non-Gaussianity measurement with Information Field Theory

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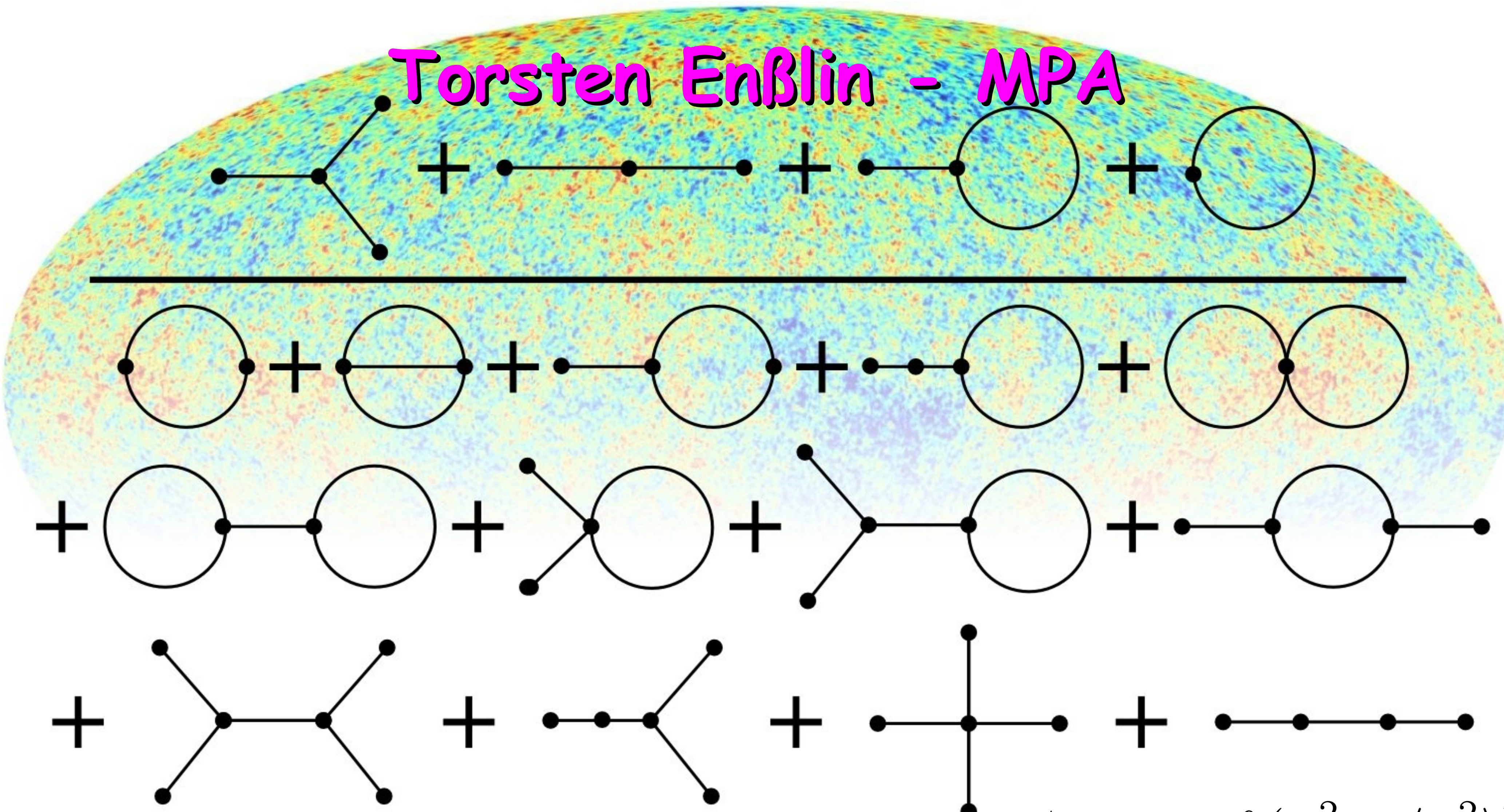


