Infrared Correlations in de Sitter Space: Field Theoretic vs. Stochastic Approach

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Outline

I. The Infrared Problem for a Scalar Field

II. Solution in the Stochastic Approach

III. Field Theory in Euclidean de Sitter Space

IV. Two Loop Analysis in Lorentzian de Sitter Space

I.

The Infrared problem for a scalar field

Motivation

- Problem: there is no physically acceptable de Sitter invariant quantum state for a free, minimally coupled massless scalar field in de Sitter space. This is due to the accumulation of superhorizon infrared (IR) modes.

- Non-interacting fields are of no physical consequence. $\longrightarrow$ Is there a dS invariant state for an interacting ($\lambda\phi^4$) field such that dS is well-defined background for Quantum Field Theory?

- Starobinsky's semiclassical stochastic answer is yes. Can we confirm this in a QFT approach & explicitly construct a dS invariant state? $\longrightarrow$ Progress toward this goal in this talk.

- Once scalar case is understood, check whether there is a dS invariant state for gravitons. Exclude/confirm whether scalar or spin-2 IR effects affect inflationary/present day acceleration.

Power Spectrum at Tree-Level

- $\mathcal{L} = \sqrt{-g}\left\{(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 - \xi R\phi^2\right\}$ $\qquad S = \int d^4x\, \mathcal{L}$

- de Sitter space (Inflation): $g_{\mu\nu} = a^2(\eta)\,\eta_{\mu\nu} = a^2(\eta)\,\mathrm{diag}(1,-1,-1,-1)$

  $a = e^{Ht} \qquad dt = a\, d\eta \qquad a(\eta) = -\frac{1}{H\eta} \qquad \eta \in\,]-\infty; 0[$

- For $m=0$, $\xi=0$: $\nabla_x^2\phi(x) = 0 \implies \left(\partial_\eta^2 + \vec{k}^2 - \frac{a''}{a}\right) a\,\phi(\vec{k},\eta) = 0$

  Bunch-Davis vacuum $\implies a\,\phi(k,\eta) = \frac{1}{\sqrt{2|\vec{k}|}}\left(1 - \frac{i}{|\vec{k}|\eta}\right) e^{-i|\vec{k}|\eta}$

- $\hat{\phi}(\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\left(e^{-i\vec{k}\cdot\vec{x}}\phi(\vec{k},\eta)a(\vec{k}) + e^{i\vec{k}\cdot\vec{x}}\phi^*(\vec{k},\eta)a^+(\vec{k})\right)$, $\quad [a(\vec{k}), a^+(\vec{k}')] = (2\pi)^3\delta^3(\vec{k}-\vec{k}')$

  $\implies \langle\hat{\phi}(\vec{k})\hat{\phi}(\vec{k}')\rangle \overset{k/a \gg H}{\approx} (2\pi)^3\delta^3(\vec{k}-\vec{k}')\frac{H^2}{2|\vec{k}|^3} =: (2\pi)^3\delta^3(\vec{k}-\vec{k}')\frac{2\pi^2}{|\vec{k}|^3}P_\phi(\vec{k})$

  $\implies P_\phi(\vec{k}) = \left(\frac{H}{2\pi}\right)^2 \longrightarrow$ finite tree-level observables despite singular ϕ for $k \to 0$ (e.g. curvature perturbation)

Position Space Analysis: Scalar Propagator

- Try to keep dS invariance manifest
- $(-\nabla_x^2 - m^2)\, i\Delta^{(0)fg}(x;x') = fg\, \delta^{fg}\, a^{-4}\, i\delta^4(x-x')$ $\quad [f,g = \pm]$

  ∇_x: covariant derivative
- Invariant length: $y(x;x') = 4\sin^2\left(\frac{1}{2} H\, \ell(x;x')\right) = H^2 \Delta x^2 a a'$

  $\Delta x^2 = (x-x')^\mu \eta_{\mu\nu} (x-x')^\nu$ $\quad$ ℓ: geodesic distance
- $z = 1 + \frac{y}{4} \Longrightarrow$

$$a^4 H^2 \left[z(1-z)\frac{d^2}{dz^2} + 4\left(\frac{1}{2} - z\right)\frac{d}{dz} - \frac{m^2}{H^2}\right] i\Delta^{(0)fg} = fg\, \delta^{fg}\, \delta^4(x-x')$$

$$\Longrightarrow$$

$$i\Delta(x;x') = \frac{\Gamma\left(3 - \frac{m}{3H^2}\right)\Gamma\left(\frac{m^2}{3H^2}\right)}{(4\pi)^2} H^2\, {}_2F_1\left(3 - \frac{m^2}{3H^2}, \frac{m^2}{3H^2}, 2, 1 + \frac{y}{4}\right)$$

- no de Sitter invariant solution* for $m = 0$

  [Allen (1985); Allen & Folacci (1987)]

