances **from us** are given by $r_H - r_i$.
- **We can see things only within this cone. Hence the name "light cone"**

Light Cone, Explained



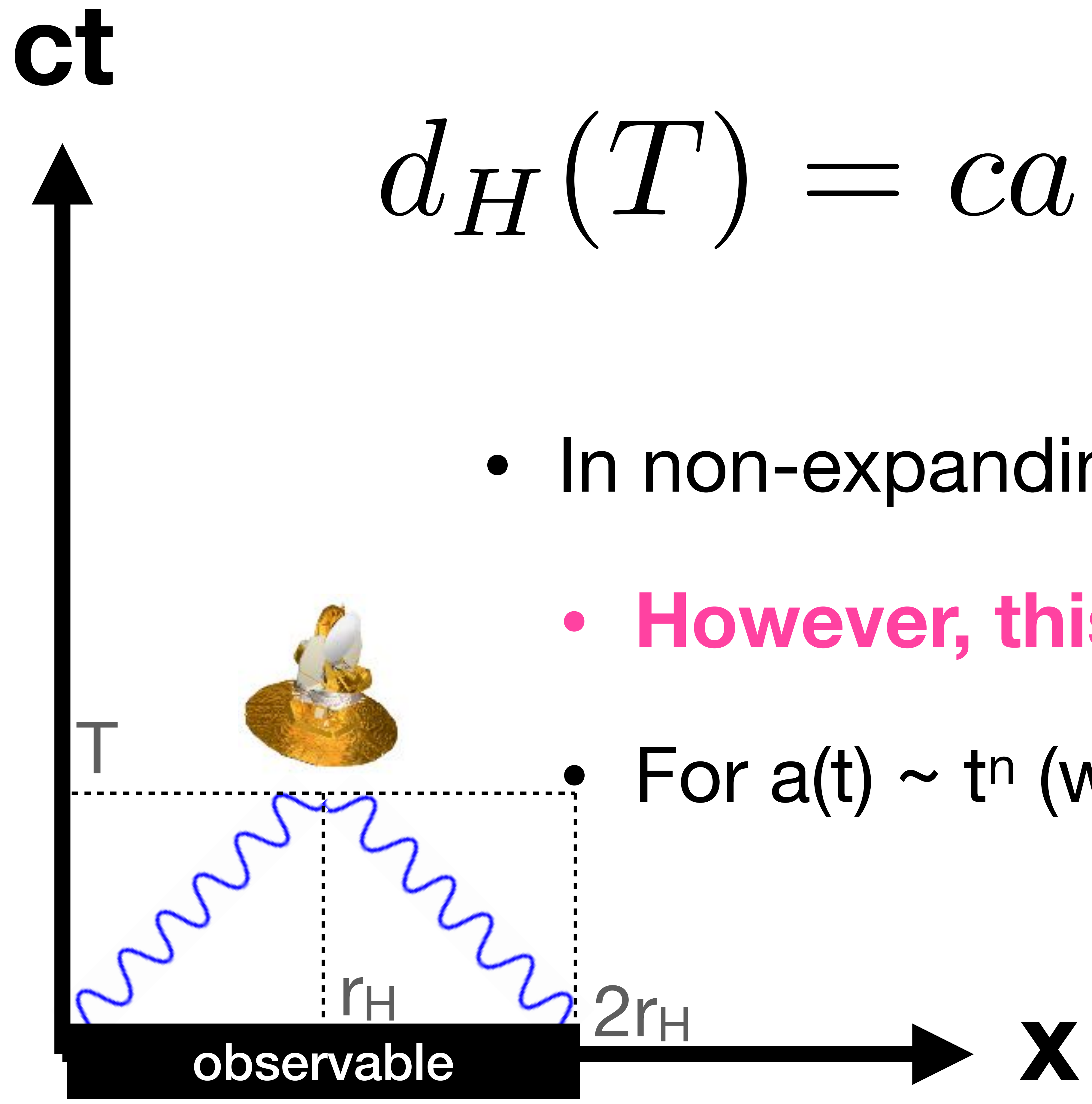

Dark Energy
Accelerated Expansion
Afterglow Light
Pattern
375,000 yrs.
Dark Ages
Development of
Galaxies, Planets, etc.
Inflation
WMAP
Quantum
Fluctuations
1st Stars
about 400 million yrs.
Big Bang Expansion
13.77 billion years
NASA/WMAP Science Team

Horizon can be larger than the naive expectation

Do you think $d_H(T)=cT$? You would be surprised…

The ***physical*** horizon distance:

$$d_H(T) = ca(T)\int_0^T \frac{dt}{a(t)}$$



- In non-expanding space, $d_H(T) = cT$, as you expect.
- **However, this relation does not hold in expanding space!**
- For $a(t) \sim t^n$ (with $n<1$; decelerating Universe), we obtain