$$\phi = \varphi + f\left(\varphi^2 - \langle \varphi^2 \rangle\right)$$

Non-Gaussianity in Garching, 2012-11-06

Information Theory

$s = signal$
$d = data$

posterior

likelihood

prior

$$P(s|d) = \frac{P(d|s)\,P(s)}{P(d)}$$

evidence

inference problem as information field theory

Free Theory

Gaussian signal & noise, linear response

signal :

$$P(s) = \mathcal{G}(s,S) = \frac{1}{|2\pi\, S|^{1/2}} \exp\left(-\frac{1}{2}\, s^\dagger S^{-1} s\right)$$

$$j^\dagger s = \int dx\, j^*(x) s(x)$$

$$S = \langle s\, s^\dagger \rangle_{(s)}$$

data :

$$d = R\, s + n,\ P(d|s) = P(n = d - R\, s)$$

noise :

$$P(n) = \mathcal{G}(n,N),\ N = \langle n\, n^\dagger \rangle_{(n)}$$

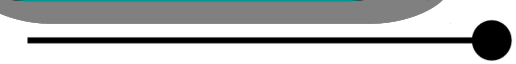
Wiener filter theory

known for 60 years

$$
\begin{aligned}
H(s) &= -\log P(d,s) = -\log P(d|s) - \log P(s) \\
&= \frac{1}{2}(d - R\,s)^\dagger N^{-1}(d - R\,s) + \frac{1}{2}\,s^\dagger S^{-1} s + \text{const} \\
&= \frac{1}{2}\,s^\dagger \underbrace{\left(S^{-1} + R^\dagger N^{-1} R\right)}_{\equiv D^{-1}} s + s^\dagger \underbrace{R^\dagger N^{-1} d}_{\equiv j} + \text{const} \\
&= \frac{1}{2}\,s^\dagger D^{-1}\,s + s^\dagger j + H_0
\end{aligned}
$$

information source

information propagator

mean: $m = \langle s \rangle_{(s|d)} = D\,j =$ 

uncertainty: $\langle (s-m)\,(s-m)^\dagger \rangle_{(s|d)} = D$

Interacting Theory

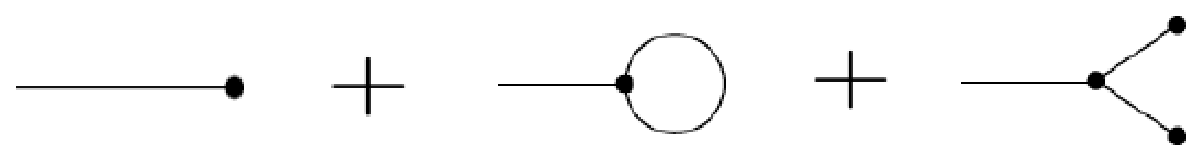
non-Gaussian signal, noise, or non-linear response

$$H[s] = \frac{1}{2}\, s^\dagger D^{-1}\, s - j^\dagger s + H_0 + \sum_{n=3}^{\infty} \frac{1}{n!}\, \Lambda^{(n)}_{x_1 \ldots x_n}\, s_{x_1} \cdots s_{x_n}$$

Taylor-Fréchet expansion of Hamiltonian

Use expansion into Feynman diagrams

$$\langle s \rangle_{(s|d)} =$$



$+ \quad \ldots$

$$= D_{xy}\, j_y \; - \frac{1}{2} D_{xy}\, \Lambda^{(3)}_{yzu}\, D_{zu}$$

$$- \frac{1}{2} D_{xy}\, \Lambda^{(3)}_{yuz}\, D_{zz'}\, j_{z'}\, D_{uu'}\, j_u \quad + \quad \ldots$$

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Information field theory for cosmological perturbation reconstruction and nonlinear signal analysis

