Dark Matter with Phase Space Elements

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Abel, Hahn, Kaehler (2012), MNRAS
Kaehler, Hahn, Abel (2012), IEEE TVCG
Hahn, Abel, Kaehler (2013), MNRAS
Angulo, Hahn, Abel (2013), MNRAS
What is Dark Matter?

**microscopic**
proton = 1GeV, WIMP 100GeV? -> $10^{21}$/g

**cold** (or at most lukewarm)
e.g. thermally produced at very early times, cooled since then

**negligible cross-section**
weak-scale or even weaker

continuum limit

$\nu_{\text{thermal}} \ll \nu_{\text{bulk}}$

$\sigma_{\text{DM}} \ll \sigma_{\text{em}}$
collisionless

...and also the dominant gravitating component (~80%)
at first order, structure formation is well described by assuming all matter is dark matter
Dark Matter - properties on small scales

- $P(k)$
- CMB
- galaxy clustering
- decr. particle mass
- CDM
- dynamic range of simulation ICs

- hot
- warm
- cold

- 1+1D
- 1D

LSS conference
Oliver Hahn
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1D behaviour under self-gravity

cusp forms, shell-crossing, but no shock!

Vanishing collision-term
⇒ not in hydro limit
⇒ velocity can be multi-valued
⇒ cannot stop at low order moments
⇒ have to discretize distribution function
⇒ singular caustics emerge
Dark Matter - fluid flow

Lagrangian description, evolution of fluid element

\[ Q \subset \mathbb{R}^3 \rightarrow \mathbb{R}^6 : \mathbf{q} \mapsto (\mathbf{x}_q(t), \mathbf{v}_q(t)) \]

Density constant

\[ \rho = m_{DM} \left| \frac{\partial x_i}{\partial q_j} \right|^{-1} \]

For DM, motion of any point \( \mathbf{q} \) depends only on gravity

\[ (\dot{\mathbf{x}}_q, \dot{\mathbf{v}}_q) = (\mathbf{v}_q, -\nabla \phi) \]

Unlike hydro, no internal temperature, entropy, pressure

So the quest is to solve Poisson's equation

\[ \Delta \phi = 4\pi G \rho \]
N-body vs. continuum approximation

The N-body approximation:

\[ i \in \{1 \ldots N\} \mapsto (x_i, v_i) \]

\[ \Rightarrow \text{EoM are just Hamiltonian N-body eq. (method of characteristics)} \]

for small N, density field is poorly estimated,

\[ \rho = m_p \sum \delta_D(x - x_i) \otimes W \]

continuum structure is given up, but ‘easy’ to solve for forces

**hope that as** \( N \rightarrow \text{very large numbers, approach collisionless continuum} **
Problems of the N-body method

Discreteness effects with some influence of softening

Most obvious for non-CDM simulations!
(e.g. Centrella&Melott 1983, Melott&Shandarin 1989, Wang&White 2007)
Problems of the N-body method: multi-fluid

Main Problem: two-body effects couple particles!

two fluids, coupled only through gravity:

\[
\frac{\partial f_{1,2}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{x} f_{1,2} - \nabla \phi \cdot \nabla \mathbf{v} f_{1,2} = 0
\]

\[\Delta \phi = 4\pi G (\rho_1 + \rho_2)\]

very sensitive to spurious coupling!

Problem for precision predictions of high-z baryon distribution
Dark Matter - fluid flow, full manifold description

Lagrangian description, evolution of fluid element

\[ Q \subset \mathbb{R}^3 \rightarrow \mathbb{R}^6 : q \mapsto (x_q(t), v_q(t)) \]

Describe map between Lagrangian and Eulerian space by
(infinite dimensional) space of tri-polynomials

\[ Q \in P_k = \{ \pi(q) \mid \pi(q) = \sum_{\alpha,\beta,\gamma=0}^{k} a_{\alpha\beta\gamma} q_0^\alpha q_1^\beta q_2^\gamma \} \]

Exact for \( k \rightarrow \infty \), manifold tracking instead of particles
Equations of motion:

N-body characteristics
\[ \dot{x}_i = v_i, \quad \text{and} \quad \dot{v}_i = -\nabla_x \phi|_{x_i}, \quad \text{with} \ i \in \mathbb{N} \]

Characteristics on Lagrangian manifold
\[ \dot{x}_q = v_q, \quad \text{and} \quad \dot{v}_q = -\nabla_x \phi|_{x_q}, \quad \text{with} \ q \in \mathcal{Q} \]