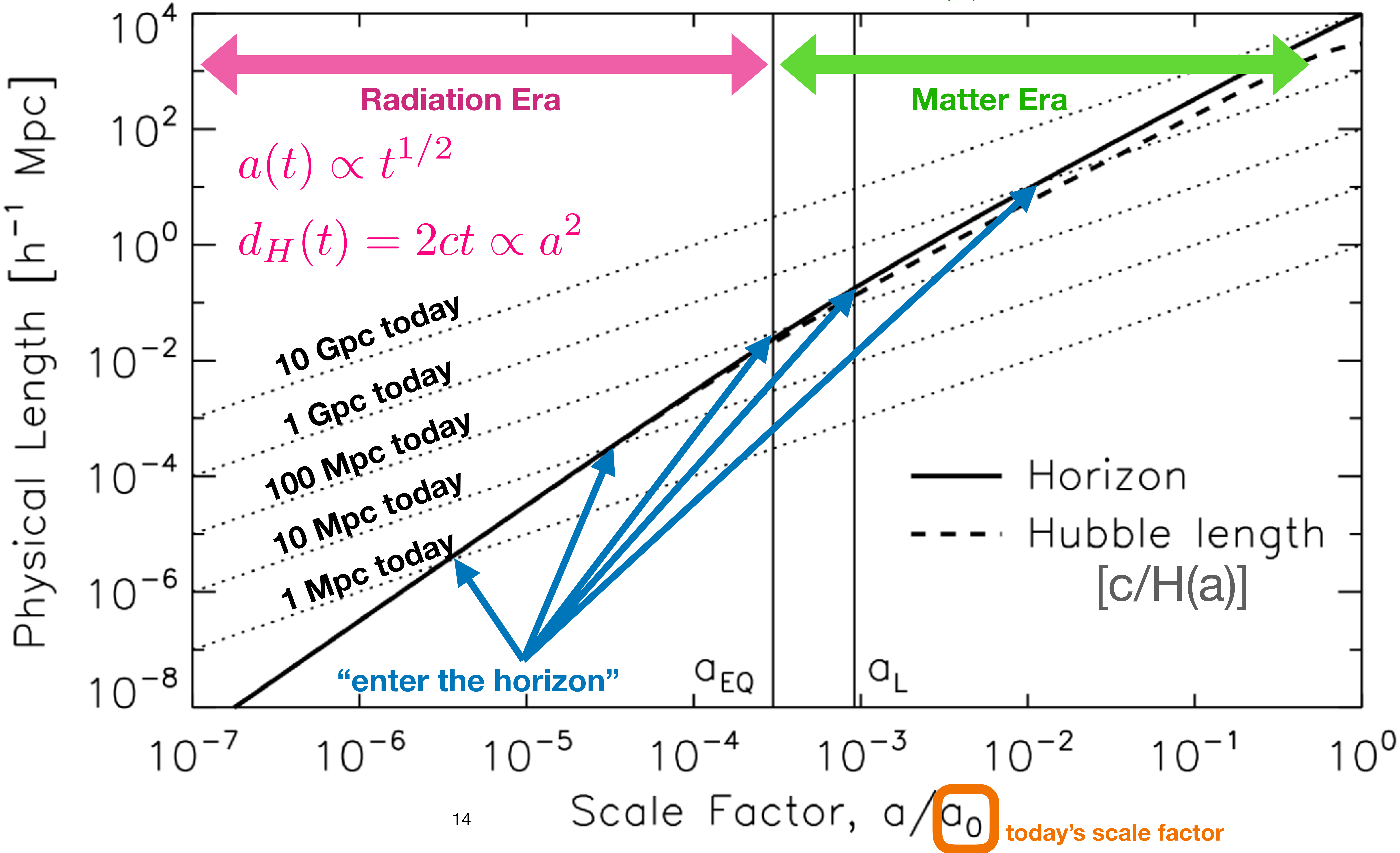
$$d_H(T) = \frac{cT}{1-n} > cT$$

The horizon grows faster than cT!

"Entering the Horizon"

- In a decelerating Universe, we can see more of the Universe as time goes by.
- If we wait long enough, we can see the entire Universe!
    - But in reality, today's Universe is accelerating, so this won't happen in future unfortunately.

$$a(t) \propto t^{2/3}$$
$$d_H(t) = 3ct \propto a^{3/2}$$



Accelerating Universe

$$a(t) \propto t^{2/3}$$

$$d_H(t) = 3ct \propto a^{3/2}$$

$$a(t) \propto t^{1/2}$$

$$d_H(t) = 2ct \propto a^2$$

For a(t) ~ t^n,

$$d_H(t) = ca(t)\int_{t_*}^{t} \frac{dt'}{a(t')}$$

$$= \left[\left(\frac{t}{t_*}\right)^n - \frac{t}{t_*}\right]\frac{ct_*}{n-1}$$

$$\propto a(t) \quad \text{for } n > 1$$

Part II: Light propagation in a clumpy universe

Introducing the curved space

Non-Euclidean space

- Writing the distance between two points in Euclidean space:

$$ds^2 = a^2(t)(dx^2 + dy^2 + dz^2)$$

- …in a fancy way:

$$ds^2 = a^2(t) \sum_{i=1}^{3} \sum_{j=1}^{3} \delta_{ij} \, dx^i dx^j$$

$$\delta_{ij} = \begin{cases} 1 \text{ for i=j} \\ 0 \text{ otherwise} \end{cases}$$

Introducing the curved space

Non-Euclidean space

- Writing the distance between two points in Euclidean space:

$$ds^2 = a^2(t)(dx^2 + dy^2 + dz^2)$$

- …in a fancy way:

