Lecture notes:

https://wwwmpa.mpa-garching.mpg.de/~komatsu/lectures--reviews.html

Primordial Gravitational Waves from Inflation

Eiichiro Komatsu
[Max Planck Institute for Astrophysics]

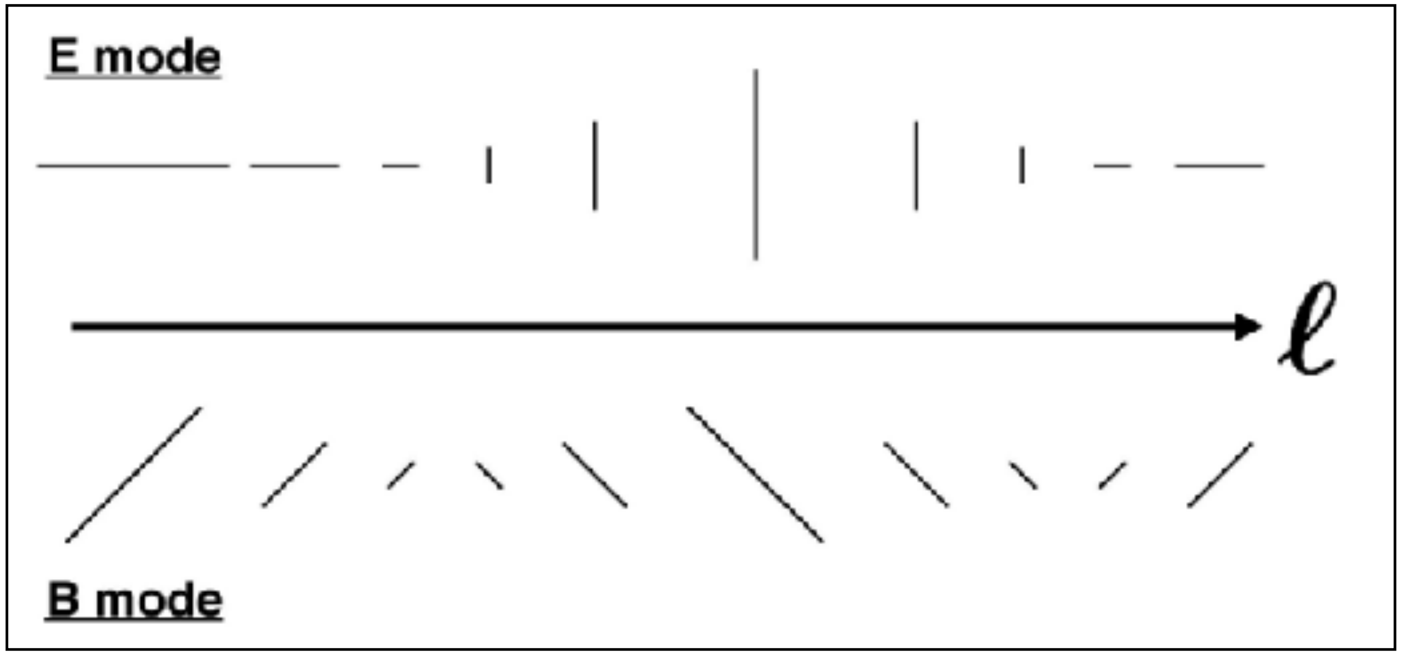
University of Amsterdam
February 27, March 5, and 19, 2020

Plan

- **Today**: Vacuum Fluctuation

$$\Box h_{ij} = 0$$

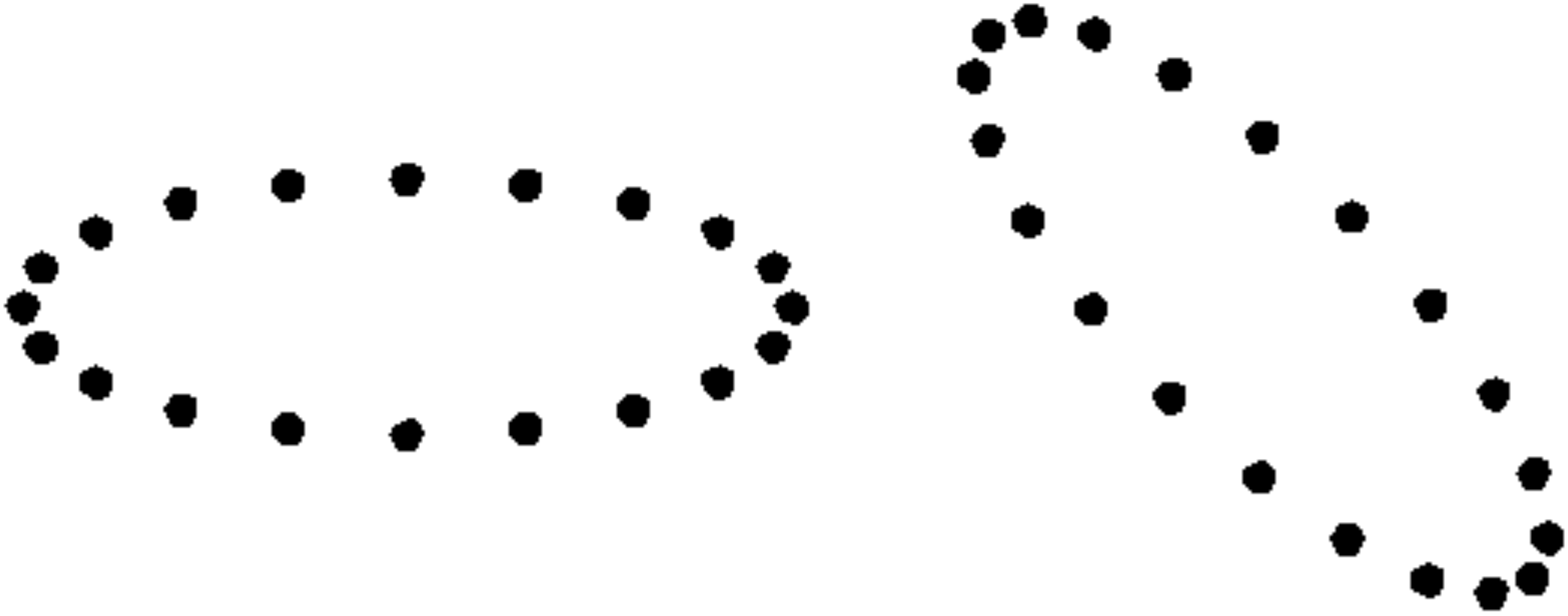
- **March 5**: Polarisation of the cosmic microwave background



- **March 19**: Sourced Contribution

$$\Box h_{ij} = -16\pi G \pi_{ij}^{GW}$$

GW = Area-conserving distortion of distances between two points



Distance between two points in space

- Static (i.e., non-expanding) Euclidean space
    - In Cartesian coordinates $\boldsymbol{x} = (x, y, z)$

$$ds^2 = dx^2 + dy^2 + dz^2$$

Distance between two points in space

- Homogeneously expanding Euclidean space
  - In Cartesian **comoving** coordinates $\boldsymbol{x} = (x, y, z)$

$$ds^2 = \boxed{a^2(t)}(dx^2 + dy^2 + dz^2)$$

“scale factor”

Distance between two points in space

- Homogeneously expanding Euclidean space
    - In Cartesian **comoving** coordinates $\boldsymbol{x} = (x, y, z)$

$$ds^2 = \underbrace{\boxed{a^2(t)}}_{\text{“scale factor”}} \sum_{i=1}^{3} \sum_{j=1}^{3} \delta_{ij} \, dx^i dx^j$$

δ_{ij} **=1 for i=j**

=0 otherwise

Distance between two points in space

- Inhomogeneous curved space
  - In Cartesian **comoving** coordinates $\boldsymbol{x} = (x, y, z)$

$$ds^2 = a^2 \sum_{i=1}^{3} \sum_{j=1}^{3} (\delta_{ij} + \boxed{h_{ij}}) dx^i dx^j$$

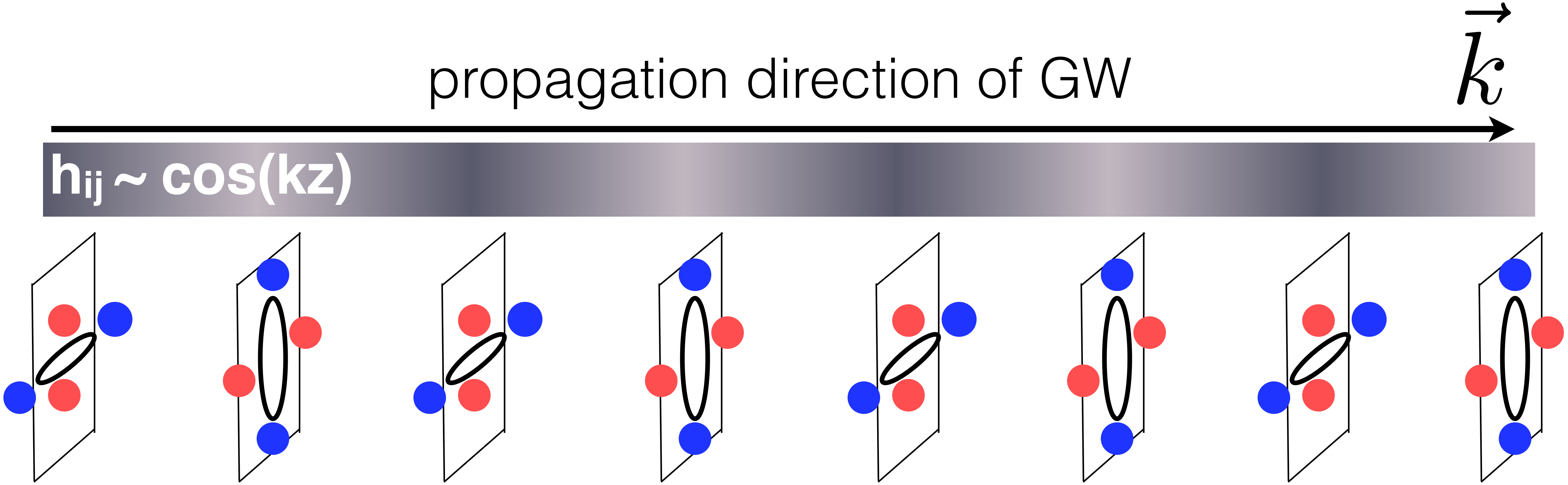
"**metric perturbation**"

-> CURVED SPACE!