  * with physical light-cone singularities

Scalar Propagator, de Sitter Breaking, Small Mass

- Inflation begins & ends → not exact de Sitter
- de Sitter breaking propagator

$$i\Delta(y) = \frac{H^2}{4\pi^2}\left\{\frac{1}{y} + \frac{1}{2}\log(-y) + \frac{1}{2}\log(aa')\right\}$$

Tsamis & Woodard (1996)

secularly growing term
$\sim$ # of e-folds

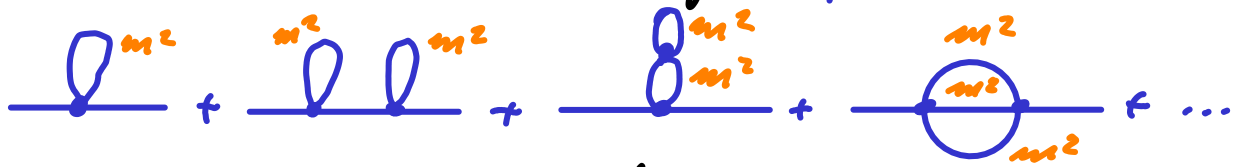
- occurs in graviton propagator as well (→ "eternal Inflation"?)
- de Sitter invariant propagator when assuming small ($\ll H$) mass m (⟶ expansion in m^2/H^2):

$$i\Delta(y) = \frac{H^2}{4\pi^2}\left\{\frac{1}{y} + \frac{1}{2}\log(-y) - \frac{3}{2}\frac{H^2}{m^2} + \dots\right\}$$

Prokopec & Puchwein (2003)

IR Problem: Field Theory Consequences & Possible Solution

- Non-existence of propagator ($\hat{=}$ quantum state) of a non-interacting field not even of academic interest.
  $\longrightarrow$ However: Interactions (e.g. $\lambda \phi^4$) lead to loop effects:

$m^2 =$ 

"Schwinger-Dyson Equation"

  These diagrammatic contributions grow with increasing IR enhancement
  $\longrightarrow$ breakdown of loop expansion (perturbation theory)

- However, this sickness may already be the remedy:
  The loops induce dynamical self-masses that self-consistently regulate the IR divergences [Sloth & Riotto (2008) @ one loop;
  incomplete two loop result: BG & Rigopoulos (2011)]

- It was then shown that the dynamical mass correctly regulates the field theory in Euclidean de Sitter space [Rajaraman (2010);
  Beneke & Moch (2012)]

- Consistency between Euclidean & Stochastic approach demonstrated

Goal: Achieve same thing for QFT in Lorentzian de Sitter space

II. The (Well Known) Solution in the Stochastic Approach

Starobinsky's Stochastic Approach

- Canonically quantised scalar

$$\phi(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} \left(e^{-i\vec{k}\cdot\vec{x}} \phi(\vec{k},\eta) a(\vec{k}) + e^{i\vec{k}\cdot\vec{x}} \phi^*(\vec{k},\eta) a^\dagger(\vec{k}) \right), \quad \left[a(\vec{k}), a^\dagger(\vec{k}')\right] = (2\pi)^3 \delta^3(\vec{k}-\vec{k}')$$

$$a\,\phi(k,\eta) = \frac{1}{\sqrt{2|\vec{k}|}} \left(1 - \frac{i}{|\vec{k}|\eta}\right) e^{-i|\vec{k}|\eta}$$

cancellations!

$$\Pi = \frac{\delta L}{\delta \partial_\eta \phi} = a^2 \phi' \Longrightarrow \left[\phi(\vec{x}), \Pi(\vec{x}')\right] = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} = i\,\delta^3(\vec{x}-\vec{x}')$$

$$\{\phi(\vec{x}), \Pi(\vec{x}')\} \sim \int \frac{d^3k}{(2\pi)^3} \left(\#1 + \#\frac{1}{k\eta} + \#\frac{1}{k^2\eta^2}\right) e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}$$

- For superhorizon modes ($k\eta \ll 1$) the anticommutator terms are much larger than commutator terms $\longrightarrow$ treat these classical, i.e. replace QM expectation values with probability distribution functions (PDFs).

Starobinsky's Stochastic Approach

- Basically the reason, why we perceive quantum fluctuations from inflation as classical.
- But is this mathematically rigorous & consistent with QFT? Moreover, in exact de Sitter space, the separation in sub-/superhorizon is observer dependent
- More concretely, we decompose $\phi = \bar{\phi} + \phi_q$

$$\phi = \bar{\phi} + \int \frac{d^3k}{(2\pi)^3}\, \vartheta(k - \varepsilon a H)\left[a(\vec{k})\,\phi(\vec{k},\eta)\,e^{-i\vec{k}\cdot\vec{x}} + a^\dagger(\vec{k})\,\phi(\vec{k},\eta)\,e^{i\vec{k}\cdot\vec{x}}\right]$$

i.e. we separate at a fixed (superhorizon) physical momentum scale $\frac{k}{a} = \varepsilon H$