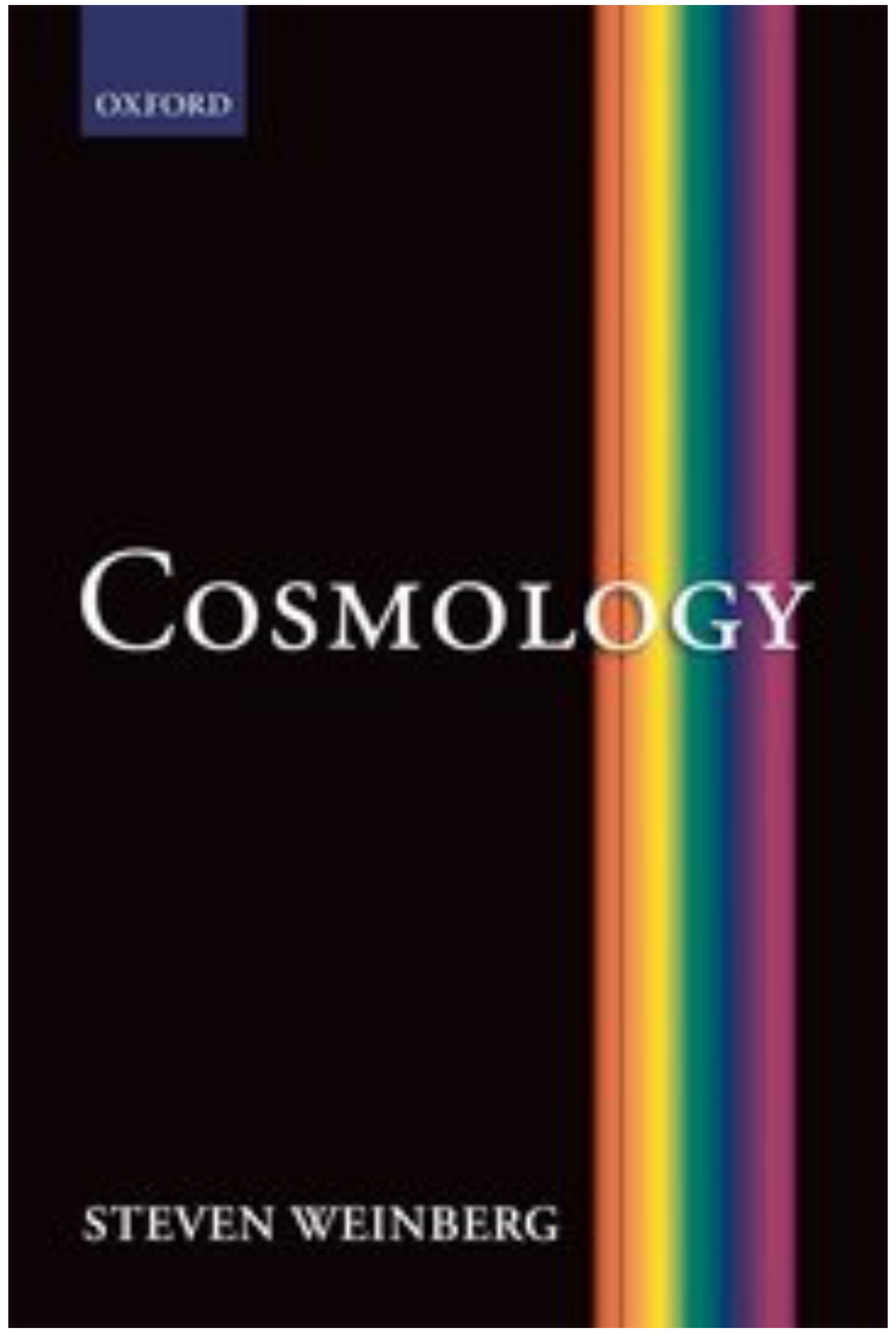
$$ds^2 = a^2 \sum_{i=1}^{3} \sum_{j=1}^{3} (\delta_{ij} + \boxed{h_{ij}}) dx^i dx^j$$

metric perturbation
-> CURVED SPACE!

Notation

From now on…

- The notation in my lectures follows mainly "*Cosmology*" by Steven Weinberg.



Spatial curvature: Ψ and D_{ij}

Non-Euclidean space

- Re-writing this fancy expression:

$$ds^2 = a^2 \sum_{i=1}^{3} \sum_{j=1}^{3} (\delta_{ij} + \boxed{h_{ij}}) dx^i dx^j$$

metric perturbation
-> CURVED SPACE!

- …in a fancier way:

$$ds^2 = a^2 \exp(-2\Psi) \sum_{i=1}^{3} \sum_{j=1}^{3} [\exp(D)]_{ij} dx^i dx^j$$

Spatial curvature: Ψ and D_{ij}

Non-Euclidean space

- Re-writing this fancy expression:

$$ds^2 = a^2 \sum_{i=1}^{3} \sum_{j=1}^{3} (\delta_{ij} + h_{ij}) dx^i dx^j$$

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- …in a fancier way:

$$ds^2 = a^2 \exp(-2\Psi) \sum_{i=1}^{3} \sum_{j=1}^{3} [\exp(D)]_{ij} dx^i dx^j$$