Four conditions

- Gravitational waves shall be:
  - **Transverse**: the direction of the oscillation of space is perpendicular to the propagation direction $\vec{k}$
    - This means $\sum_{i=1}^{3} k^i h_{ij} = 0$

3 conditions for h_{ij}

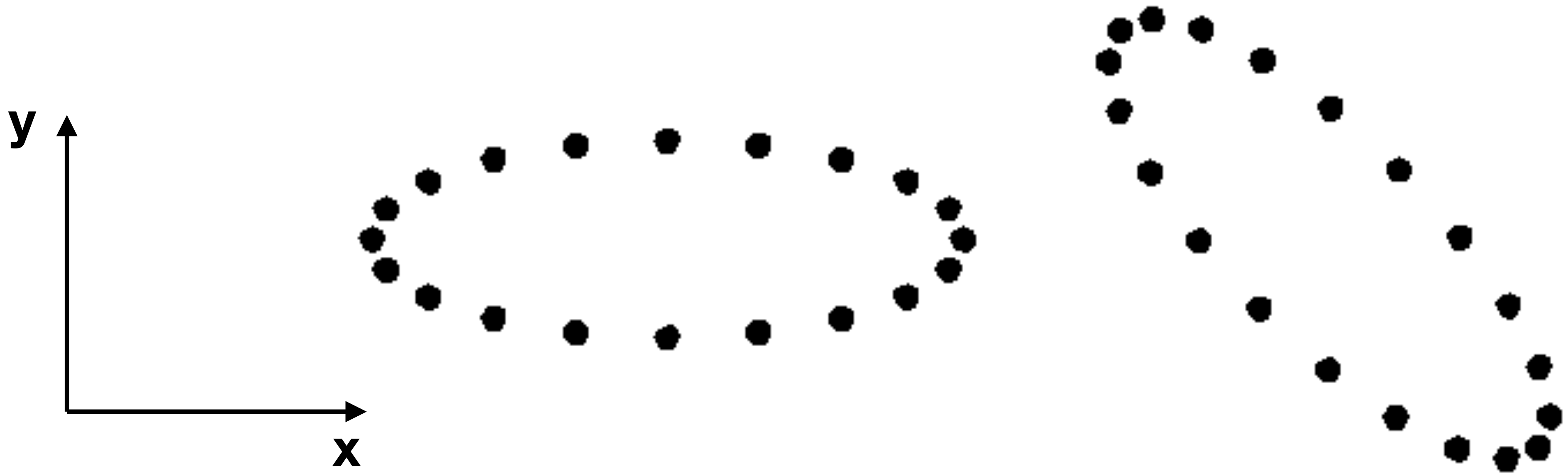


propagation direction of GW

$\vec{k}$

$h_{ij} \sim \cos(kz)$

Four conditions



- **Area-conserving**: the determinant of the distortion in space remains unchanged
  - This means that the trace vanishes: $\sum_{i=1}^{3} h_{ii} = 0$

1 condition for h_{ij}

Four conditions

- Gravitational waves shall be:
  - **Transverse**: the direction of the oscillation of space is perpendicular to the propagation direction $\vec{k}$
    - This means $\sum_{i=1}^{3} k^i h_{ij} = 0$

      3 conditions for h_{ij}
  - **Area-conserving**: the determinant of the distortion in space remains unchanged
    - This means that the trace vanishes: $\sum_{i=1}^{3} h_{ii} = 0$

      1 condition for h_{ij}

6 components of h_{ij} minus 4 conditions = 2 degrees of freedom

More precisely:

- We should start with a space-time distance with a 4-by-4 metric tensor, $g_{\mu\nu}$ [μ,ν=0,1,2,3]
  - It has 10 components:

$$ds_4^2 = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} g_{\mu\nu} dx^\mu dx^\nu$$

with

$$dx^\mu = (dt, dx^i)$$

- Coordinate condition eliminates 4 degree of freedom (DoF)

More precisely:

- We should start with a space-time distance with a 4-by-4 metric tensor, $g_{\mu\nu}$ [$\mu,\nu=0,1,2,3$]
    - It has 10 components:

$$ds_4^2 = \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g_{\mu\nu} dx^\mu dx^\nu \qquad \textbf{with} \;\; dx^\mu = (dt, dx^i)$$

$$= (-1+h_{00})dt^2 + a(t)\sum_{i=1}^{3} h_{0i} dt dx^i + a^2(t)\sum_{i=1}^{3}\sum_{j=1}^{3}(\delta_{ij}+h_{ij})dx^i dx^j$$

- Coordinate condition eliminates 4 degree of freedom (DoF)
    - leaving 6 DoF: This is where we started; **in this lecture we started with $h_{00}=0$ and $h_{0i}=0$** (called “synchronous gauge”)

More precisely:

- We should start with a space-time distance with a 4-by-4 metric tensor, $g_{\mu\nu}$ [$\mu,\nu=0,1,2,3$]

  - It has 10 components:

$$ds_4^2 = \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g_{\mu\nu}dx^\mu dx^\nu \quad \textbf{with} \quad dx^\mu = (dt, dx^i)$$

$$= (-1+h_{00})dt^2 + a(t)\sum_{i=1}^{3} h_{0i}dtdx^i + a^2(t)\sum_{i=1}^{3}\sum_{j=1}^{3}(\delta_{ij}+h_{ij})dx^i dx^j$$

6 DoF = 2 scalar, 2 vector, 2 tensor DoF

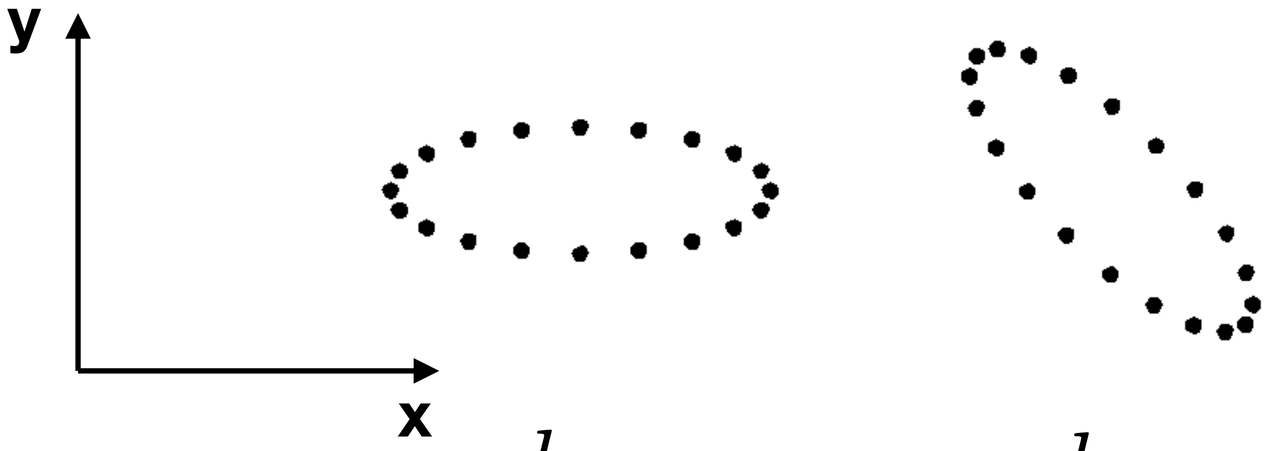
This is GW, which can be extracted by imposing transverse and traceless conditions

  - leaving 6 DoF: This is where we started; **in this lecture we started with $h_{00}=0$ and $h_{0i}=0$** (called "synchronous gauge")

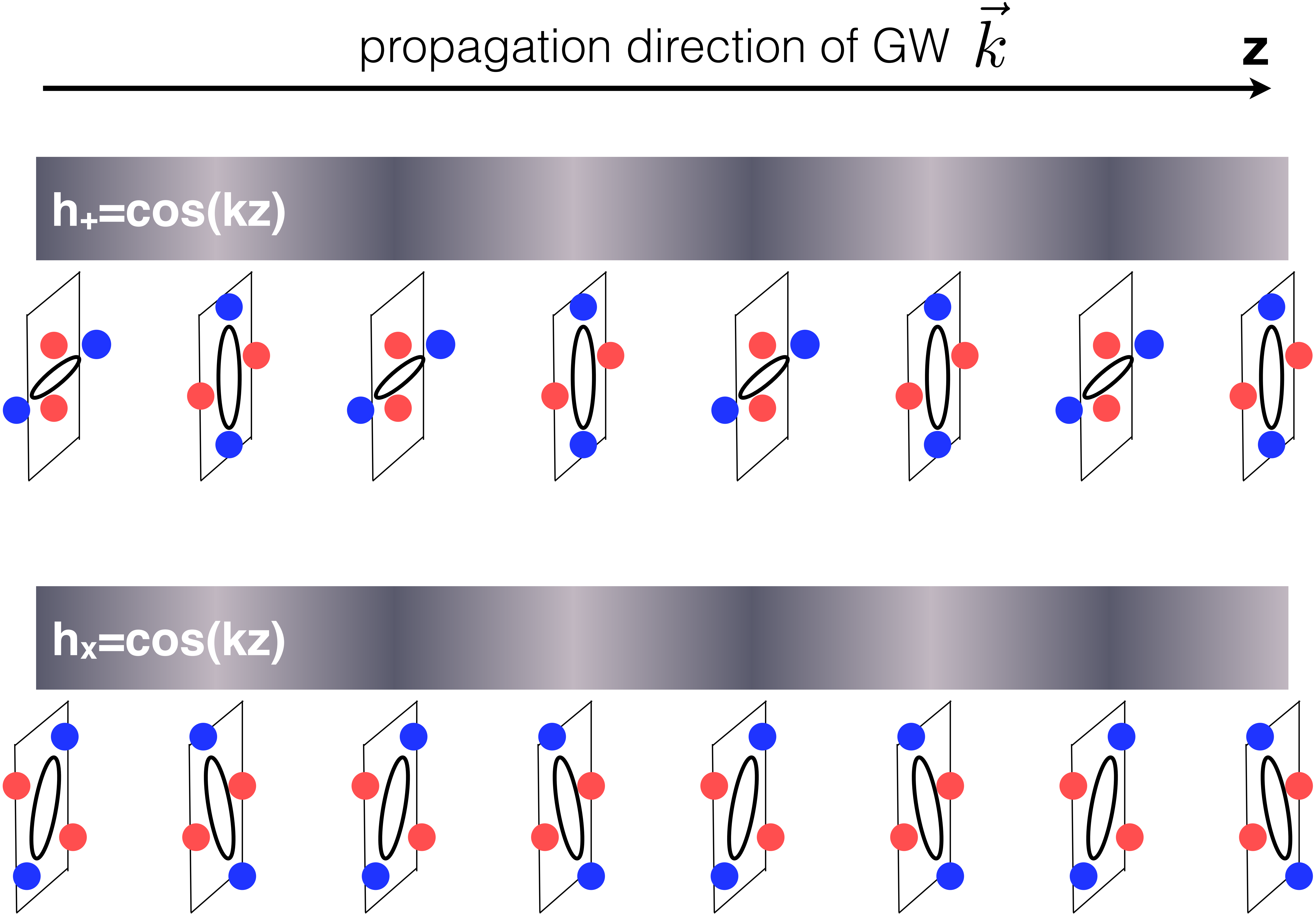
+ and x modes

- If the GW is propagating in the z (i=3) direction, we can write

$$h_{ij} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



h_+ $h_\times$


propagation direction of GW $\vec{k}$
z
$h_+=\cos(kz)$
$h_\times=\cos(kz)$

Equation of motion (EoM)

- Writing Einstein's gravitational field equation with

$$ds^2 = a^2 \sum_{i=1}^{3} \sum_{j=1}^{3} (\delta_{ij} + h_{ij}) dx^i dx^j$$

- and $\sum_{i=1}^{3} k^i h_{ij} = 0, \quad \sum_{i=1}^{3} h_{ii} = 0;$ We obtain

$$a^2 \Box h_{ij} = -16\pi G T_{ij}^{GW}$$

stress-energy source of GW

EoM in a non-expanding space

$$a^2 \Box h_{ij} = -16\pi G T_{ij}^{GW}$$

stress-energy source of GW

$$\Box = -\frac{\partial^2}{\partial t^2} + \nabla^2 = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} \eta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu}$$

with $\eta^{00} = -1, \quad \eta^{0i} = 0, \quad \eta^{ij} = \delta^{ij}$

EoM in an expanding Universe

$$a^2 \Box h_{ij} = -16\pi G T^{GW}_{ij}$$

stress-energy source of GW

$$\Box \equiv \frac{1}{\sqrt{-g}} \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} \frac{\partial}{\partial x^\mu} \left(\sqrt{-g} g^{\mu\nu} \frac{\partial}{\partial x^\nu} \right)$$

with $g^{00} = -1, \quad g^{0i} = 0, \quad g^{ij} = a^{-2}(t)\delta^{ij}, \quad \sqrt{-g} = a^3(t)$

$$\Box = -\frac{\partial^2}{\partial t^2} - 3\frac{\dot{a}}{a}\frac{\partial}{\partial t} + \frac{1}{a^2}\nabla^2$$

EoM in an expanding Universe

$$a^2\Box h_{ij} = -16\pi G T_{ij}^{GW}$$

stress-energy source of GW

$$\Box \equiv \frac{1}{\sqrt{-g}} \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} \frac{\partial}{\partial x^\mu} \left(\sqrt{-g} g^{\mu\nu} \frac{\partial}{\partial x^\nu} \right)$$

with $g^{00} = -1, \quad g^{0i} = 0, \quad g^{ij} = a^{-2}(t)\delta^{ij}, \quad \sqrt{-g} = a^3(t)$

$$\Box = -\frac{\partial^2}{\partial t^2} - 3\frac{\dot{a}}{a}\frac{\partial}{\partial t} - \frac{k^2}{a^2}$$