$\bar{\phi}$ is now treated as a classical stochastic variable

$$\dot{\phi} = \dot{\bar{\phi}} + \dot{\phi}_q + \frac{V_\phi}{3H} = 0 \longrightarrow$$

Langevin equation, provided $\dot{\phi}_q$ is evaluated as below

↳ noise term

$$\langle \dot{\phi}_q(t)\,\dot{\phi}_q(t') \rangle = \frac{H^3}{4\pi^2}\,\delta(t-t') \longrightarrow$$

- independent of ε
- treat this quantum fluctuation as a classical noise

Solution to Starobinsky-Langevin Equation

- Solve Langevin equation in terms of a PDF $\varrho(\phi)$
  $\longrightarrow$ Smoluchowsky equation:

$$\dot{\varrho}(\bar{\phi},t) = -\frac{\partial}{\partial\bar{\phi}}\frac{1}{3H}V_\phi\,\varrho(\bar{\phi},t) + \frac{H^3}{8\pi^2}\frac{\partial^2}{\partial\bar{\phi}^2}\varrho(\bar{\phi},t)$$

- Stationary ($t \longrightarrow \infty$) solution:

$$\varrho(\bar{\phi}) = \underbrace{\mathcal{N}}_{\text{normalisation}}\, e^{-\frac{8\pi^2 V(\phi)}{3H^4}}$$

- Field fluctuation

$$\langle\bar{\phi}^2\rangle = \frac{\int d\bar{\phi}\,\bar{\phi}^2\,\varrho(\bar{\phi})}{\int d\bar{\phi}\,\varrho(\bar{\phi})} = \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\frac{3H^2}{\pi\sqrt{\lambda}}$$ [Starobinsky & Yokoyama (1994)]

- Note: Energy density $\sim H^4 \longrightarrow$ negligible backreaction
- A free propagator with $m^2 = \frac{3H^4}{8\pi^2\langle\bar{\phi}^2\rangle} = \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}\frac{\sqrt{\lambda}\,H^2}{8\pi}$ has the same two-point correlation.

Breakdown of Loop Expansion

- Naïve counting:
  In a Feynman diagram, each new vertex gives factor $\sim \lambda$ but at the same time two more propagators $\sim \sqrt{\lambda}^{\,2}$.
  $\longrightarrow$ Each diagram contributes at the same order
  $\longrightarrow$ Non-diagrammatic calculation or all-order summation necessary

- This naïve counting argument applies to diagrammatic representation of classical stochastic expectation values & to diagrams in Euclidean QFT (see next Section).
  A bit more tricky in Lorentzian dS (see discussion below).

Field Theory in Euclidean de Sitter

Scalar Field in Euclidean dS

- D-dimensional Euclidean dS has the metric of a D-sphere embedded in $D+1$ dimensions
- Solutions to the Laplace Equation are given in terms of D-dimensional spherical harmonics $Y_{\vec{L}}$
- Can expand $\phi = \sum_{\vec{L}} \phi_{\vec{L}} Y_{\vec{L}}$ (discrete expansion, because compact space)
- Two-point function still divergent for $m \longrightarrow 0$.
  Affects constant part of propagator ("zero mode")
- $\longrightarrow$ Dominant Loop effects from zero mode $\longrightarrow$ self-regulation neglecting contributions from non-zero modes

Zero mode action: $S_0 = \frac{\lambda}{4!} \phi_{\vec{0}}^4 Y_{\vec{0}}^4 * Vol_{dS} \overset{D=4}{=} \frac{\lambda}{4!} \phi_{\vec{0}}^4 Y_{\vec{0}}^4 \frac{8\pi^2}{3H^4}$

$$Y_{\vec{0}}^2 \langle \phi_{\vec{0}}^2 \rangle = \frac{\int \mathcal{D}\phi_{\vec{0}} Y_{\vec{0}}^2 \phi_{\vec{0}}^2 e^{-S_0}}{\int \mathcal{D}\phi_{\vec{0}} e^{-S_0}} = \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \frac{3H^2}{\pi\sqrt{\lambda}}$$

$$Vol_{dS} = \frac{1}{Y_{\vec{0}}^2 H^D} = \frac{2\pi^{\frac{D+1}{2}}}{\Gamma(\frac{D+1}{2}) H^D}$$

[Rajaraman (2010)] $\longrightarrow$ Just the same result as in stochastic approach!