Scalar curvature perturbation

Tensor perturbation = Gravitational wave

Tensor perturbation D_{ij}

Area-conserving deformation of space = Gravitational wave

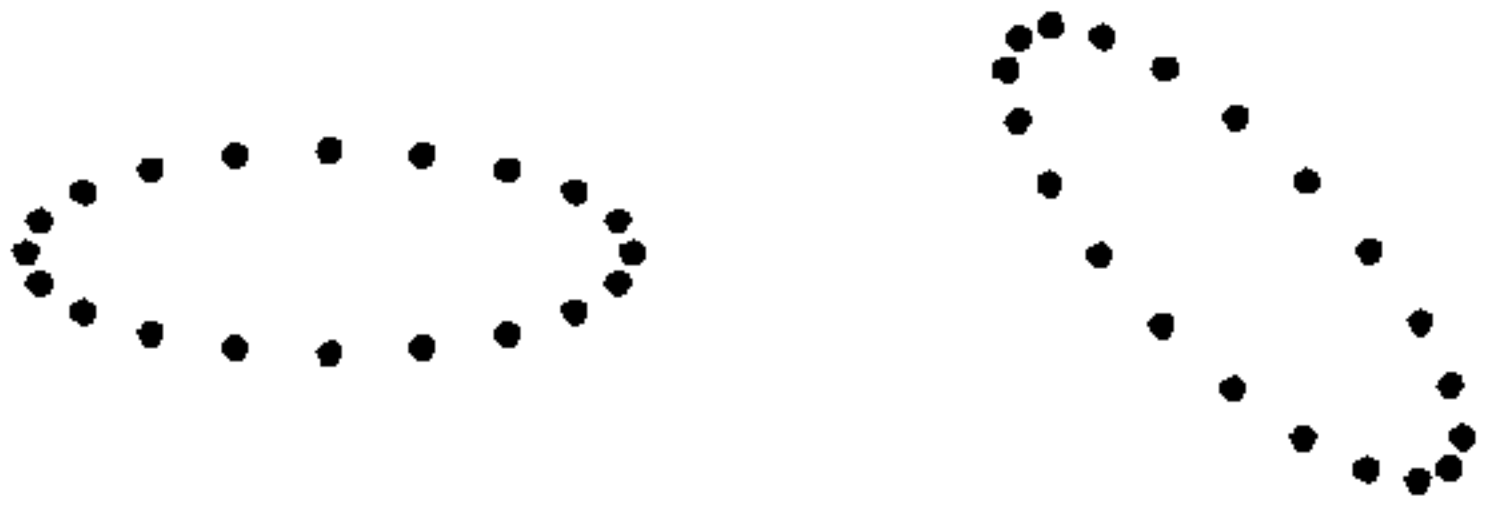
- Determinant of the matrix

$$[\exp(D)]_{ij} \equiv \delta_{ij} + D_{ij} + \frac{1}{2}\sum_{k=1}^{3} D_{ik}D_{kj} + \frac{1}{6}\sum_{km} D_{ik}D_{km}D_{mj} + \cdots$$

  is given by $\exp(\sum_i D_{ii})$.

- **Thus, D_{ij} must be trace-less:** $\sum_i D_{ii} = 0$ **,**

  so that the determinant is not modified by D_{ij}.



Not just space…

Distance between two points in space AND TIME

- The four-dimensional space-time distance in non-expanding Euclidean space is given by

$$ds_4^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

- The light path is given by $\mathbf{ds_4^2=0}$ (hence called “*null*”).

- Let us write this in a compact matrix form with 4-dimensional coordinates:

$$x^\mu = (ct, x^i)$$

called the “metric tensor”

$$ds_4^2 = \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g_{\mu\nu} dx^\mu dx^\nu$$

For the example above, we have

$$g_{00} = -1, \quad g_{ij} = \delta_{ij}$$

Not just space…

Distance between two points in space AND TIME

- The four-dimensional space-time distance in non-expanding Euclidean space is given by

$$ds_4^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

- The light path is given by **$ds_4^2=0$** (hence called “*null*”).

- Including the expanding space and curvature in space AND TIME, we write

$$ds_4^2 = -c^2 \exp(2\Phi) dt^2 + a^2 \exp(-2\Psi) \sum_{i=1}^{3} \sum_{j=1}^{3} [\exp(D)]_{ij} dx^i dx^j$$

Newtonian gravitational potential

Not just space…

Distance between two points in space AND TIME

- The four-dimensional space-time distance in non-expanding Euclidean space is given by

$$ds_4^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

- The light path is given by **ds_4^2=0** (hence called "*null*").

- Including the expanding space and curvature in space AND TIME, we write

$$ds_4^2 = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} g_{\mu\nu} dx^\mu dx^\nu \quad \text{with} \begin{cases} g_{00} = -\exp(2\Phi), \quad g_{0i} = 0, \\ g_{ij} = a^2 \exp(-2\Psi)[\exp(D)]_{ij} \end{cases}$$