Polynomial expansion of EoM leads to EoM for coefficients
\[ \dot{x}_{\alpha\beta\gamma} = v_{\alpha\beta\gamma}, \quad \dot{v}_{\alpha\beta\gamma} = -\rho^{-1}f_{\alpha\beta\gamma}, \quad \alpha, \beta, \gamma \in \mathbb{N} \]

finite expansion at order \( k \) leads to the following truncation error:
\[ \Delta \dot{v} = -\rho^{-1} \sum_{\alpha,\beta,\gamma=k+1}^{\infty} f_{\alpha\beta\gamma} q_0^\alpha q_1^\beta q_2^\gamma \]

sourced by high order derivatives of the force field across the element
-> need to keep bounded to keep energy conservation bounded
-> refinement essential!
Lagrangian elements of order $k$

Finite order maps:

- $k=1$: bi-linear
- $k=2$: bi-quadratic
- Affine map: tetrahedral

Cost: truncation error in EoM!
Describing the density field & softening I

\[ \rho = m_p \sum \delta_D(x - x_i) \otimes W \]

\[ \rho = m_p \sum_{\text{streams}} \left| \det \frac{\partial x_i}{\partial q_j} \right|^{-1} \]
analysis
Three dimensions

Same simulation data! (Abel, Hahn, Kaehler 2012)

rendering points for particles.

rendering tetrahedral phase space cells.
Derivatives of the bulk velocity field

- Discontinuities make ordinary derivatives ill-defined without coarse-graining!

- Away from discontinuities:
  Need to explicitly evaluate action of derivative on **projected** field:

  \[
  \nabla \cdot \langle \mathbf{v} \rangle = \langle (\nabla \log \rho) \cdot (\mathbf{v} - \langle \mathbf{v} \rangle) \rangle + \langle \nabla \cdot \mathbf{v} \rangle \\
  \nabla \times \langle \mathbf{v} \rangle = \langle (\nabla \log \rho) \times (\mathbf{v} - \langle \mathbf{v} \rangle) \rangle + \langle \nabla \times \mathbf{v} \rangle
  \]

- Vorticity for std. gravity pure multi-stream phenomenon!!

- At discontinuities:
  Derivatives are singular, but have finite measure.

**compressive singularities at caustics (=motion of caustics)**
Properties of the cosmic velocity field II

Hahn et al. 2014a
Faster convergence (for WDM: convergence!)
Better small scale properties
simulations
Describing the density field & softening II

\[ \rho = m_p \sum \delta_D (x - x_i) \otimes W \]

need softening, no knowledge what it should be (empirical?)

\[ \rho = m_p \sum_{\text{streams}} \left| \det \frac{\partial x_i}{\partial q_j} \right|^{-1} \]

self-adaptive

what are the evolution equations for \( W \)?
= evolution of the local manifold!
300eV toy WDM problem

**fixed** mass resolution, varying force resolution:

- **std PM**
- **sheet monopole**
- **sheet quadrupole**

![Graphs showing mass resolution and force resolution](image)

- Force resolution: features become sharper, fragmentation appears.
- Sheet tessellation based method cures artificial fragmentation.

But halos become too dense!
refinement + higher order!

hi-res N-body + able to track fine-grained phase space

tesselated cube orbiting in non-harmonic potential

adaptively refined tri-quadratic phase-space element

first alternative to N-body in highly non-linear regime!

Hahn & Angulo 2015
Final results with refinement

i. quadratic interpolant for refinement
ii. quartic interpolant for refinement
iii. high-res N-body solution

\[ \Delta E/E = 5 \times 10^{-4} \]
\[ \Delta E/E = 2 \times 10^{-3} \]

\[ t = 24, \ \Theta = 0.1 \]

Hahn & Angulo 2015
How noisy are N-body sims?

- a. N-body $32^3$
- b. tetrahedra $32^3$
- c. tri-linear $32^3$
- d. tri-quadratic $32^3$
- e. high-res N-body $512^3$

Hahn & Angulo 2015
cosmological simulations w/ refinement

Hahn & Angulo 2015
a = 0.015625000
First determination of WDM halo mass function!

Angulo, Hahn & Abel 2013
Towards the WDM mass function...

...halo finding becomes challenging

Very dense cores of filaments, linking the halo structures

More work has to be done to understand structure formation. what do baryons do in such a universe? we don’t know yet!
Structures at different masses...

Are at different stages of formation...
Conclusions

• Lagrangian elements can give new insights into existing simulations (density/velocity fields, multi-stream analysis,…)
• Provide also self-consistent simulation technique. (functional when using high-order and adaptive refinement)
• Solves fragmentation problems of N-body
• requires refinement to ensure energy conservation
• First methodological test of N-body in deeply non-linear regime
• Stay tuned for halo properties…