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$$\phi \hookleftarrow P(\phi) = \mathcal{G}(\phi, \Phi) \qquad \Phi = \langle \phi \phi^\dagger \rangle_{(\phi)}$$

$$\varphi(x) = \phi(x) + f_{\mathrm{nl}}(\phi^2(x) - \langle \phi^2(x) \rangle_{(\phi)})$$

$$d \equiv \delta T^{\{\mathrm{I,E}\}}_{\mathrm{obs}} / T_{\mathrm{CMB}} = R\varphi + n$$

$$d = R(\phi + f(\phi^2 - \hat{\Phi})) + n \qquad N = \langle n n^\dagger \rangle_{(n)} \qquad P(s|d) = \frac{P(d|s)\,P(s)}{P(d)} \equiv \frac{1}{Z}\, e^{-H[s]}$$

$$H_f[d, \phi] = -\log(\mathcal{G}(\phi, \Phi)\mathcal{G}(d - R(\phi + f(\phi^2 - \hat{\Phi})), N)) \qquad M = R^\dagger N^{-1} R$$

$$= \frac{1}{2}\phi^\dagger D^{-1}\phi + H_0 - j^\dagger \phi + \sum_{n=0}^{4} \frac{1}{n!}\Lambda^{(n)}[\phi, \dots, \phi], \quad \text{with} \quad D^{-1} = \Phi^{-1} + R^\dagger N^{-1} R \equiv \Phi^{-1} + M,$$

$$j = R^\dagger N^{-1} d, \qquad \Lambda^{(0)} = j^\dagger (f\hat{\Phi}) + \frac{1}{2}(f\hat{\Phi})^\dagger M (f\hat{\Phi}), \qquad \Lambda^{(1)} = -(f\hat{\Phi})^\dagger M \quad \text{and} \quad j' = j - \Lambda^{(1)\dagger},$$

$$\Lambda^{(2)} = -2\widehat{f j'}, \qquad \Lambda^{(3)}_{xyz} = (M_{xy} f_y \delta_{yz} + 5 \text{ permutations}), \qquad \Lambda^{(4)}_{xyzu} = \frac{1}{2}(f_x \delta_{xy} M_{yz} \delta_{zu} f_u + 23 \text{ permutations}),$$

$$m_0 = Dj.$$

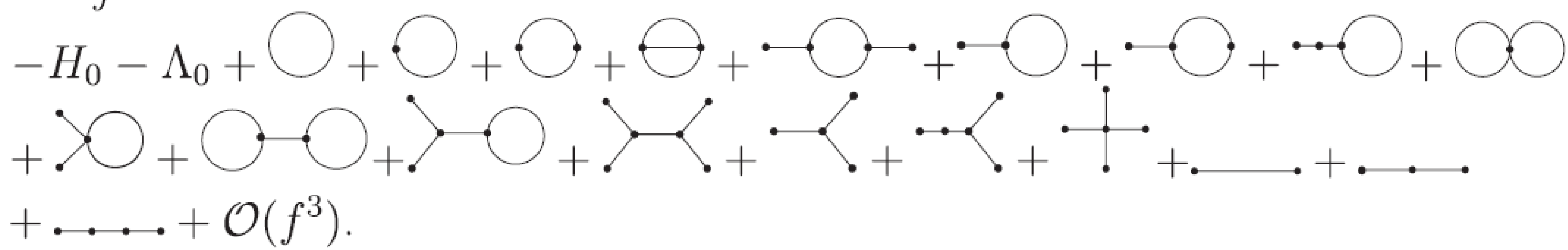
Feynman expansion

$\log P(d|f)$ =

$$\log Z_f[d] = \log \int \mathcal{D}\phi \, P(d,\phi|f)$$

$$= \log \int \mathcal{D}\phi \, e^{-H_f[d,\phi]}$$

$$= -H_0 - \Lambda_0 + \ldots$$



$$+ \mathcal{O}(f^3).$$

$$H_d[f] \equiv -\log(P(d|f)P(f))$$

$$= \tilde{H}_0 + \tfrac{1}{2} f^\dagger \tilde{D}^{-1} f + \tilde{j}^\dagger f + \mathcal{O}(f^3)$$

$$\tilde{j}_{(lm)} = \left. \frac{dH_d[f]}{df_{lm}} \right|_{f=0}, \quad \text{and}$$

$$\tilde{D}^{-1}_{(lm)(l'm')} = \left. \frac{d^2 H_d[f]}{df_{lm} df_{l'm'}} \right|_{f=0}.$$

$$
\begin{aligned}
H_d[f] &\equiv -\log(P(d|f)P(f)) \\
&= \tilde{H}_0 + \tfrac{1}{2} f^\dagger \tilde{D}^{-1} f + \tilde{j}^\dagger f + \mathcal{O}(f^3)
\end{aligned}
$$