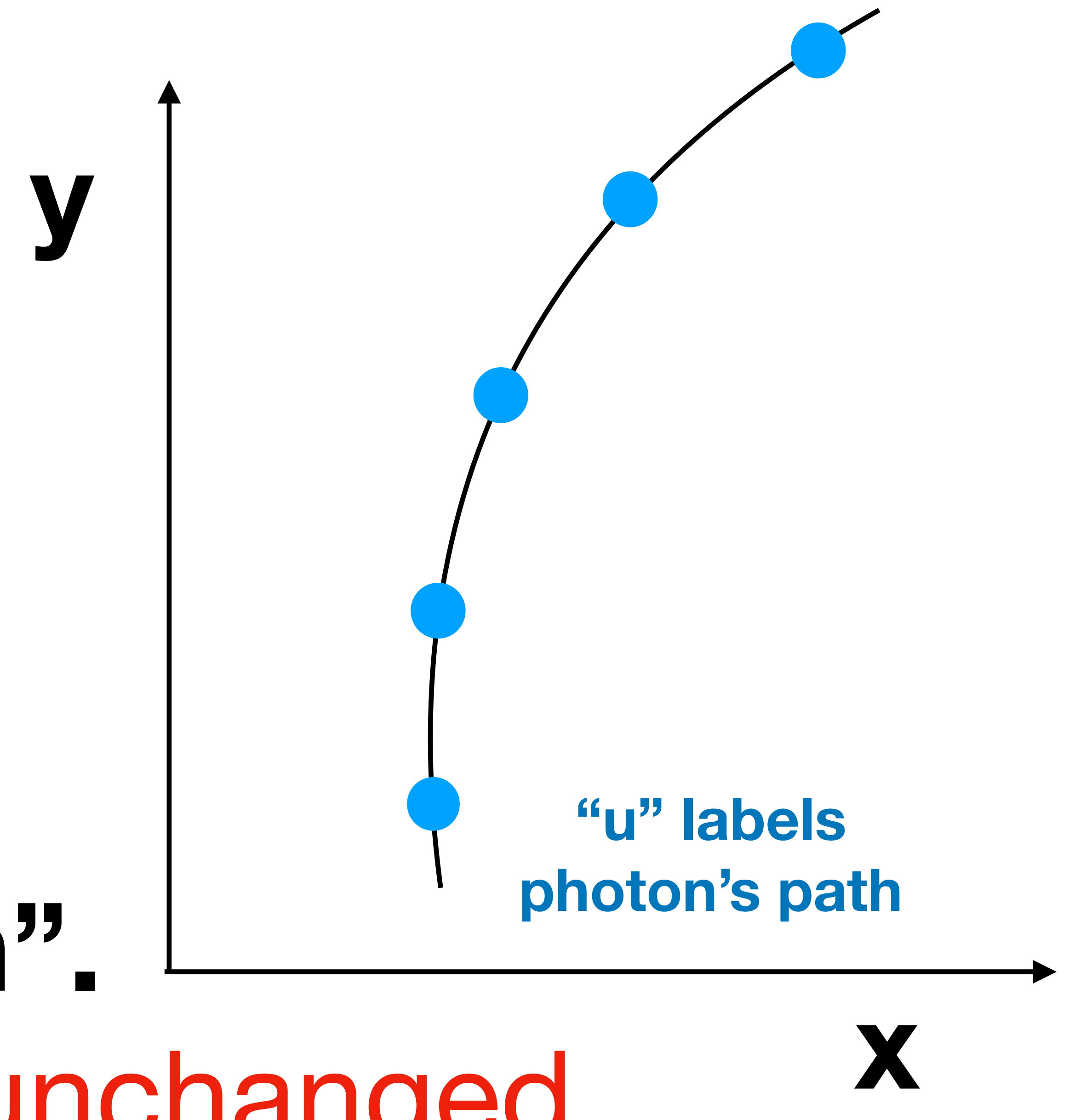
From now on, I will set the speed of light to be unity: $c=1$

Equation of motion for photons

Evolution of photon's *coordinates*

- Photon's path is determined such that the distance traveled by a photon between two points is minimised. This yields the equation of motion for photon's coordinates $x^\mu = (t, x^i)$:

$$\frac{d^2 x^\lambda}{du^2} + \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{du} \frac{dx^\nu}{du} = 0$$



This equation is known as the "geodesic equation".