In Fourier space

$$\nabla^2 e^{i\mathbf{k}\cdot\mathbf{x}} = -k^2 e^{i\mathbf{k}\cdot\mathbf{x}}$$

EoM in an expanding Universe

$$a^2\Box h_{ij} = -16\pi G T_{ij}^{GW}$$

stress-energy source of GW

$$\Box \equiv \frac{1}{\sqrt{-g}} \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} \frac{\partial}{\partial x^\mu}\left(\sqrt{-g}\, g^{\mu\nu} \frac{\partial}{\partial x^\nu}\right)$$

with $g^{00} = -1, \quad g^{0i} = 0, \quad g^{ij} = a^{-2}(t)\delta^{ij}, \quad \sqrt{-g} = a^3(t)$

k: *comoving* wavenumber

$$\Box = -\frac{\partial^2}{\partial t^2} - 3\frac{\dot{a}}{a}\frac{\partial}{\partial t} - \frac{\boxed{k}^2}{a^2}$$

In Fourier space

$$\nabla^2 e^{i\mathbf{k}\cdot\mathbf{x}} = -k^2 e^{i\mathbf{k}\cdot\mathbf{x}}$$

EoM in an expanding Universe

$$a^2\Box h_{ij} = -16\pi G T_{ij}^{GW}$$

stress-energy source of GW

$$\Box \equiv \frac{1}{\sqrt{-g}}\sum_{\mu=0}^{3}\sum_{\nu=0}^{3}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}g^{\mu\nu}\frac{\partial}{\partial x^\nu}\right)$$

with $g^{00} = -1, \quad g^{0i} = 0, \quad g^{ij} = a^{-2}(t)\delta^{ij}, \quad \sqrt{-g} = a^3(t)$

k/a: *physical* wavenumber

$$\Box = -\frac{\partial^2}{\partial t^2} - 3\frac{\dot{a}}{a}\frac{\partial}{\partial t} - \frac{\boxed{k}^2}{\boxed{a}^2}$$

In Fourier space

$$\nabla^2 e^{i\mathbf{k}\cdot\mathbf{x}} = -k^2 e^{i\mathbf{k}\cdot\mathbf{x}}$$

EoM in an expanding Universe

$$a^2\Box h_{ij} = -16\pi G T_{ij}^{GW}$$

stress-energy source of GW

$$\Box = -\frac{\partial^2}{\partial t^2} - 3\frac{\dot{a}}{a}\frac{\partial}{\partial t} - \frac{k^2}{a^2}$$

In Fourier space

We define:

$$\boxed{T_{ij} = a^2\pi_{ij}}$$

$$\ddot{h}_{ij} + \frac{3\dot{a}}{a}\dot{h}_{ij} + \frac{k^2}{a^2}h_{ij} = 16\pi G\pi_{ij}^{GW}$$

EoM in an expanding Universe

$$a^2\Box h_{ij} = -16\pi G T_{ij}^{GW}$$

stress-energy source of GW

$$\Box = -\frac{\partial^2}{\partial t^2} - 3\frac{\dot{a}}{a}\frac{\partial}{\partial t} - \frac{k^2}{a^2}$$

In Fourier space

We define:

$$\boxed{T_{ij} = a^2\pi_{ij}}$$

$$\ddot{h}_{ij} + \boxed{\frac{3\dot{a}}{a}}\dot{h}_{ij} + \frac{k^2}{a^2}h_{ij} = 16\pi G\underline{\pi_{ij}^{GW}}$$

expansion of the Universe affects h_{ij}

Let’s solve EoM

$$\ddot{h}_{ij} + \boxed{\frac{3\dot{a}}{a}}\dot{h}_{ij} + \frac{k^2}{a^2}h_{ij} = 16\pi G\pi_{ij}^{GW}$$

expansion of the Universe affects h_{ij}

- Two tricks:

(1) Define “conformal time”

$$\eta = \int \frac{dt}{a(t)}$$

and use this instead of time derivatives

$$a(t)\frac{\partial}{\partial t} = \frac{\partial}{\partial \eta}$$

Let’s solve EoM

$$h''_{ij} + \boxed{\frac{2a'}{a}} h'_{ij} + k^2 h_{ij} = 16\pi G a^2 \pi_{ij}^{GW}$$

Primes =
conformal time derivatives

- <u>Two tricks:</u>

(1) Define “conformal time”

$$\eta = \int \frac{dt}{a(t)}$$

and use this instead of time derivatives

$$a(t)\frac{\partial}{\partial t} = \frac{\partial}{\partial \eta}$$

Let’s solve EoM

$$h''_{ij} + \boxed{\frac{2a'}{a}} h'_{ij} + k^2 h_{ij} = 16\pi G a^2 \pi_{ij}^{GW}$$

- Two tricks:

(2) Multiply h_{ij} by the scale factor and define

$$u_{ij} = a h_{ij}$$

Let’s solve EoM

$$u_{ij}'' + \left(k^2 - \boxed{\frac{a''}{a}}\right) u_{ij} = 16\pi G a^3 \pi_{ij}^{GW}$$

- Two tricks:

(2) Multiply h_{ij} by the scale factor and define

$$u_{ij} = a h_{ij}$$

Let’s solve EoM

$$u''_{ij} + \left(k^2 - \boxed{\frac{a''}{a}}\right) u_{ij} = 16\pi G a^3 \pi^{GW}_{ij}$$

Defining

$$m^2(\eta) = -\frac{a''}{a}$$

effect of the expansion of the Universe

We obtain a harmonic oscillator with a time-dependent mass term!

$$u''_{ij} + \left[k^2 + m^2(\eta)\right] u_{ij} = 16\pi G a^3 \pi^{GW}_{ij}$$

Propagation of GW in vacuum: Two regimes

$$u''_{ij} + \left[k^2 + m^2(\eta)\right] u_{ij} = 0$$

- Two regimes:

  1. *Short wavelength* ($k >> |m|$)
     - $u_{ij} \sim \exp(ik\eta) \Rightarrow h_{ij} \sim a^{-1}\exp(ik\eta)$ [decaying]
  2. *Long wavelength* ($k << |m|$)
     - $u_{ij} \sim a \Rightarrow h_{ij} \sim$ **constant**

Meaning of m²

$$m^2(\eta) = -\frac{a''}{a} = -a^2(2H^2 + \dot{H})$$

Hubble's expansion rate

$$H = \frac{\dot{a}}{a}$$

- The inverse of the expansion rate, $(aH)^{-1}$, gives an estimate of the (comoving) size of the observable Universe, or "*horizon*"

- So, k << |m| is the "**super-horizon**" mode, and k >> |m| is the "**sub-horizon**" mode

Horizon Distance

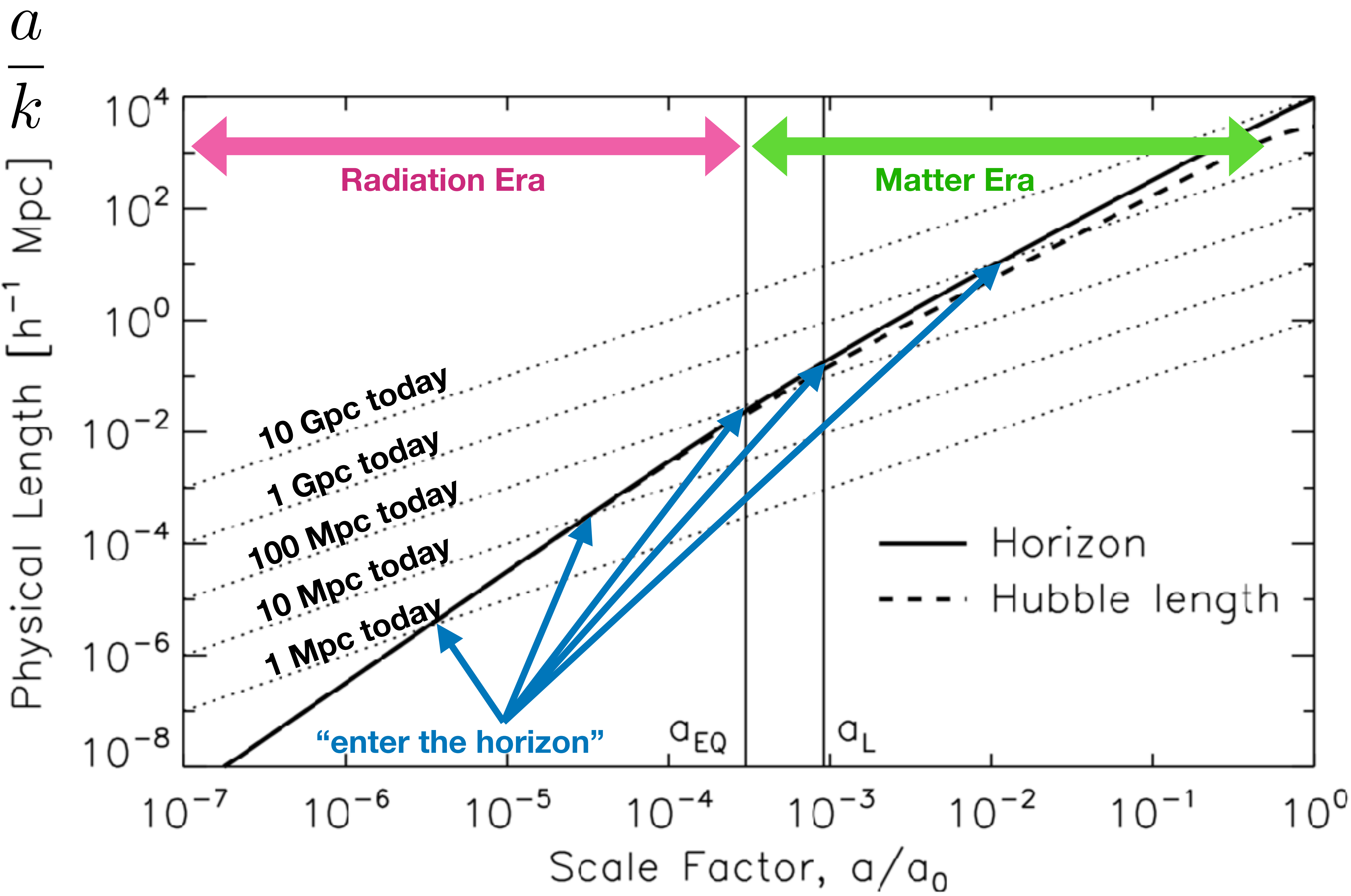
- **Horizon** = the physical distance traveled by a photon
  - The (unperturbed) photon path in the radial direction is given by $ds_4^2 = -dt^2 + a^2(t)dr^2 = 0$
  - Integrating it, we obtain the physical distance traveled by a photon, d_{horizon}, as

$$d_{\text{horizon}} = a(t)r = a(t)\int_0^t \frac{dt'}{a(t')}$$

- Hubble length is given by H^{-1}, which is on the same order of magnitude as d_{horizon}. **Comoving Hubble length is $(aH)^{-1}$, which is on the same order of magnitude as $d_{\text{horizon}}/a(t)=r$**

GW “entering the horizon”

- This is a tricky concept, but it is important
- Suppose that GWs exist at all wavelengths
  - Let’s not **yet** ask the origin of these “super-horizon GW”, but assume their existence
- As the Universe expands, the horizon size grows and we can see longer and longer wavelengths
  - **Fluctuations “entering the horizon”**


$\frac{a}{k}$
Radiation Era
Matter Era
10 Gpc today
1 Gpc today
100 Mpc today
10 Mpc today
1 Mpc today
"enter the horizon"
a_{EQ}
a_L
Horizon
Hubble length
Physical Length [h^{-1} Mpc]
10^4
10^2
10^0
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-7}
10^{-6}
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^0
Scale Factor, a/a_0

GW Evolution: Summary

- **Super-horizon scales [$k << aH$]**
    - The amplitude of GW is **conserved** **(i.e., h_{ij} = constant)**
- **Sub-horizon scales [$k >> aH$]**
    - The amplitude of GW decays **(i.e., $h_{ij} \sim 1/a$)**

Therefore, the long-wavelength GW preserves the initial condition: the beginning of the Universe!