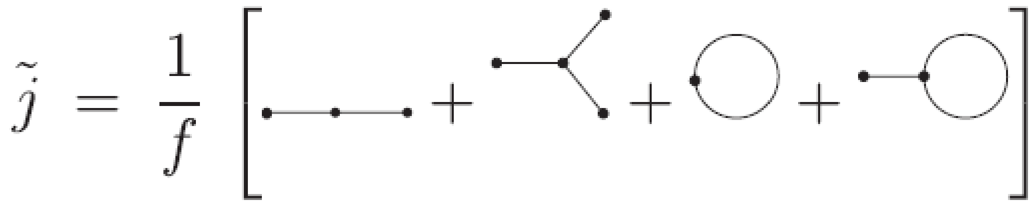
where we collected the linear and quadratic coefficients into $\tilde{j}$ and $\tilde{D}^{-1}$. It is obvious that the optimal f estimator to lowest order is therefore

$$
m_f = \langle f \rangle_{(\phi, f|d)} = \tilde{D}\,\tilde{j}, \tag{135}
$$

and its uncertainty variance is just

$$
\langle (f - m_f)(f - m_f)^\dagger \rangle_{(\phi, f|d)} = \tilde{D}. \tag{136}
$$

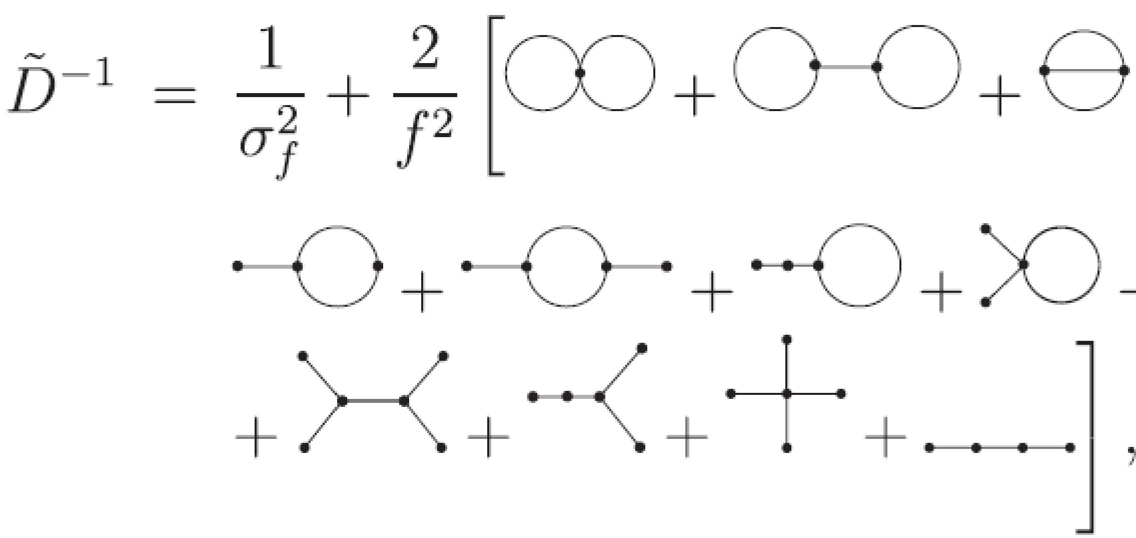
Information source and propagator



$$= m_0^\dagger \Phi^{-1} m_0^2 + m_0^\dagger \left[\Phi^{-1}\widehat{D} - 2\,\widehat{M}\widehat{D}\right]$$