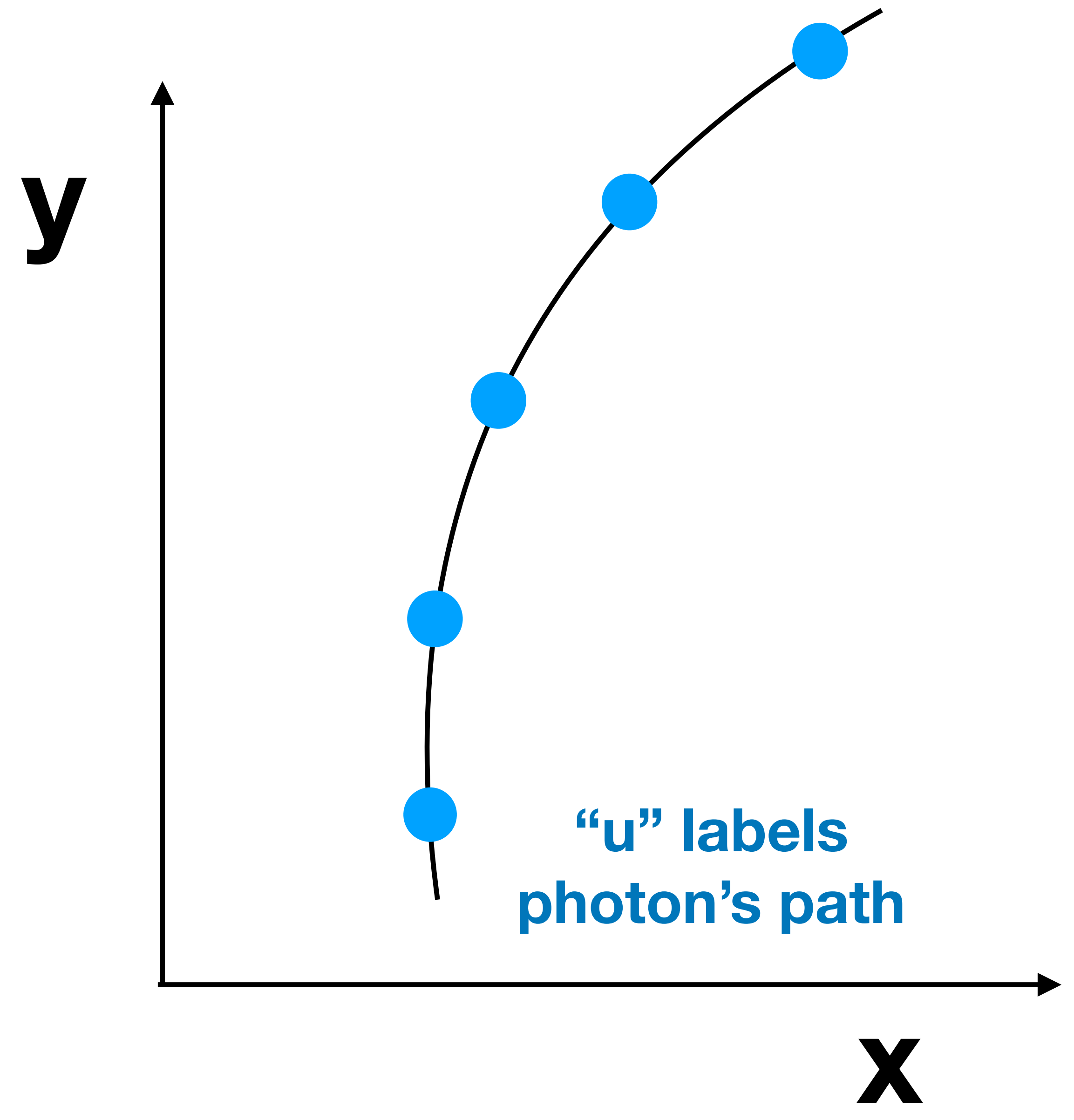
The second term is needed to keep the form of the equation unchanged under general coordinate transformation => **GRAVITATIONAL EFFECT!**

Equation of motion for photons

Evolution of photon's *momentum*

- It is more convenient to write down the geodesic equation in terms of **photon's momentum**: $p^\mu \equiv \frac{dx^\mu}{du}$

$$\frac{dp^\lambda}{dt} + \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} \Gamma^\lambda_{\mu\nu}\frac{p^\mu p^\nu}{p^0} = 0$$



- The magnitude of photon's momentum is equal to photon's energy: $p^2 \equiv \sum_{i=1}^{3}\sum_{j=1}^{3} g_{ij}p^i p^j$

Some calculations…

$$\frac{dp^\lambda}{dt} + \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} \Gamma^\lambda_{\mu\nu} \frac{p^\mu p^\nu}{p^0} = 0$$

With $ds_4^2 = \sum_{\mu\nu} g_{\mu\nu} dx^\mu dx^\nu$ $\left(\begin{array}{l} g_{00} = -\exp(2\Phi), \quad g_{0i} = 0, \\ g_{ij} = a^2 \exp(-2\Psi)[\exp(D)]_{ij} \end{array} \right)$

$$\Gamma^\lambda_{\mu\nu} \equiv \frac{1}{2} \sum_{\rho=0}^{3} g^{\lambda\rho} \left(\frac{\partial g_{\rho\mu}}{\partial x^\nu} + \frac{\partial g_{\rho\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\rho} \right)$$

Scalar perturbation [valid to all orders]

$$\Gamma^0_{00} = \dot{\Phi}, \quad \Gamma^0_{0i} = \frac{\partial \Phi}{\partial x^i}, \quad \Gamma^i_{00} = \exp(2\Phi) \sum_j g^{ij} \frac{\partial \Phi}{\partial x^j},$$

$$\Gamma^i_{0j} = \left(\frac{\dot{a}}{a} - \dot{\Psi} \right) \delta^i_j, \quad \Gamma^0_{ij} = \exp(-2\Phi) \left(\frac{\dot{a}}{a} - \dot{\Psi} \right) g_{ij},$$

$$\Gamma^k_{ij} = \delta_{ij} \sum_\ell \delta^{k\ell} \frac{\partial \Psi}{\partial x^\ell} - \delta^k_i \frac{\partial \Psi}{\partial x^j} - \delta^k_j \frac{\partial \Psi}{\partial x^i},$$

Tensor perturbation [valid to 1st order in D]

$$\Gamma^i_{0j} = \frac{\dot{a}}{a} \delta^i_j + \frac{1}{2} \sum_k \delta^{ik} \dot{D}_{kj}, \quad \Gamma^0_{ij} = \frac{\dot{a}}{a} g_{ij} + \frac{a^2}{2} \dot{D}_{ij},$$

$$\Gamma^k_{ij} = \frac{1}{2} \sum_\ell \delta^{k\ell} \left(\frac{D_{i\ell}}{\partial x^j} + \frac{D_{\ell j}}{\partial x^i} - \frac{D_{ij}}{\partial x^\ell} \right),$$

Recap so far

Math may be messy but the concept is transparent!