Diagrammatic / Dynamical Mass Approach

- Loop diagrams in Lorentzian dS take the effect of a dynamical self-mass ⟶ self consistent mass gap equation [BG, Rigopoulos (2011) -incomplete result]
- For Euclidean diagrams, it can be shown that the loop expansion corresponds to a divergent series. This can be resummed to obtain a mass gap equation that yields a result with the stochastic approach / zero-mode path integral. [Beneke, Moch (2012)]
- ⟶ Provided, we can show that the Lorentzian diagrams agree at all orders (in the leading IR approximation) with the Euclidean / stochastic diagrams, the stochastic approach and QFT in Lorentzian de Sitter space are consistent.
- Have shown agreement to two-loop order [BG, Rigopoulos, Zhu (2013)] ⟶ next Section

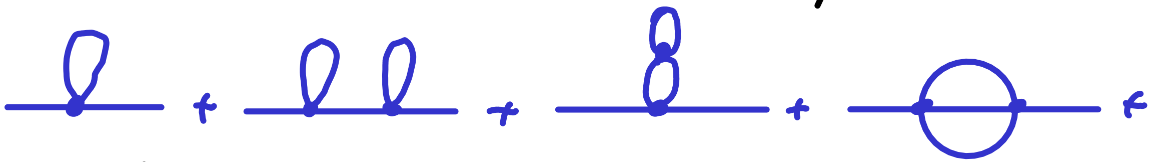
IV Two-Loop Calculation in Lorentzian de Sitter Space

Light Massive ϕ^4 Theory in de Sitter Space

- Work in a parametric regime where infrared effects dominate the loops, but yet, perturbation theory is valid:

  $$\sqrt{\lambda}\,H^2 \ll m^2 \ll H^2$$

- Calculate corrections to two-loop order:

- One loop has been done already, self energy $\Pi^{1\text{-loop}}(x;x') \propto \delta^4(x;x')$ is manifestly local and can be interpreted as a self-mass [Riotto & Sloth (2008)].

- Non-local two-loop contribution takes the effect of an effective ("dynamical") mass [BG, Rigopoulos (2011)].

- Calculate this in perturbative parametric regime & compare with stochastic approach [BG, Rigopoulos, Zhu (2013)].

- QFT in curved space-time dealt with using Closed Time Path (CTP) approach. $\longrightarrow$ Explain on the coming slides.

Time-Ordered vs. General Expectation Values

- Two-point functions (Green functions, propagators) in the canonical formalism:

$$\langle \phi(x)\phi(y)\rangle \sim \lim_{t\to\infty} \langle U(t,x^0)\,\phi_I(x)\,U(x^0,y^0)\,\phi_I(y)\,U(y^0,-t)\rangle$$

$$U(t,t') = T\, e^{-i\int_{t'}^{t} dt''\, H_I(t'')}$$

H_I ⟶ interaction picture Hamiltonian

- Simplifies for time-ordered product:

$$\langle T\phi(x)\phi(y)\rangle = \lim_{t\to\infty} \langle T\, \phi_I(x)\,\phi_I(y)\,U(t,-t)\rangle$$

⟶ Wick's theorem ⟶ Feynman rules

- Functional formalism: perhaps easier & more elegant approach for computing general expectation values.

Functional Approach (in-out)

- In-out generating functional for time ordered expectation values:

$$Z[J] = \mathcal{N}^{-1} \langle vac_{out} | vac_{in} \rangle_J = \int \mathcal{D}\phi \, e^{i\int d^4x \left(\mathcal{L} + J(x)\phi(x)\right)}$$

$$\langle T[\phi(x)\phi(y)] \rangle = -\frac{\delta^2}{\delta J(x)\,\delta J(y)} \log Z[J] \Big|_{J=0}$$

- Suitable for calculation of correlations of S-matrix elements
- But what in curved-space time, where $|vac_{out}\rangle$ is a priori not known?

The Closed-Time-Path

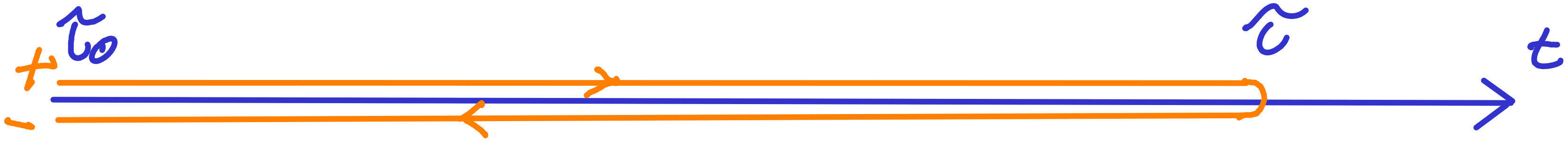
[Schwinger (1961), Keldysh (1964), Calzetta & Hu (1987, 1988)]

- **In-In** generating functional: $\phi_{in}(\vec{x}) = \phi(\vec{x}, \tau_0)$

$$Z[J_+, J_-] = \int \mathcal{D}\varphi\, \mathcal{D}\phi^-_{in}\, \mathcal{D}\phi^+_{in} \langle \phi^-_{in} | \varphi, \tau \rangle_{J_-} \langle \varphi, \tau | \phi^+_{in} \rangle_{J_+} \langle \phi^-_{in} | \varrho | \phi^+_{in} \rangle$$

$$= \int \mathcal{D}\phi^+ \mathcal{D}\phi^- \, e^{i \int d^4x \{ \mathcal{L}[\phi^+] + J_+ \phi^+ - \mathcal{L}[\phi^-] - J_- \phi^- \}} \langle \phi^-_{in} | \varrho | \phi^+_{in} \rangle$$