Source of GW in the early Universe?

$$u''_{ij} + \left(k^2 - \frac{a''}{a}\right) u_{ij} = 16\pi G a^3 \pi_{ij}^{GW}$$

- Was there any **source of GW** in the early Universe?
- Yes, in a sense that there are many papers on possible sources in the literature
  - See a recent review article by C. Caprini and D. Figueroa, *Classical and Quantum Gravity*, 35, 163001 (2018), arXiv:1801.04268

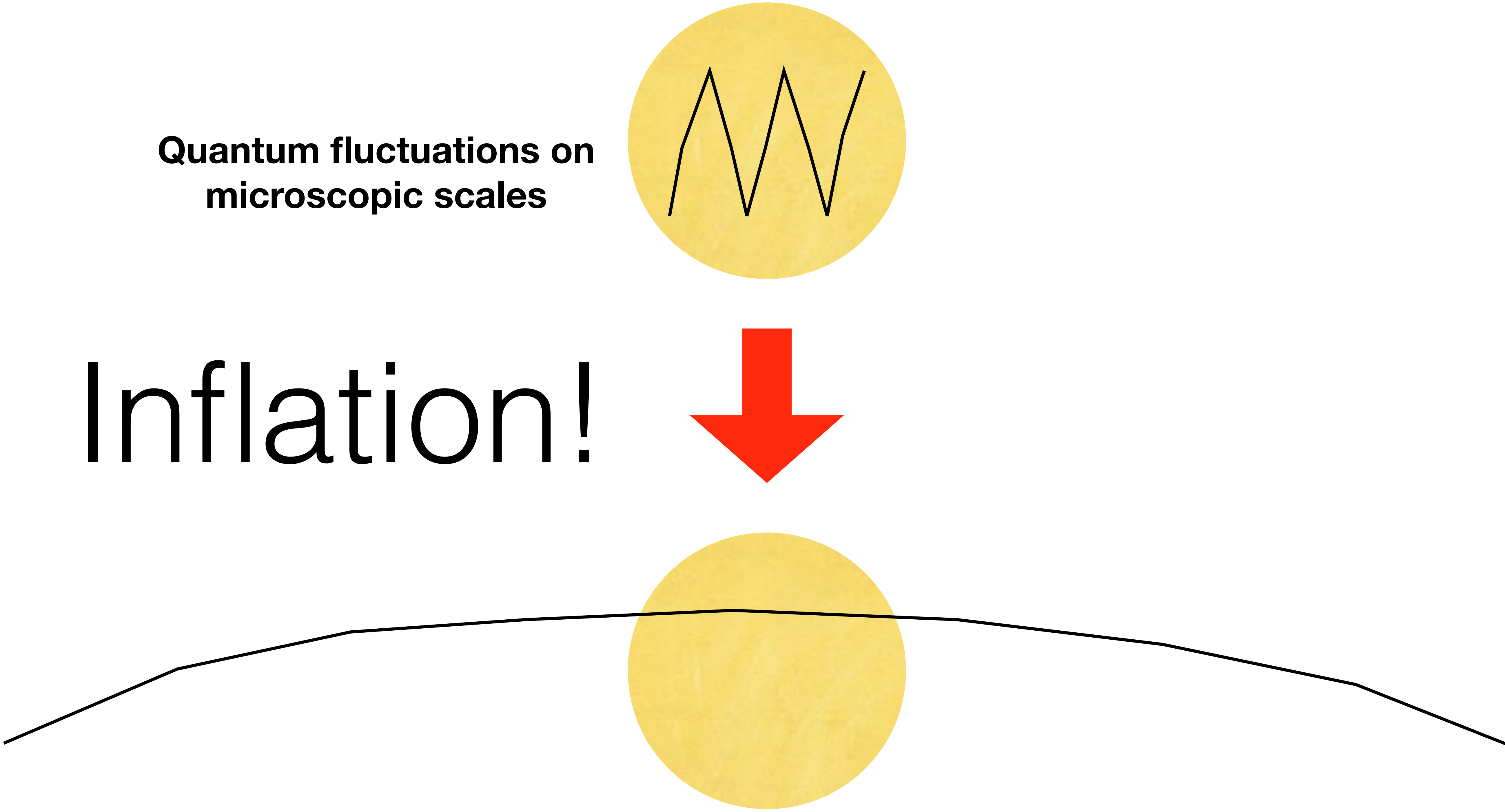
Quantum generation of GW in the early Universe!

$$u_{ij}'' + \left(k^2 - \frac{a''}{a}\right) u_{ij} = \xcancel{16\pi G a^3 \pi_{ij}^{GW}}$$

- But, even if there was no source, **GW can emerge quantum-mechanically!** ***Grishchuk (1974); Starobinsky (1979)***
  - This is the subject of today's lecture. We will talk about the right hand side on March 19
- To see this, we need to quantise the left hand side of the equation

Starobinsky (1980); Sato (1981); Guth (1981); Linde (1982); Albrecht & Steinhardt (1982)

Cosmic Inflation



Quantum fluctuations on microscopic scales

Inflation!

- Exponential expansion (inflation) stretches the wavelength of quantum fluctuations to very large scales

Starobinsky (1980); Sato (1981); Guth (1981); Linde (1982); Albrecht & Steinhardt (1982)

Cosmic Inflation

- Inflation is the accelerated, quasi-exponential expansion. Thus, we must have

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 > 0 \quad \longrightarrow \quad \epsilon \equiv -\frac{\dot{H}}{H^2} < 1$$

Actually, we rather need ε << 1, to have a sustained period of inflation. So H(t) is a slowly-varying function of time

Starobinsky (1980); Sato (1981); Guth (1981); Linde (1982); Albrecht & Steinhardt (1982)

Cosmic Inflation

- Inflation is the accelerated, quasi-exponential expansion. Thus, we must have

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 > 0 \quad \Rightarrow \quad \epsilon \equiv -\frac{\dot{H}}{H^2} < 1$$

Therefore,

$$\frac{\dot{a}}{a} = H \to a(t) = \exp\left[\int_{t_0}^{t} dt\ H(t')\right] \approx \exp[H(t - t_0)]$$

During inflation, a(t) grows exponentially in time

$$m^2(\eta) = -\frac{a''}{a} = -a^2(2H^2 + \cancel{\dot{H}})$$

GW from inflation

$$u''_{ij} + \left(k^2 - 2a^2H^2\right) u_{ij} = 0$$

- During inflation, the scale factor grows exponentially in time,

$$a(t) \propto \exp(Ht)$$

- In conformal time, this means

$$a(\eta) = -(H\eta)^{-1}$$

for $-\infty < \eta < 0$

$$\eta = \int \frac{dt}{a(t)}$$

GW from inflation

$$u''_{ij} + \left(k^2 - \frac{2}{\eta^2}\right) u_{ij} = 0$$

- During inflation, the scale factor grows exponentially in time,

$$a(t) \propto \exp(Ht)$$

- In conformal time, this means

$$a(\eta) = -(H\eta)^{-1}$$ **for** $-\infty < \eta < 0$

GW from inflation

$$u''_{ij} + \left(k^2 - \frac{2}{\eta^2}\right) u_{ij} = 0$$

- The solution is

$$u_{ij} = A_{ij}\left[\cos(k\eta) - \frac{\sin(k\eta)}{k\eta}\right] + B_{ij}\left[\frac{\cos(k\eta)}{k\eta} + \sin(k\eta)\right]$$

- How do we fix the integration constants, A_{ij} and B_{ij}? **We need QM!**
  - We find A_{ij} and B_{ij}, such that the u_{ij} coincides with the known flat-space (Minkowski) results for the quantum fluctuation in vacuum

Second-order Action

- The action that gives Einstein's field equations is the so-called "Einstein-Hilbert action", given by the Ricci scalar R:

$$I_{GR} = \int \sqrt{-g} d^4x \left(\frac{1}{2} M_{\rm pl}^2 R \right) \qquad \text{with} \quad \begin{array}{l} M_{\rm pl} = (8\pi G)^{-1/2} \\ \sqrt{-g} = a^3 \end{array}$$

- Expanding this to second-order in h_{ij}, we obtain the action that gives the equation of motion for h_{ij}:

$$I_{GR}^{(2)} = \int a^3 d^4x \; \frac{1}{4} M_{\rm pl}^2 \left(\frac{1}{2} \dot{h}_{ij}^2 - \frac{(\nabla h_{ij})^2}{2a^2} \right) \quad \text{with} \quad h_{ij} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \int a^3 d^4x \; \frac{1}{2} M_{\rm pl}^2 \sum_{\lambda=+,\times} \left(\frac{1}{2} \dot{h}_\lambda^2 - \frac{(\nabla h_\lambda)^2}{2a^2} \right)$$

Second-order Action

- The action that gives Einstein’s field equations is the so-called “Einstein-Hilbert action”, given by the Ricci scalar R:

$$I_{GR} = \int \sqrt{-g} d^4x \left(\frac{1}{2} M_{\rm pl}^2 R \right)$$

with

$$M_{\rm pl} = (8\pi G)^{-1/2}$$
$$\sqrt{-g} = a^3$$

- Expanding this to second-order in h_{ij}, we obtain the action that gives the equation of motion for h_{ij}:

$$I_{GR}^{(2)} = \int a^3 d^4x \, \frac{1}{4} M_{\rm pl}^2 \left(\frac{1}{2} \dot{h}_{ij}^2 - \frac{(\nabla h_{ij})^2}{2a^2} \right)$$

with $h_{ij} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$= \int \boxed{a^3} d^4x \boxed{\frac{1}{2} M_{\rm pl}^2} \sum_{\lambda=+,\times} \left(\frac{1}{2} \dot{h}_\lambda^2 - \frac{(\nabla h_\lambda)^2}{2a^2} \right)$$

unwanted pre-factor

“Canonically-normalised” mode function

$$I_{GR}^{(2)} = \int a^3 d^4x \, \frac{1}{4} M_{\rm pl}^2 \left(\frac{1}{2} \dot{h}_{ij}^2 - \frac{(\nabla h_{ij})^2}{2a^2} \right)$$

$$= \int \underbrace{\boxed{a^3} d^4x \boxed{\frac{1}{2} M_{\rm pl}^2}}_{\text{unwanted pre-factor}} \sum_{\lambda=+,\times} \left(\frac{1}{2} \dot{h}_{\lambda}^2 - \frac{(\nabla h_{\lambda})^2}{2a^2} \right)$$

- Two tricks again:
  - (1) Use the conformal time: $a^3 d^4x = a^4 d\eta d^3x$
  - (2) Define: $u_\lambda = \frac{M_{\rm pl}}{\sqrt{2}} a h_\lambda$

“Canonically-normalised” mode function

$$I_{GR}^{(2)} = \int a^3 d^4x \, \frac{1}{4} M_{\rm pl}^2 \left(\frac{1}{2} \dot{h}_{ij}^2 - \frac{(\nabla h_{ij})^2}{2a^2} \right)$$

$$= \int d\eta d^3x \sum_{\lambda=+,\times} \left(\frac{1}{2} {u'_\lambda}^2 - \frac{1}{2} (\nabla u_\lambda)^2 + \frac{a''}{2a} u_\lambda^2 \right)$$

- Two tricks again:
  - (1) Use the conformal time: $a^3 d^4x = a^4 d\eta d^3x$
  - (2) Define: $u_\lambda = \frac{M_{\rm pl}}{\sqrt{2}} a h_\lambda$

This is the correct (“canonical”) normalisation!