- Requiring **photons to travel between two points in space-time with the minimum path length**, we obtained the geodesic equation.
- The geodesic equation contains $\Gamma^{\lambda}_{\mu\nu}$ that is required to make **the form of the equation unchanged under general coordinate transformation.**
- Expressing $\Gamma^{\lambda}_{\mu\nu}$ in terms of the metric perturbations, we can obtain the desired result - **the equation that describes the rate of change of the photon energy.**

Rewrite $\frac{dp^{\lambda}}{dt} + \sum_{\mu=0}^{3}\sum_{\nu=0}^{3}\Gamma^{\lambda}_{\mu\nu}\frac{p^{\mu}p^{\nu}}{p^{0}} = 0$ in terms of $p^2 \equiv \sum_{i=1}^{3}\sum_{j=1}^{3} g_{ij}p^i p^j$

The Result

Sachs & Wolfe (1967)

γ^i is a unit vector of the direction of photon's momentum: $\sum_i (\gamma^i)^2 = 1$

$$\frac{1}{p}\frac{dp}{dt} = -\frac{\dot{a}}{a} + \dot{\Psi} - \frac{1}{a}\sum_i \frac{\partial \Phi}{\partial x^i}\gamma^i - \frac{1}{2}\sum_{ij}\dot{D}_{ij}\gamma^i\gamma^j$$

- Let's interpret this equation *physically*. Explain each term in words!

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- **Cosmological redshift**
- Photon's wavelength is stretched in proportion to the scale factor, and thus the photon energy decreases as

$$p \propto a^{-1}$$

The Result

Sachs & Wolfe (1967)

γ^i is a unit vector of the direction of photon's momentum: $\sum_i (\gamma^i)^2 = 1$

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- **Cosmological redshift - part II**
- The spatial metric is given by $ds^2 = a^2(t)\exp(-2\Psi)d\mathbf{x}^2$
- Thus, locally we can define a new scale factor: $\tilde{a}(t,\mathbf{x}) = a(t)\exp(-\Psi)$
- Then the photon energy decreases as

$$p \propto \tilde{a}^{-1}$$

The Result

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γ^i is a unit vector of the direction of photon's momentum: $\sum_i (\gamma^i)^2 = 1$

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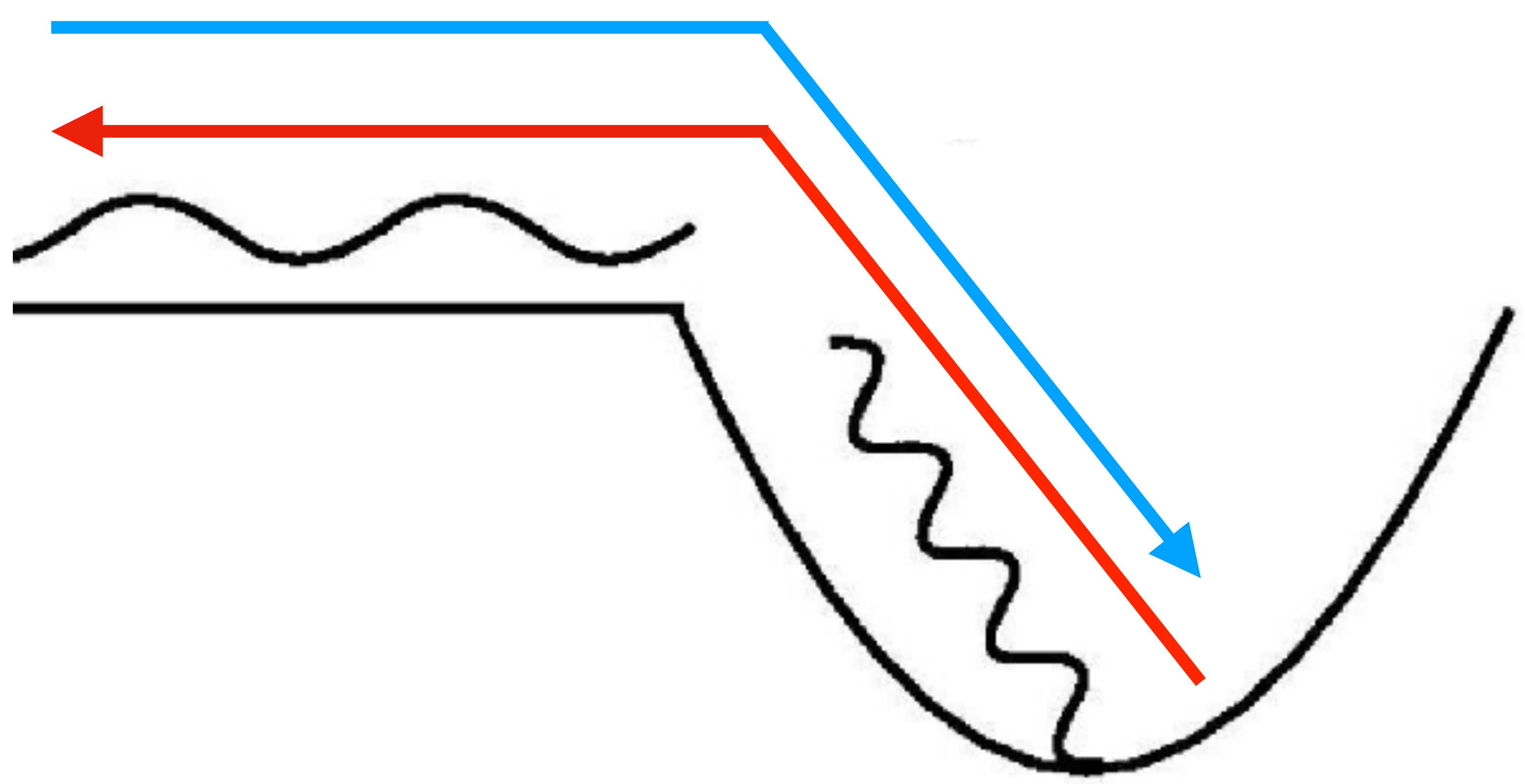
- **Gravitational blue/redshift (Scalar)**