$$M = D^{-1} - \Phi^{-1}$$

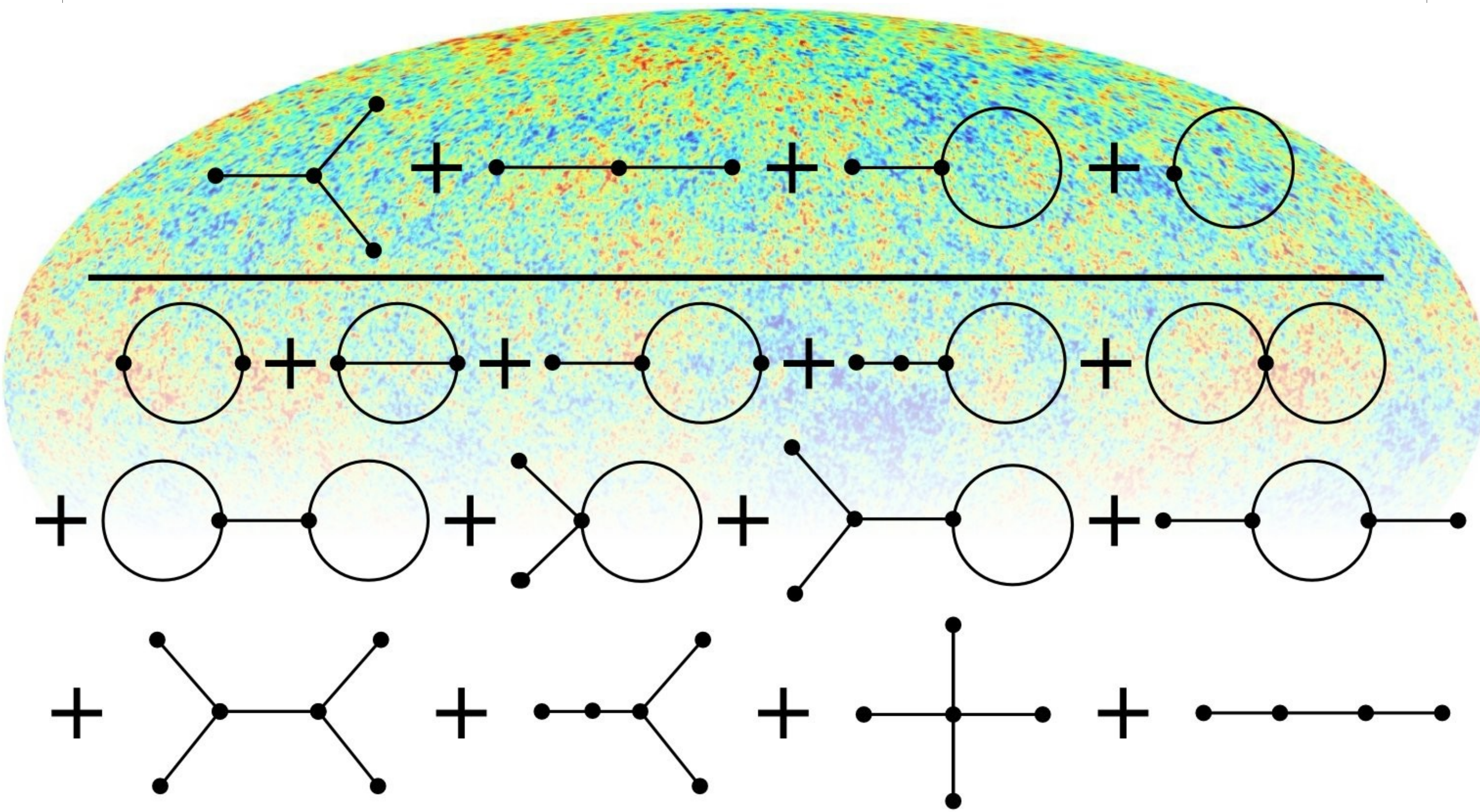
$$m_0 = Dj.$$



(142)