The Closed Time Path:



- Path ordered Green functions:

$$i \Delta^{ab}_{\phi}(u, v) = - \frac{\delta^2}{\delta J_a(u)\, \delta J_b(v)} \log Z[J_+, J_-] \Big|_{J_\pm = 0}$$

$$= i \langle \mathcal{C}[\phi^a(u)\, \phi^b(v)] \rangle$$

Feynman Rules

- Vertices either $+$ or $-$
- Connect vertices $a=\pm$ and $b=\pm$ with $i\Delta^{ab}$
- Factor -1 for each $-$ vertex
- ε – prescriptions in position space:

$$\Delta x(x;x') = x - x'$$

$$\Delta x^2_{++} = \left(|\Delta x^0| - i\varepsilon\right)^2 - |\Delta\vec{x}|^2 \qquad \Delta x^2_{-+} = \left(\Delta x^0 - i\varepsilon\right)^2 - |\Delta\vec{x}|^2$$

$$\Delta x^2_{+-} = \left(\Delta x^0 + i\varepsilon\right)^2 - |\Delta\vec{x}|^2 \qquad \Delta x^2_{--} = \left(|\Delta x^0| + i\varepsilon\right)^2 - |\Delta\vec{x}|^2$$

- Various Green functions: (cf. slide on scalar propagator)

time-ordered : $i\Delta^T = i\Delta^{++}$

anti-time ordered : $i\Delta^{\bar{T}} = i\Delta^{--}$

Wightman: $\begin{cases} i\Delta^> = i\Delta^{-+} \\ i\Delta^< = i\Delta^{+-} \end{cases}$

retarded : $i\Delta^R = i\Delta^T - i\Delta^<$

advanced : $i\Delta^A = i\Delta^T - i\Delta^>$

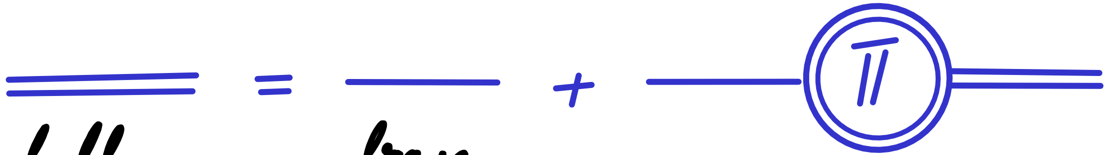
- Causality from "retarded logarithms":

$$\log(-\Delta x^2_{++}) - \log(-\Delta x^2_{+-}) = 2\pi i\, \underbrace{\vartheta}_{\text{step function}}(\Delta x^0)\, \vartheta(|\Delta x^0| - |\Delta\vec{x}|)$$

Schwinger-Dyson Equations

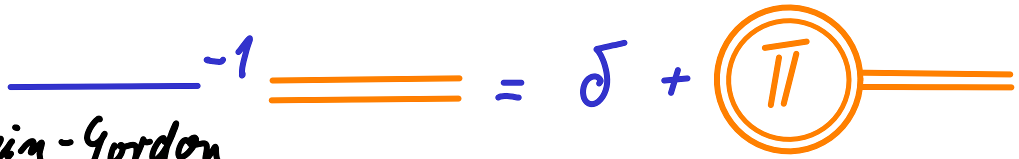
- $i\Delta^{ab} = i\Delta^{(0)ab} + c\,d\, i\Delta^{(0)ac} \odot \Pi^{cd} \odot \Delta^{db}$

$$A(x,w) \odot B(w,y) = \int d^4w\, A(x,w)\, B(w,y)$$



full propagator — bare propagator — self energy: 1PI, full propagators

- Alternatively:



Klein-Gordon Operator

functions of m_{dyn}

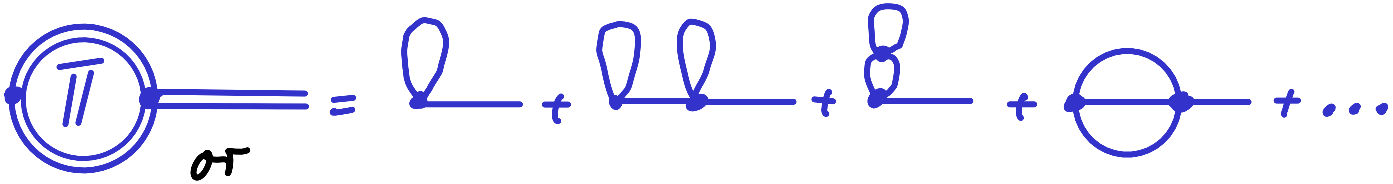
$$a^4(-\nabla_x^2 - m^2)\, i\Delta^{ab}(x;x') + ic \int d^4w\, i\Pi^{ac}(x;w)\, i\Delta^{cb}(w;x') = \delta^{ab}\, i\delta^4(x-x')$$