GW from inflation

$$u_{ij}'' + \left(k^2 - \frac{2}{\eta^2}\right) u_{ij} = 0$$

- The solution is

$$u_{ij} = A_{ij}\left[\cos(k\eta) - \frac{\sin(k\eta)}{k\eta}\right] + B_{ij}\left[\frac{\cos(k\eta)}{k\eta} + \sin(k\eta)\right]$$

- How do we fix the integration constants, A_{ij} and B_{ij}? **We need QM!**
  - We find A_{ij} and B_{ij}, such that the u_{ij} coincides with the known flat-space (Minkowski) results for the quantum fluctuation in vacuum

GW from inflation

$$u_\lambda'' + \left(k^2 - \frac{2}{\eta^2}\right) u_\lambda = 0$$

- The solution is

$$u_\lambda = A_\lambda \left[\cos(k\eta) - \frac{\sin(k\eta)}{k\eta}\right] + B_\lambda \left[\frac{\cos(k\eta)}{k\eta} + \sin(k\eta)\right]$$

- In the very short wavelength limit, kη->∞, we want to reproduce the quantum field theory result in the flat (Minkowski) space, which is

$$u_\lambda \to \frac{\exp(-ik\eta)}{\sqrt{2k}}$$

GW from inflation

$$u_\lambda'' + \left(k^2 - \frac{2}{\eta^2} \right) u_\lambda = 0$$

- The solution is

$$u_\lambda = \underset{\substack{\Big\downarrow \\ (2k)^{-1/2}}}{A_\lambda} \left[\cos(k\eta) - \frac{\sin(k\eta)}{k\eta} \right] + \underset{\substack{\Big\downarrow \\ -i(2k)^{-1/2}}}{B_\lambda} \left[\frac{\cos(k\eta)}{k\eta} + \sin(k\eta) \right]$$

- In the very short wavelength limit, kη->∞, we want to reproduce the quantum field theory result in the flat (Minkowski) space, which is

$$u_\lambda \to \frac{\exp(-ik\eta)}{\sqrt{2k}}$$

GW from inflation

$$u_\lambda'' + \left(k^2 - \frac{2}{\eta^2}\right) u_\lambda = 0$$

- The solution is

$$u_\lambda = \frac{1}{\sqrt{2k}} \left(e^{-ik\eta} - \frac{i}{k\eta} e^{-ik\eta}\right)$$

This term dominates in the super-horizon mode!
"Particle Production by Inflation"

GW from inflation

$$u_\lambda'' + \left(k^2 - \frac{2}{\eta^2}\right) u_\lambda = 0$$

- The **super-horizon** solution is

$$u_\lambda \to -\frac{i}{\sqrt{2k^3\eta}} e^{-ik\eta}$$

GW from inflation

$$u_\lambda'' + \left(k^2 - \frac{2}{\eta^2}\right) u_\lambda = 0$$

- The **super-horizon** solution is

$$u_\lambda \to -\frac{i}{\sqrt{2k^3\eta}} e^{-ik\eta}$$

Since $u_\lambda = \frac{M_{\rm pl}}{\sqrt{2}} a h_\lambda$ **and** $a(\eta) = -(H\eta)^{-1}$**…**

GW from inflation

$$u_\lambda'' + \left(k^2 - \frac{2}{\eta^2}\right) u_\lambda = 0$$

- The **super-horizon** solution is

$$h_\lambda \to \frac{i\,H}{\sqrt{k^3} M_{\rm pl}} e^{-ik\eta}$$

GW from inflation

$$u''_\lambda + \left(k^2 - \frac{2}{\eta^2}\right) u_\lambda = 0$$

- The **super-horizon** solution is

$$h_\lambda \to \frac{i\boxed{H}}{\sqrt{k^3}M_{\rm pl}} e^{-ik\eta}$$

The amplitude of GW on super-horizon scale is proportional to H!

Quantum fluctuations during inflation are proportional to H

- **THE KEY RESULT**: The earlier the fluctuations are generated, the more its wavelength is stretched, and thus the bigger the angles they subtend in the sky.

- **We can map H(t) by measuring fluctuations over a wide range of wavelengths**

  - Earlier time -> Larger angular scales

  - Late time -> Smaller angular scales

Total Variance of GW

$$\sum_{ij}\langle h_{ij}(\mathbf{x})h_{ij}(\mathbf{x})\rangle = \sum_{ij}\int \frac{d^3k}{(2\pi)^3}\ \langle h_{ij}(\mathbf{k})h^*_{ij}(\mathbf{k})\rangle$$

$$= 2\sum_{\lambda=+,\times}\int \frac{d^3k}{(2\pi)^3}\ \langle |h_\lambda(\mathbf{k})|^2\rangle$$

$$\boxed{h_\lambda \to \frac{iH}{\sqrt{k^3}M_{\rm pl}}e^{-ik\eta}}$$

$$= 4\int \frac{4\pi k^2 dk}{(2\pi)^3}\ \frac{H^2}{k^3 M_{\rm pl}^2}$$

$$= 4\int \frac{dk}{k}\ \frac{H^2}{2\pi^2 M_{\rm pl}^2}$$

$$= \int \frac{dk}{k}\ \frac{8}{M_{\rm pl}^2}\left(\frac{H}{2\pi}\right)^2$$

GW variance per log(k)

$$\frac{k^3}{2\pi^2}\sum_{ij}\langle h_{ij}h^*_{ij}\rangle = \frac{8}{M^2_{\rm pl}}\left(\frac{H}{2\pi}\right)^2$$

- Variance per log(k) depends only on H; thus,
  - It is scale-invariant if H is constant during inflation; or
  - It is **nearly** scale-invariant if H changes slowly during inflation
- In general, H is a decreasing function of time; thus,
  - **The variance of GW is smaller at shorter wavelengths.** This is the key prediction of GW from the *vacuum fluctuation* during inflation

Energy Density of GW

$$\rho_{\rm GW}(t) = \frac{1}{4} M_{\rm pl}^2 \sum_{ij} \langle \dot{h}_{ij}(t, \mathbf{x}) \dot{h}_{ij}(t, \mathbf{x}) \rangle$$

$$= \frac{1}{2} M_{\rm pl}^2 \sum_{\lambda=+,\times} \langle \dot{h}_{\lambda}^2(t, \mathbf{x}) \rangle$$

Solution of EoM

$$\dot{h}_{ij} \propto a^{-2}(t) \Longrightarrow \rho_{\rm GW}(t) \propto a^{-4}(t)$$

As expected, because GW is radiation

- During the radiation era,

$$\Omega_{\rm GW}(t) = \rho_{\rm GW}(t)/\rho_{\rm total}(t) = \text{constant}$$

- During the matter era,

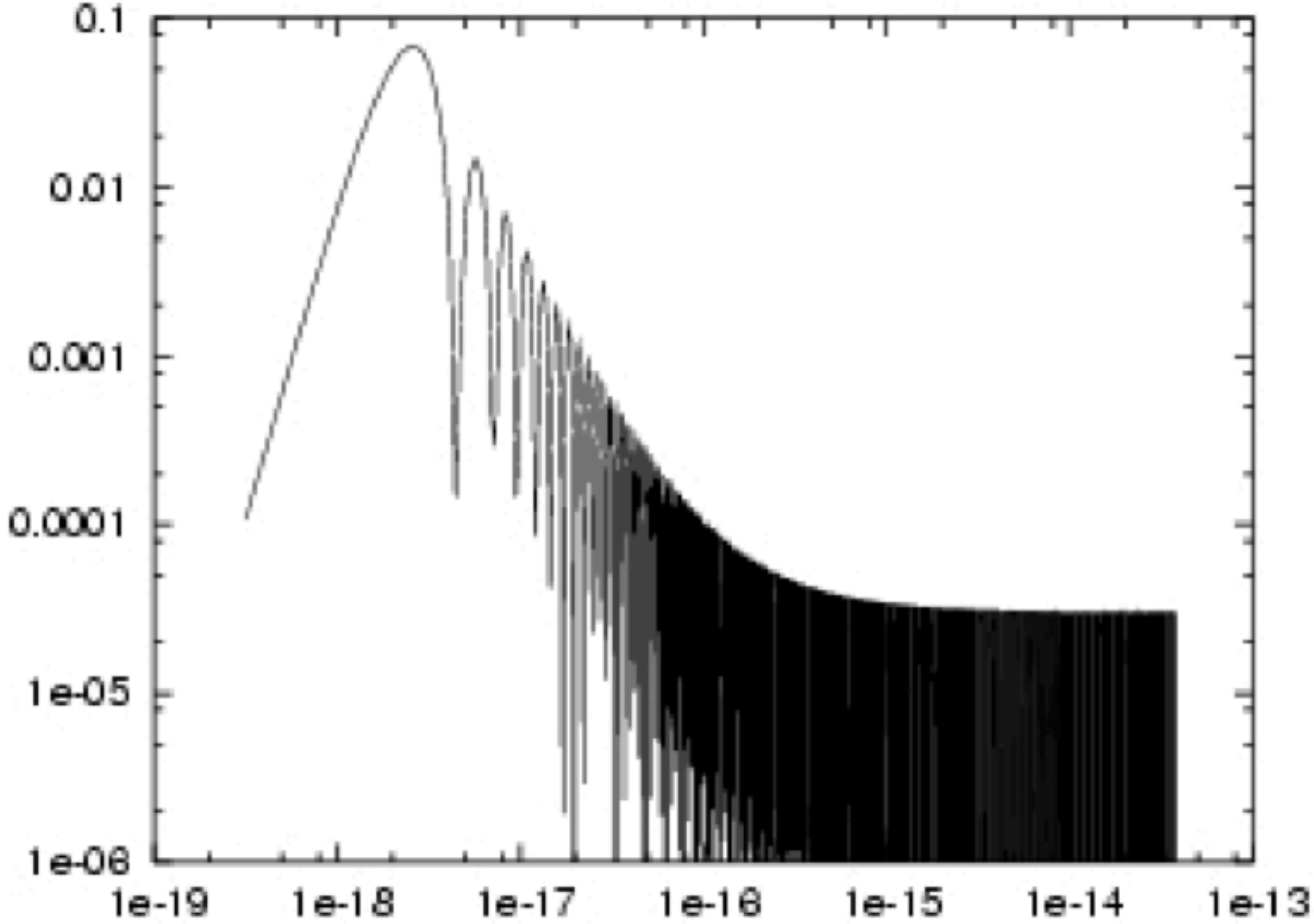
$$\Omega_{\rm GW}(t) = \rho_{\rm GW}(t)/\rho_{\rm total}(t) = a^{-1}(t)$$

Watanabe & EK (2006)