Primordial non-Gaussianity measurement

$$\phi = \varphi + f\left(\varphi^2 - \langle \varphi^2 \rangle\right)$$



$$\tilde{j} = \frac{1}{f}\Big[$$ $+$  $+$  $+$  $\Big]$

$$= m_0^\dagger \Phi^{-1} m_0^2 + m_0^\dagger \left[\Phi^{-1}\widehat{D} - 2\,\widehat{MD}\right]$$

 $= \frac{1}{2}\log|2\pi\, D| = \frac{1}{2}\,\mathrm{Tr}(\log(2\pi\, D))$

 $= -\mathrm{Tr}\left[D^2\, M\right] - \frac{1}{2}\widehat{D}^\dagger M\,\widehat{D},$

 $= \frac{1}{2}\left[2\widehat{DM} + \widehat{D}M\right]^\dagger D\left[2\widehat{MD} + M\widehat{D}\right],$

 $= \mathrm{Tr}\left[D^2\, M\, D\, M\right]$

$$+2M_{xy}D_{yy'}M_{y'x'}D_{x'y}D_{xx'}\,,$$

 $= j^\dagger D^2 j$

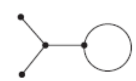
 $= -2\,m^\dagger M\,D^2 j - 4\,\mathrm{Tr}\left[\widehat{m}\,D\,\widehat{j}\,D\,M\right],$

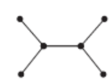
 $= m^\dagger M\,D^2 M\,m + 4\,\mathrm{Tr}\left[\widehat{m}\,D\,\widehat{Mm}\,D\,M\right]$

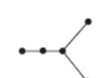
$$+2\,\mathrm{Tr}\left[\widehat{m}\,D\,(\widehat{m}\,M + M\,\widehat{m})\,D\,M\right],$$

 $= -2\left[2\widehat{DM} + \widehat{D}M\right]^\dagger D\,\widehat{j}\,m\,,$

 $= -{m^2}^\dagger M\,\widehat{D} - 2\,\mathrm{Tr}\left[\widehat{m}\,M\,\widehat{m}\,D\right],$

 $= \left[2\widehat{DM} + \widehat{D}M\right]^\dagger D\left[2\widehat{m}Mm + Mm^2\right],$

 $= \frac{1}{2}\left[2\widehat{m}Mm + Mm^2\right]^\dagger D\left[2\widehat{m}Mm + Mm^2\right.$

 $= -2\,(m\,j)^\dagger D\,(M\,m^2 + 2\,\widehat{m}\,M\,m),$

 $= -\frac{1}{2}\,{m^2}^\dagger M\,m^2\,,$

 $= 2\,(m\,j)^\dagger D\,(j\,m).$

Traditional Estimator

The traditional estimator is usually written as

$$\varepsilon = \frac{1}{\mathcal{N}} \int dx\, A(x) B^2(x) = \frac{1}{\mathcal{N}} m_0^\dagger \Phi^{-1} m_0^2, \qquad (139)$$

where $B = Dj = m_0$ is the Wiener-filter reconstruction of the gravitational potential, $A = \Phi^{-1} B$ is the same, just additionally filtered by the inverse power spectrum, and $\mathcal{N}$ is a normalization constant [202]. This is fixed by the condition that the estimator should be unbiased with respect to all gravitational potential and noise realizations,

$$\begin{aligned} \mathcal{N} &= \langle m_0^\dagger \Phi^{-1} m_0^2 \rangle_{(d,\phi|f=1)} \\ &= B^{(\varphi)}_{xyz}|_{f=1} [(MD)_{xu} \Phi^{-1}_{uv} (DM)_{vy} (DM)_{vz}] \\ &= 2[\Phi_{xy}\Phi_{yz} + \Phi_{yz}\Phi_{zx} + \Phi_{zx}\Phi_{xy}] \\ &\quad \times [(MD)_{xu} \Phi^{-1}_{uv} (DM)_{vy} (DM)_{vz}]. \end{aligned} \qquad (140)$$