- ansatz:

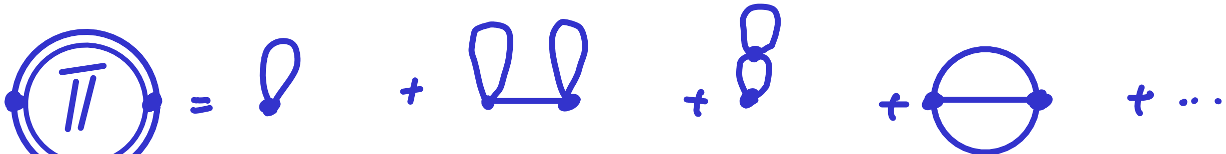
$$a^4(-\nabla_x^2 - m_{dyn}^2)\, i\Delta^{ab}(x;x') = \delta^{ab}\,\delta^4(x-x')$$

Simplification in Perturbative Domain

- When perturbation theory is valid, do not need to use m_{dyn} on the Schwinger-Dyson Equations, but can expand in terms of tree-level propagators:



or



Integrate vertices over $\sqrt{-g}\, d^4x$.

- Useful linear combination of the Schwinger-Dyson Equations:

$$a^4(-\nabla_x^2 - m^2)\, i\Delta^{<,>}(x;x') = -i\int d^4w\; i\Pi^R(x;w)\, i\Delta^{<,>}(w;x') - i\int d^4w\; i\Pi^{<,>}(x;w)\, i\Delta^A(w;x')$$

$$a^4(-\nabla_x^2 - m^2)\, i\Delta^{R,A}(x;x') = i\delta^4(x;x') - i\int d^4w\; i\Pi^{R,A}(x;w)\, i\Delta^{R,A}(w;x')$$

- The first of these are known in finite-temperature field theory as the celebrated Kadanoff-Baym equations.
  Here, we use these, because $i\Delta^{<,>}$ directly contains information about $\langle\phi^2\rangle$, whereas $i\Delta^{R,A}$ do not.

A Problem With Convolution Integrals

- Because $i\Pi^R_{sg}(x;x') = \frac{\lambda}{2} a^2(\eta)\,\delta^4(x-x')\,i\Delta^{++}(x;x)$, the convolution
  ↳ "seagull" one-loop
  integrals at one-loop order are trivial & the interpretation of the self energy as a self-mass is obvious (local effect!)

- Convolution over two-loop sunset self-energy yields non-local contributions.
  Integration over constant term in

$$i\Delta(y) = \frac{H^2}{4\pi^2}\left\{\frac{1}{y} + \frac{1}{2}\log(-y) - \frac{3}{2}\frac{H^2}{m^2} + \dots\right\}$$

  yields result proportional to the volume of space-time, which is infinite in Lorentzian dS.

- This is an artefact, because above approximation for the propagator does not account for the decay of correlations way beyond the horizon.

Improved Expansion of the Propagator

- Improved Expansion of hypergeometric solution yields [BG, Rigopoulos, Zhu (2013)]

$$i\Delta(y) = \frac{H^2}{4\pi^2}\left[-\frac{1}{y} + \frac{3H^2}{2m^2}\left(-\frac{1}{y}\right)^{\frac{1}{3}\frac{m^2}{H^2}} + \mathcal{O}\left(y^{-2}\frac{m^2}{H^2}\right)\right]$$

- Note : $(-y)^{-\frac{1}{3}\frac{m^2}{H^2}} = \left(1 - \frac{i}{3}\frac{m^2}{H^2}\arg(-y)\right)|y|^{-\frac{1}{3}\frac{m^2}{H^2}} + \mathcal{O}\left(\frac{m^2}{H^2}\right)$

$$|y|^{-\frac{1}{3}\frac{m^2}{H^2}} = 1 - \frac{1}{3}\frac{m^2}{H^2}\log|y| + \mathcal{O}\left(\left(\frac{1}{3}\frac{m^2}{H^2}\log|y|\right)\right)$$

- Hence, we recover the previously used propagator and find relevant decay on scales $> H^{-1}e^{\frac{3H^2}{m^2}}$.

- This mild decay also regulates the convolution integrals, e.g.

$$-i\int d^4w\, \Pi_{SS}^{<,>}(x;w)\, i\Delta^A(w;x) = -i\int d^4w\, \frac{\lambda^2}{6}a^4(x)\left(i\Delta^{<,>}(x;w)\right)^3 a^4(w)\, i\Delta^A(w;x)$$

Sunset

$$= -\frac{9\lambda^2H^{14}}{2^{12}\pi^8m^6}a^4(x)\int d^4w\,|y(x;w)|^{-\frac{4}{3}\frac{m^2}{H^2}}\,\pi\,\vartheta\left((w^0-x^0)^2-(\vec{w}-\vec{x})^2\right)\vartheta(w^0-x^0)$$

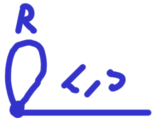
$$\approx -a^4(x)\frac{9\lambda^2H^{12}}{2^{12}\pi^6m^8}$$

Note that we obtain additional IR enhancement from the slow decay of the integrand.