Theoretical energy density

Spectrum of GW today

$$d\Omega_{\rm GW}(t_0)/d\ln k$$



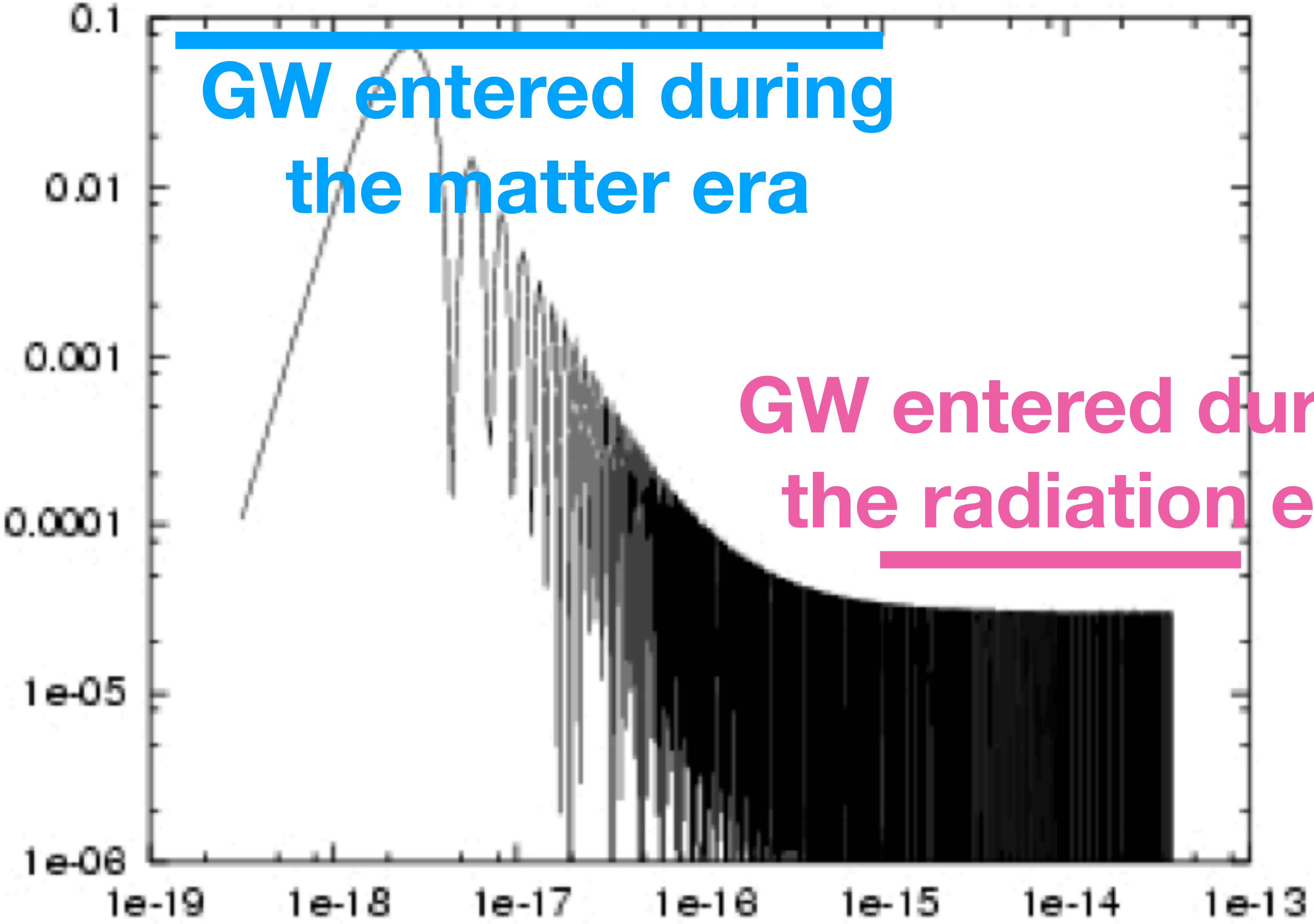
Frequency, f=kc [Hz]

Watanabe & EK (2006)

Theoretical energy density

Spectrum of GW today

$$d\Omega_{\rm GW}(t_0)/d\ln k$$



GW entered during the matter era

GW entered during the radiation era

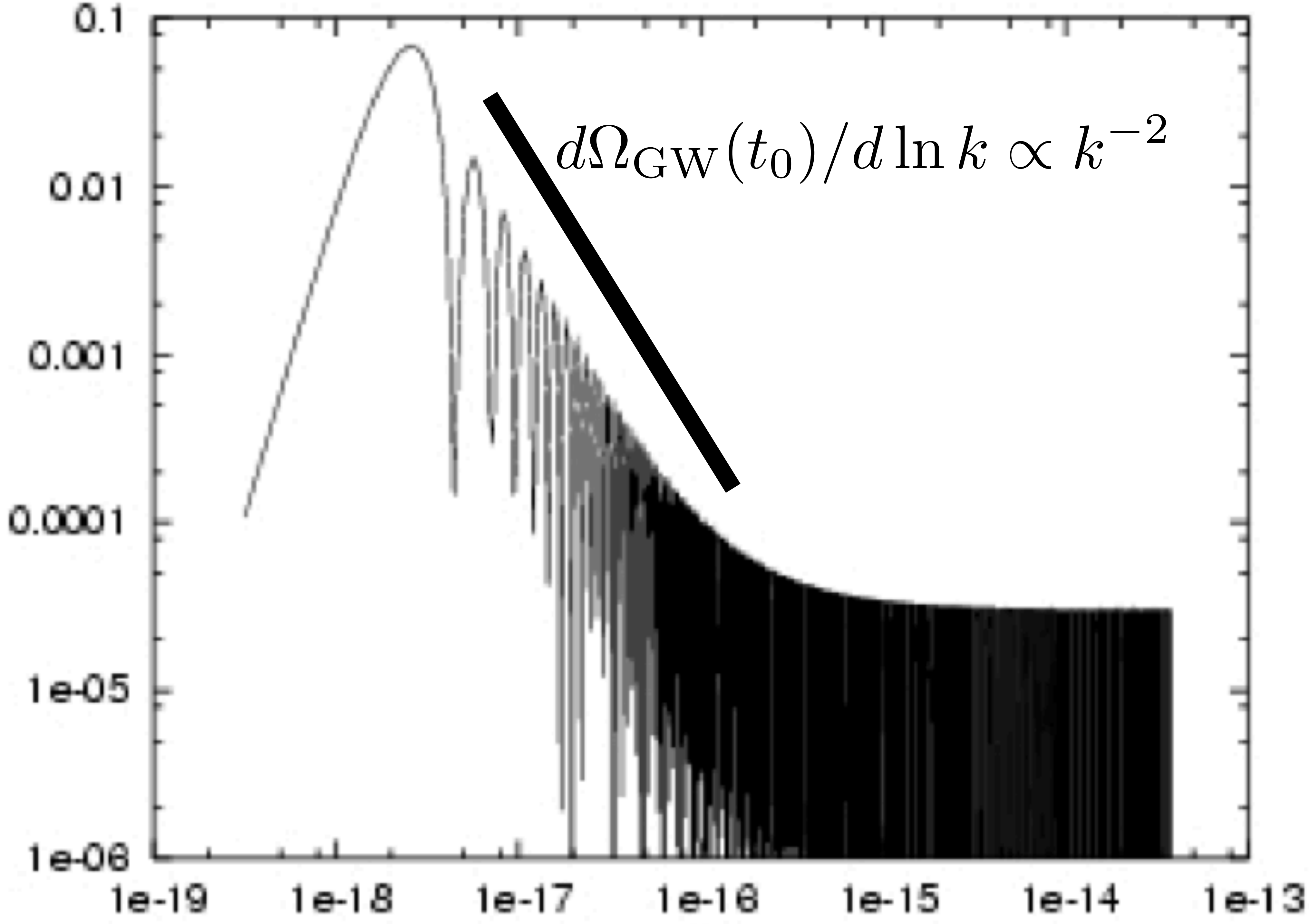
Frequency, f=kc [Hz]

Watanabe & EK (2006)

Theoretical energy density

Spectrum of GW today

$$d\Omega_{\rm GW}(t_0)/d\ln k$$



Frequency, f=kc [Hz]

Why k^{-2} ?

$$\Omega_{\rm GW}(t_0) = \Omega_{\rm GW}(t_{\rm horizon}) \frac{a(t_{\rm horizon})}{a(t_0)}$$ **during the matter era**

with

$$\begin{aligned} k &= a(t_{\rm horizon}) H(t_{\rm horizon}) \\ &\propto a a^{-3/2} \\ &\propto a^{-1/2}(t_{\rm horizon}) \end{aligned}$$

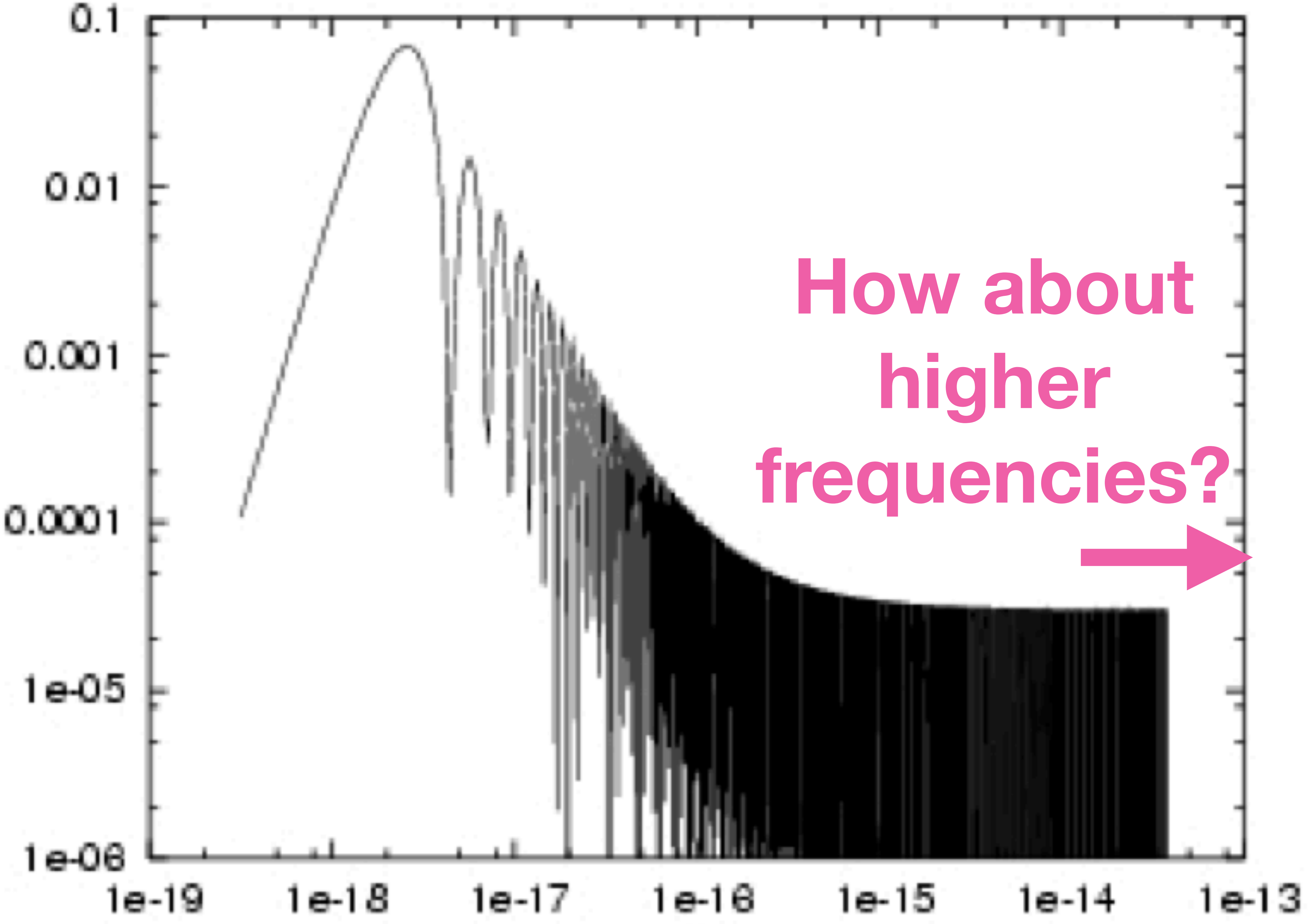
- Therefore,

$$\Omega_{\rm GW}(t_0) \propto \Omega_{\rm GW}(t_{\rm horizon}) k^{-2}$$

Watanabe & EK (2006)