Potential well ($\phi < 0$)

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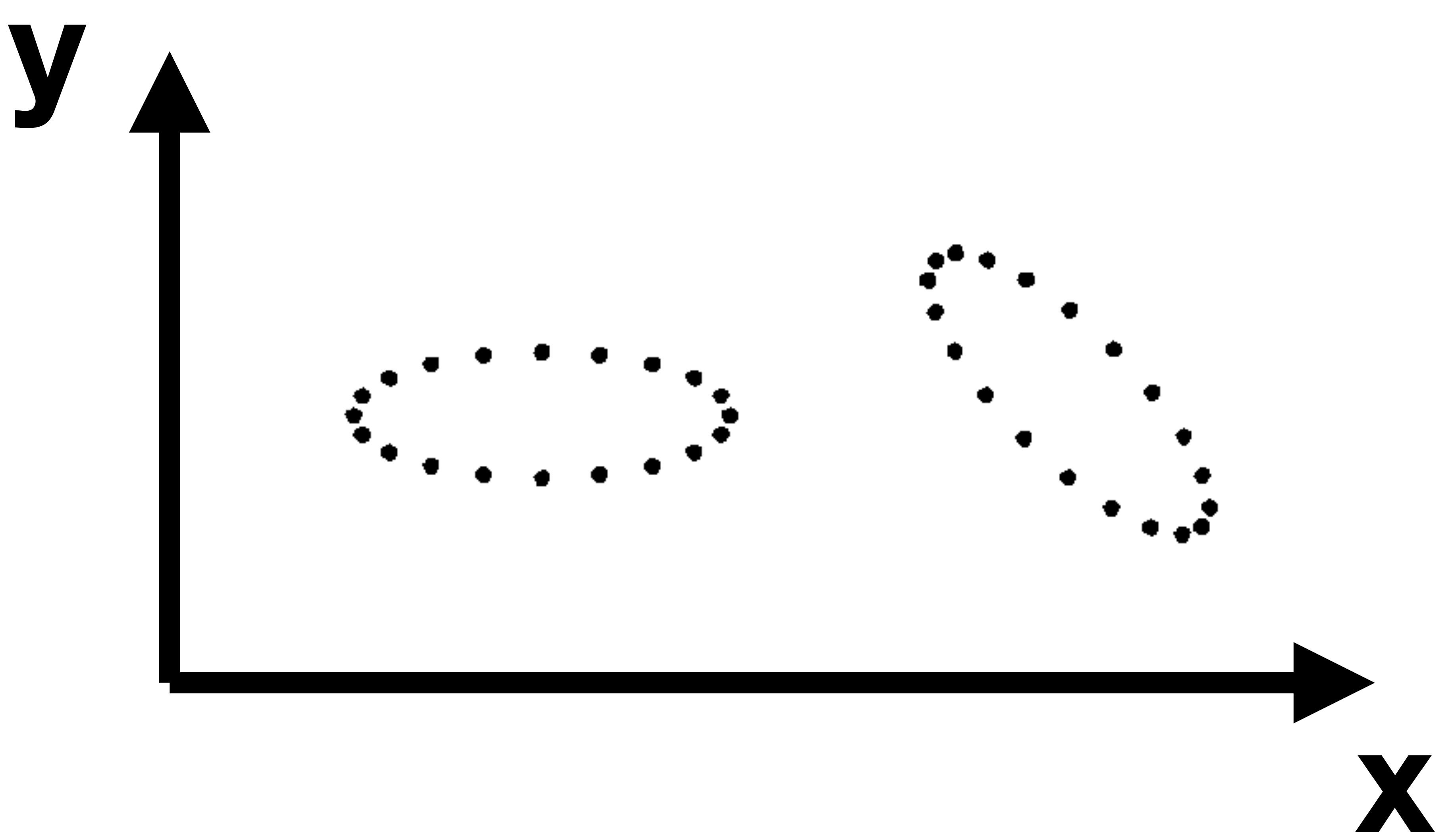
- **Gravitational blue/redshift (Tensor)**

$$D_{ij} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



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