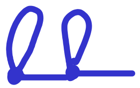
similarly $-i\int d^4w\, \Pi_{SS}^R(x;w)\, i\Delta^A(w;x) = -a^4(x)\frac{27\lambda^2H^{12}}{2^{12}\pi^6m^8}$

Summary of QFT Result

- Individual diagrammatic contributions:

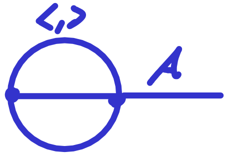
$$a^{-4}$$ 

$$= -\lambda \frac{9H^8}{128\pi^4 m^4}$$

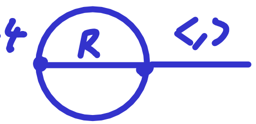
$$a^{-4}$$ 
$$= \lambda^2 \frac{27H^{12}}{2^{12}\pi^6 m^8}$$

$$a^{-4}$$ 

$$= \lambda^2 \frac{27H^{12}}{2^{11}\pi^6 m^8}$$

$$a^{-4}$$ 

$$= \frac{9\lambda^2 H^{12}}{2^{12}\pi^6 m^8}$$

$$a^{-4}$$ 

$$= \frac{27\lambda^2 H^{12}}{2^{12}\pi^6 m^8}$$

- These add up to $\mu \equiv$ r.h.s. of Kadanoff-Baym equations

$$a^4(-\nabla_x^2 - m^2)\, i\Delta^{<,>}(x;x') = \mu \implies i\delta\Delta^{<,>} = \frac{\mu}{m^2}$$

$$\implies i\Delta^{<,>}(x;x') = \frac{3H^4}{8\pi^2 m^2} - \frac{9\lambda H^8}{128\pi^4 m^6} + \frac{9\lambda^2 H^{12}}{256\pi^6 m^{10}} + \dots$$

$y(x;x') \sim 1$

- Can also express this as "dynamical mass"

$$m_{dyn}^2 = \frac{3H^4}{8\pi^2\, i\Delta^{<,>}} = m^2 + \frac{3\lambda H^4}{16\pi^2 m^2} - \frac{15\lambda^2 H^8}{256\pi^5 m^6} + \dots$$

Diagrammatic Expansion of Stochastic Expectation Values

- We can also "perturbatively" expand the stochastic expectation values

$$\langle \bar\phi^2 \rangle = \frac{\int d\bar\phi\, \bar\phi^2\, e^{-\frac{8\pi^2}{3H^4}\left(\frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\right)}}{\int d\bar\phi\, e^{-\frac{8\pi^2}{3H^4}\left(\frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\right)}}$$

according to

$$e^{-\frac{8\pi^2}{3H^4}\left(\frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\right)} = e^{-\frac{8\pi^2}{3H^4}\frac{1}{2}m^2\phi^2} \times \left[1 + \frac{8\pi^2}{3H^4}\frac{\lambda}{4!}\phi^4 + \frac{1}{2!}\left(\frac{8\pi^2}{3H^4}\frac{\lambda}{4!}\phi^4\right)^2 + \dots\right]$$

- Result agrees with what is obtained from Field Theory.

  Diagrammatically, $\langle \bar\phi^2 \rangle$ corresponds to the sum of all connected two-point functions with "propagators" $\frac{3H^4}{8\pi^2 m^2}$ and "vertices" $\frac{8\pi^2}{3H^4}\frac{\lambda}{4!}$.

- These stochastic diagrams individually agree with the sum of the field theory diagrams of the same topology (i.e. the two QFT sunset diagrams add up to the single stochastic one).

- Note that here the IR enhancement is only from the propagators, whereas in QFT propagators and integrations give IR enhancement.

Diagrammatic Expansion of Langevin Dynamics

- We can find an even closer stochastic/QFT matching when deriving the solution to the Langevin equation in a functional/diagrammatic manner.