Theoretical energy density

Spectrum of GW today

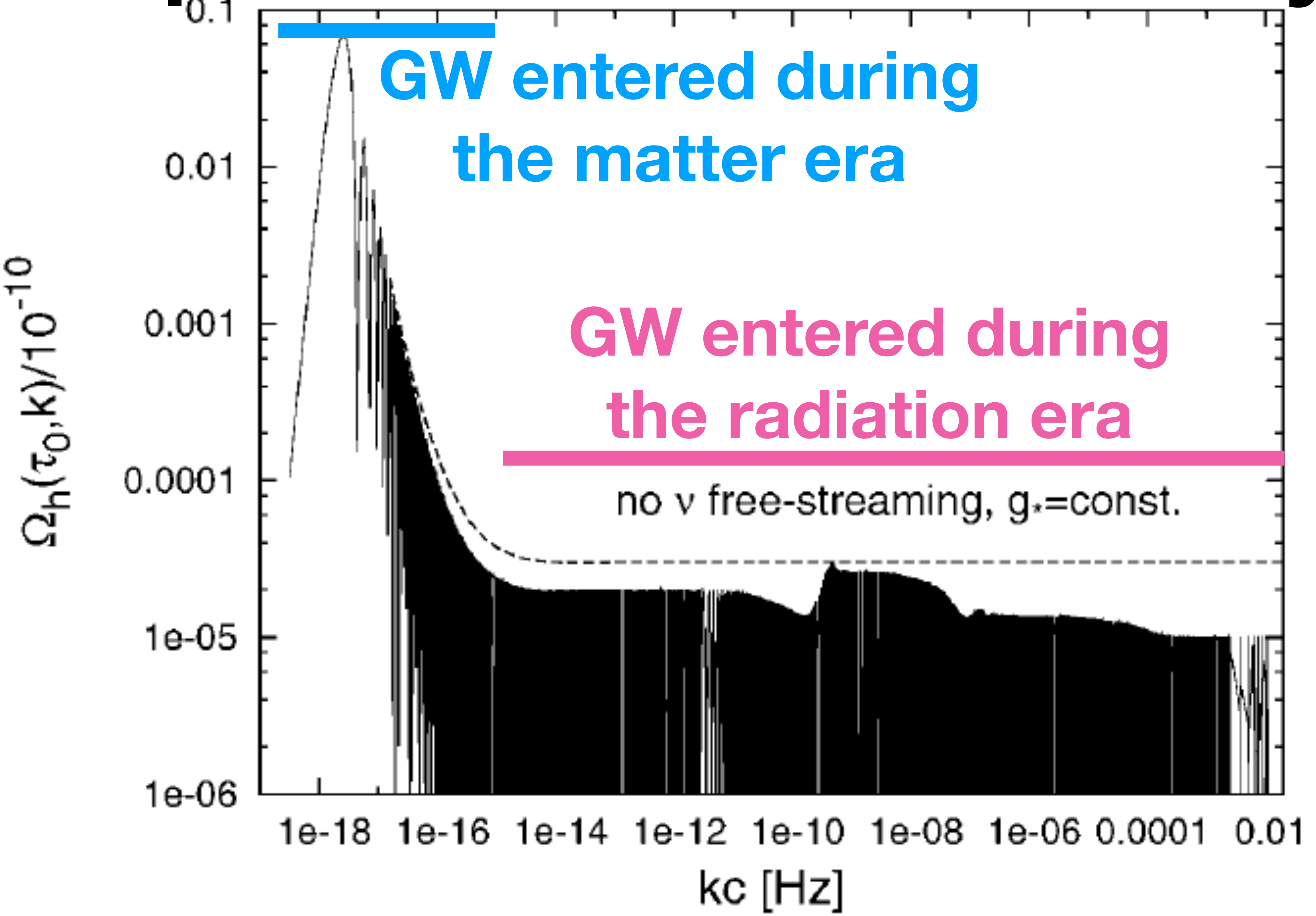


How about higher frequencies?

Watanabe & EK (2006)

Theoretical energy density

Spectrum of GW today



GW entered during the matter era

GW entered during the radiation era

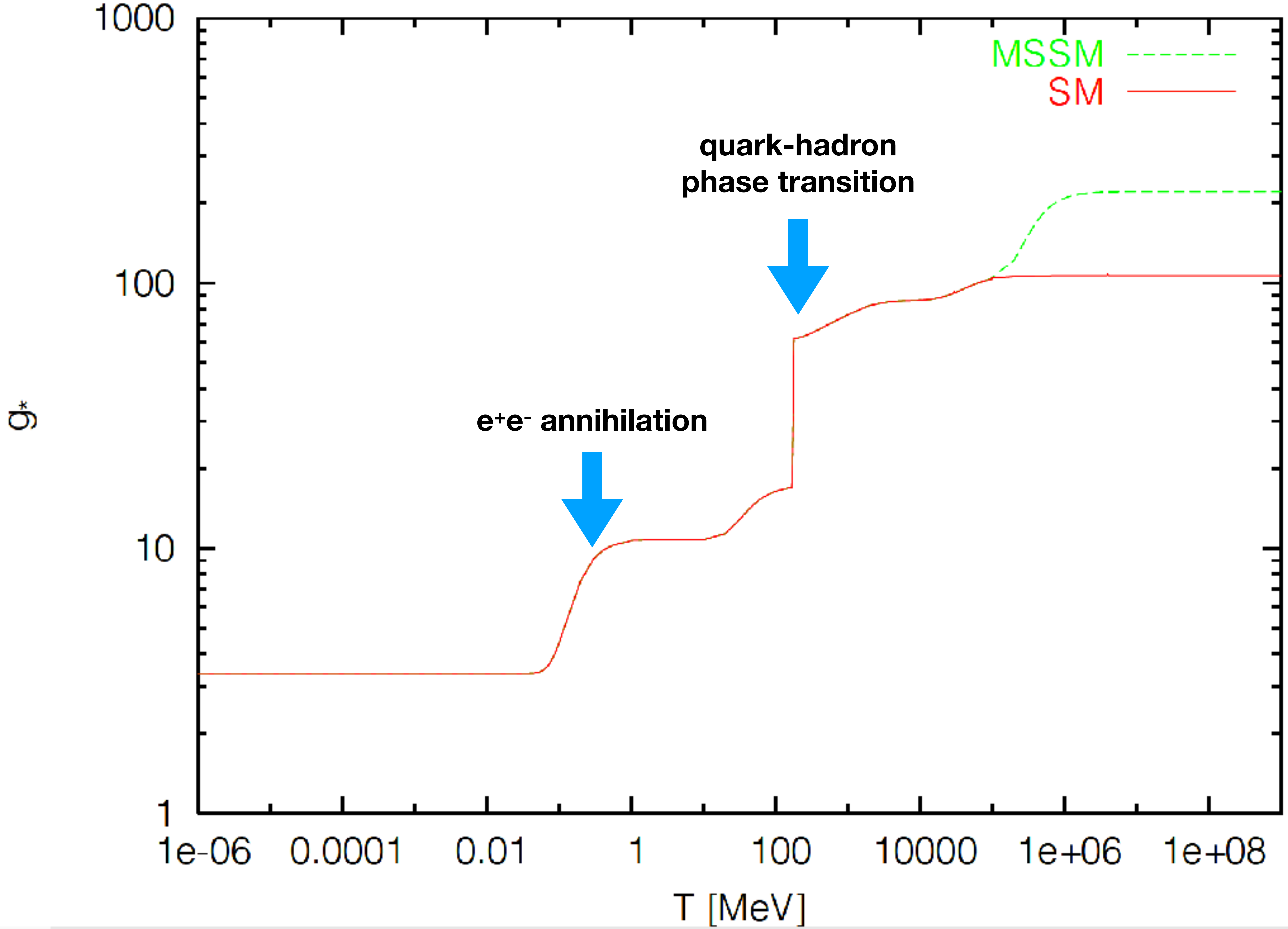
Kolb & Turner, "The Early Universe"
Sec. 3.3 & 3.4

Evolution of Radiation Density

- You might have learned in the cosmology class that the radiation density redshifts as $\rho_{\mathrm{radiation}} \propto a^{-4}$
- This is true only when the radiation content (relativistic degrees of freedom) does not change
- The correct formulae:
  - Entropy conservation: $S = sa^3 \propto g_{*s} T^3 a^3 = \mathrm{constant}$
  - Then the radiation density redshifts as

$$\rho_{\mathrm{radiation}} \propto g_* T^4 \propto g_* g_{*s}^{-4/3} a^{-4}$$

Watanabe & EK (2006)



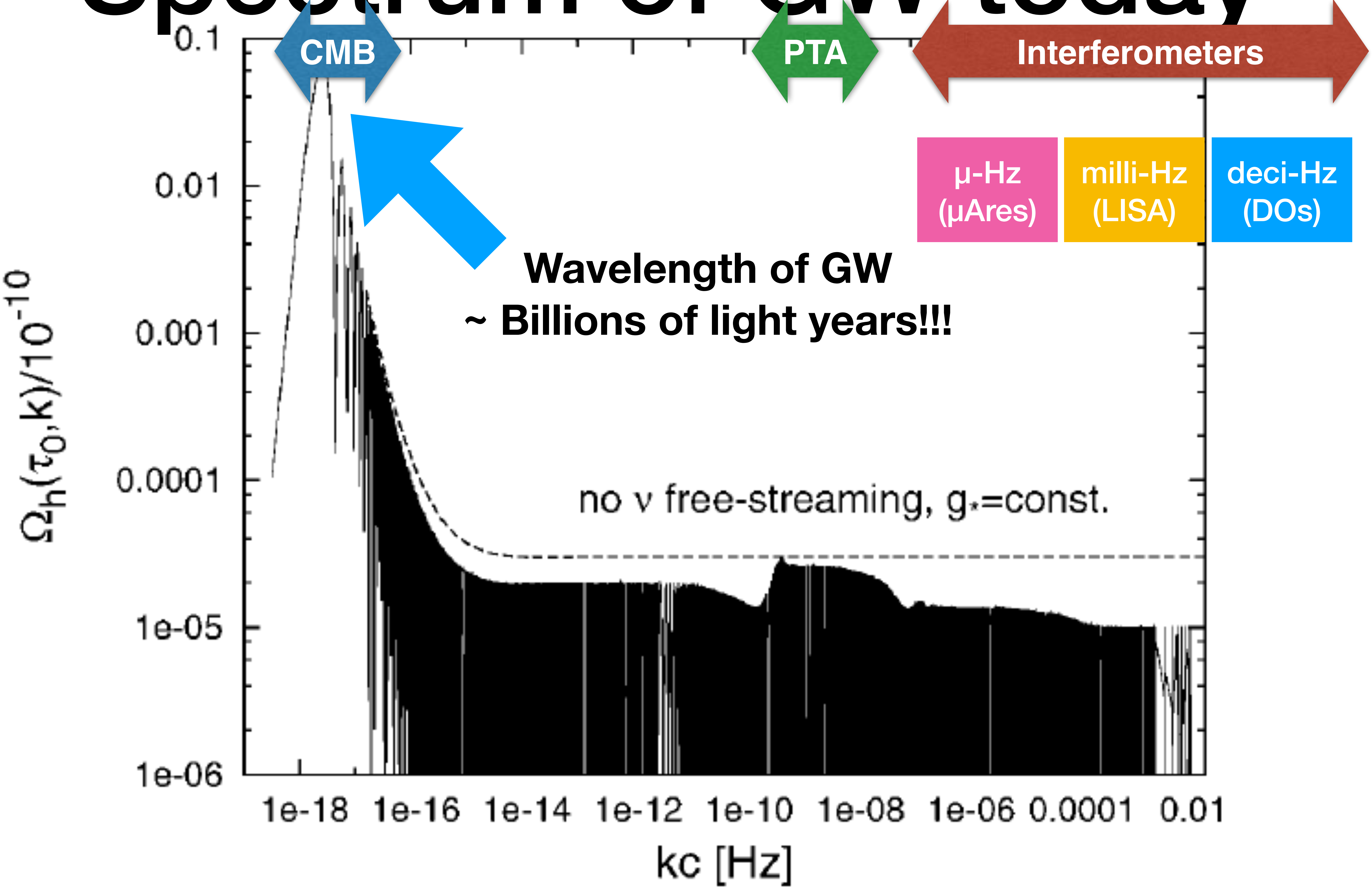
quark-hadron phase transition

e^+e^- annihilation

Watanabe & EK (2006)

Theoretical energy density

Spectrum of GW today



Wavelength of GW
~ Billions of light years!!!

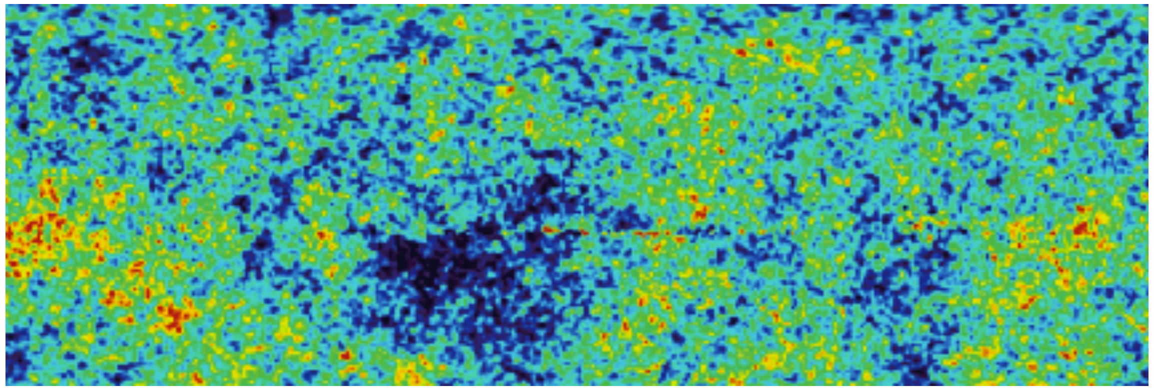
Appendix:
Scalar Perturbation

Inflationary Predictions

ζ

scalar mode

- Fluctuations we observe today in CMB and the matter distribution originate from quantum fluctuations during inflation

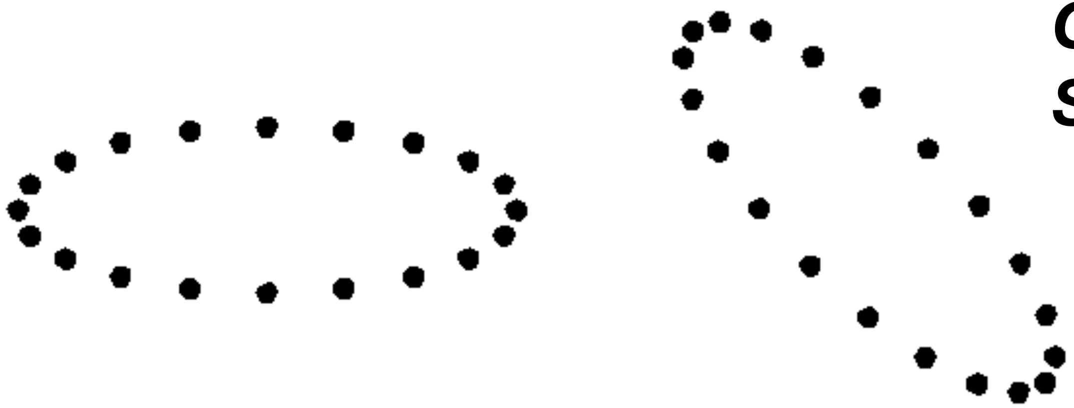


Mukhanov&Chibisov (1981)
Guth & Pi (1982)
Hawking (1982)
Starobinsky (1982)
Bardeen, Steinhardt&Turner (1983)

h_{ij}

tensor mode

- There should also be *ultra long-wavelength* gravitational waves generated during inflation



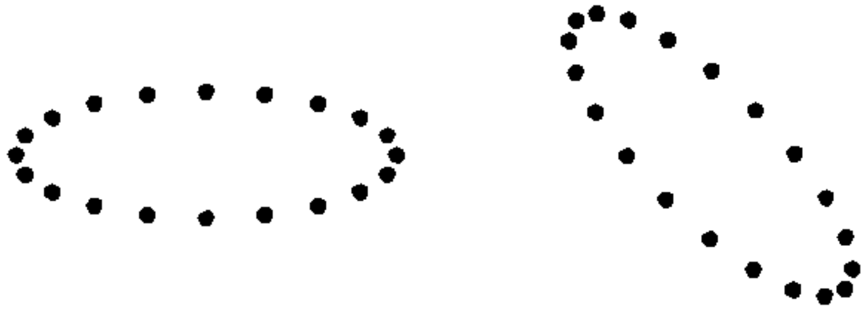
Grishchuk (1974)
Starobinsky (1979)

We measure distortions in space

- A distance between two points in space

$$d\ell^2 = a^2(t)[1+2\zeta(\mathbf{x},t)][\delta_{ij}+h_{ij}(\mathbf{x},t)]dx^i dx^j$$

- **ζ** : “curvature perturbation” (scalar mode)
  - Perturbation to the determinant of the spatial metric
- **h_{ij}** : “gravitational waves” (tensor mode)
  - Perturbation that does not alter the determinant



$$\sum_i h_{ii} = 0$$

Second-order Action for h_{ij}

- The action that gives Einstein's field equations is the so-called "Einstein-Hilbert action", given by the Ricci scalar R:

$$I_{GR} = \int \sqrt{-g} d^4x \left(\frac{1}{2} M_{\rm pl}^2 R \right)$$

with

$$M_{\rm pl} = (8\pi G)^{-1/2}$$

$$\sqrt{-g} = a^3$$

- Expanding this to second-order in h_{ij}, we obtain the action that gives the equation of motion for h_{ij}:

$$I_{GR}^{(2)} = \int a^3 d^4x \ \frac{1}{4} M_{\rm pl}^2 \left(\frac{1}{2} \dot{h}_{ij}^2 - \frac{(\nabla h_{ij})^2}{2a^2} \right)$$

with $h_{ij} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$= \int a^3 d^4x \ \frac{1}{2} M_{\rm pl}^2 \sum_{\lambda=+,\times} \left(\frac{1}{2} \dot{h}_\lambda^2 - \frac{(\nabla h_\lambda)^2}{2a^2} \right)$$

Second-order Action for ζ

- The action that gives Einstein's field equations is the so-called "Einstein-Hilbert action", given by the Ricci scalar R:

$$I_{GR} = \int \sqrt{-g} d^4 x \left(\frac{1}{2} M_{\rm pl}^2 R \right)$$

with

$$M_{\rm pl} = (8\pi G)^{-1/2}$$

$$\sqrt{-g} = a^3$$

- Expanding this to second-order in ζ, we obtain the action that gives the equation of motion for ζ:

$$I_{GR}^{(2)} = \int a^3 d^4 x \; 2\epsilon M_{\rm pl}^2 \left(\frac{1}{2} \dot{\zeta}^2 - \frac{(\nabla\zeta)^2}{2a^2} \right)$$

with

$$\epsilon \equiv -\frac{\dot{H}}{H^2} \ll 1$$

Getting this result is not as easy as it may look. See the steps leading to Eq.(2.12) of Maldacena, JHEP 0305 (2003) 013, astro-ph/0210603

Second-order Action for ζ

- The action that gives Einstein’s field equations is the so-called “Einstein-Hilbert action”, given by the Ricci scalar R:

$$I_{GR} = \int \sqrt{-g}d^4x \left(\frac{1}{2}M_{\rm pl}^2 R\right)$$

with

$$M_{\rm pl} = (8\pi G)^{-1/2}$$

$$\sqrt{-g} = a^3$$

- Expanding this to second-order in **ζ**, we obtain the action that gives the equation of motion for **ζ**:

$$I_{GR}^{(2)} = \int \boxed{a^3} d^4x \boxed{2\epsilon M_{\rm pl}^2} \left(\frac{1}{2}\dot{\zeta}^2 - \frac{(\nabla\zeta)^2}{2a^2}\right)$$

unwanted pre-factor

with

$$\epsilon \equiv -\frac{\dot{H}}{H^2} \ll 1$$

Getting this result is not as easy as it may look. See the steps leading to Eq.(2.12) of Maldacena, JHEP 0305 (2003) 013, astro-ph/0210603

Canonically-normalised mode function

$$I_{GR}^{(2)} = \int \boxed{a^3} d^4x \, \boxed{2\epsilon M_{\rm pl}^2} \left(\frac{1}{2}\dot{\zeta}^{\,2} - \frac{(\nabla\zeta)^2}{2a^2} \right)$$

unwanted pre-factor

$$= \int d\eta d^3x \left(\frac{1}{2}u'^{\,2} - \frac{1}{2}(\nabla u)^2 + \frac{a''}{2a}u^2 \right)$$

- Two tricks again:
  - (1) Use the conformal time: $a^3 d^4x = a^4 d\eta d^3x$
  - (2) Define: $u = \sqrt{2\epsilon} M_{\rm pl} a \zeta$ **This is the correct ("canonical") normalisation!**

The rest follows as before!

$$u'' + \left(k^2 - \frac{2}{\eta^2}\right) u \quad = 0$$

- The **super-horizon** solution is

$$u \quad \rightarrow -\frac{i}{\sqrt{2k^3}\eta} e^{-ik\eta}$$

Since $u = \sqrt{2\epsilon} M_{\rm pl} a \zeta$ **and** $a(\eta) = -(H\eta)^{-1}$**…**

The rest follows as before!

$$u'' + \left(k^2 - \frac{2}{\eta^2}\right)u \quad = 0$$

- The **super-horizon** solution is

$$\zeta \to \frac{i\boxed{H}}{\sqrt{4k^3\boxed{\epsilon}}M_{\rm pl}}e^{-ik\eta}$$

The amplitude of ζ on super-horizon scale is proportional to $H/\sqrt{\varepsilon}$!

Variance of ζ

$$\boxed{\zeta \to \frac{iH}{\sqrt{4k^3\epsilon M_{\rm pl}}} e^{-ik\eta}}$$

$$\begin{aligned}\langle\zeta^2(\mathbf{x})\rangle &= \int \frac{d^3k}{(2\pi)^3}\, \langle|\zeta(\mathbf{k})|^2\rangle \\ &= \int \frac{4\pi k^2 dk}{(2\pi)^3}\, \frac{H^2}{4k^3\epsilon M_{\rm pl}^2} \\ &= \int \frac{dk}{k}\, \frac{1}{2\epsilon M_{\rm pl}^2} \left(\frac{H}{2\pi}\right)^2\end{aligned}$$

- In general, H is a decreasing function of time; thus,
  - **The variance of ζ is smaller at shorter wavelengths. This has been measured from the CMB data! (Colloquium on March 5)**

Important milestone of cosmology in 2012–2013

Variance of ζ

$$\langle \zeta^2(\mathbf{x}) \rangle = \int \frac{d^3k}{(2\pi)^3} \langle |\zeta(\mathbf{k})|^2 \rangle$$

$$\boxed{\zeta \to \frac{iH}{\sqrt{4k^3\epsilon}M_{\rm pl}} e^{-ik\eta}}$$

$$= \int \frac{4\pi k^2 dk}{(2\pi)^3} \frac{H^2}{4k^3\epsilon M_{\rm pl}^2}$$

$$= \int \frac{dk}{k} \frac{1}{2\epsilon M_{\rm pl}^2} \left(\frac{H}{2\pi}\right)^2$$

Compare this with GW:

$$\sum_{ij} \int \frac{d^3k}{(2\pi)^3} \langle h_{ij}(\mathbf{k}) h^*_{ij}(\mathbf{k}) \rangle = \int \frac{dk}{k} \frac{8}{M_{\rm pl}^2} \left(\frac{H}{2\pi}\right)^2$$

Tensor-to-scalar Ratio

$$r \equiv \frac{\langle h_{ij} h^{ij} \rangle}{\langle \zeta^2 \rangle} = 16\epsilon$$

with

$$\epsilon \equiv -\frac{\dot{H}}{H^2} \ll 1$$

Super famous result for the vacuum fluctuation, which does not necessarily hold for the sourced contribution! (the topic on March 19)

Tensor-to-scalar Ratio

$$r \equiv \frac{\langle h_{ij} h^{ij} \rangle}{\langle \zeta^2 \rangle} = 16\epsilon$$

with $\epsilon \equiv -\frac{\dot{H}}{H^2} \ll 1$

ε << 1 is observationally shown already

- We really want to find this! The current upper bound is **r<0.06** (95%CL)

BICEP2/Keck Array Collaboration (2018)