- Langevin equation: $\dot{\phi} + \frac{\partial_\phi V}{3H} = \xi(t)$

$$\langle \xi(t)\xi(t')\rangle = \frac{H^3}{4\pi^2}\delta(t-t') = \int \mathcal{D}\xi\, \xi(t)\xi(t') e^{-\frac{1}{2}\int d\tau \frac{4\pi^2}{H^3}\xi^2(\tau)}$$

- Expectation values (forcing Langevin equation):

$$\langle O[\phi]\rangle = \int \mathcal{D}\xi\, e^{-\frac{1}{2}\int dt \frac{4\pi^2}{H^3}\xi^2} \int \mathcal{D}\phi\, O[\phi]\, \delta\left(\dot{\phi} + \frac{\partial_\phi V}{3H} - \xi\right)$$

- Introduce auxiliary field φ:

$$\delta\left(\dot{\phi} - \frac{\partial_\phi V}{3H} - \xi\right) = \int \mathcal{D}\varphi\, e^{-i\int dt\, \varphi\left(\dot{\phi} + \frac{\partial_\phi V}{3H} - \xi\right)}$$

Integrate out ξ:

$$\langle O[\phi]\rangle = \int \mathcal{D}\phi \mathcal{D}\varphi\, O[\phi] e^{-i\int dt\left\{\frac{1}{2}(\phi,\varphi)\begin{pmatrix} 0 & -\partial_t + \frac{m^2}{3H} \\ \partial_t + \frac{m^2}{3H} & -i\frac{H^3}{4\pi^2}\end{pmatrix}\begin{pmatrix}\phi \\ \varphi\end{pmatrix} + \frac{\lambda}{3!}\frac{1}{3H}\varphi\phi^2\right\}}$$

Stochastic Green Functions

- Invert bilinear operator

$$i\begin{pmatrix} 0 & -\partial_t + \frac{m^2}{3H} \\ \partial_t + \frac{m^2}{3H^2} & -i\frac{H^3}{4\pi^2}\end{pmatrix}\begin{pmatrix} i\Delta^{<,>}(t,t') & -iG^R(t,t') \\ -iG^A(t,t') & 0\end{pmatrix} = \delta(t-t')\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$\Longrightarrow \; G^R(t,t') = G^A(t',t) = e^{-\frac{m^2}{3H}(t-t')}\,\vartheta(t-t')$$

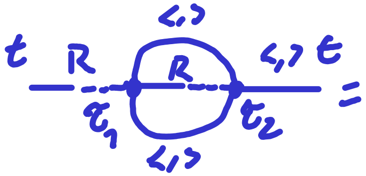
$$i\Delta^{<,>}(t,t') = \frac{3H^4}{8\pi^2 m^2}\, e^{-\frac{m^2}{3H}|t-t'|}$$

- Note: Matrix-valued Green function corresponds to Keldysh representation of CTP Green functions:

$$\begin{pmatrix} \frac{1}{2}(i\Delta^> + i\Delta^<) & -i\Delta^R \\ -i\Delta^A & 0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix} i\Delta^T & i\Delta^< \\ i\Delta^> & i\Delta^{\bar T}\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$$

$\longrightarrow$ same structure as in QFT approach, just different basis

- Again, retarded/advanced propagators exhibit no IR-enhancement, but extra IR enhancement comes from convolution integrals, e.g.



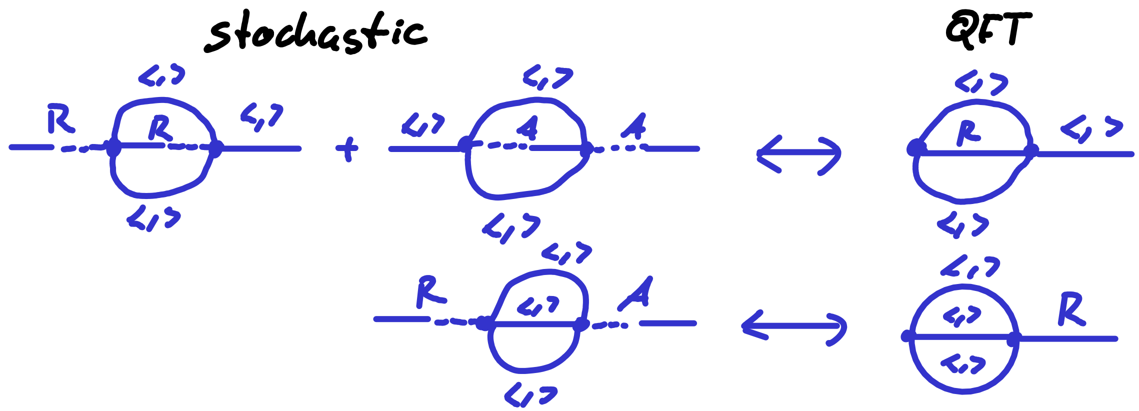
$$= \frac{1}{2}\left(\frac{3H^4}{8\pi^2 m^2}\right)^3\left(\frac{\lambda}{3H}\right)^2\int_{-\infty}^{t} d\tau_1 \int_{-\infty}^{\tau_1} d\tau_2\, e^{-\frac{m^2}{3H}(t-\tau_1)}\, e^{-\frac{3m^2}{3H}(\tau_1-\tau_2)}\, e^{-\frac{m^2}{3H}(t-\tau_2)}$$

$t > \tau_1 > \tau_2$

$$= \frac{27 H^{12}}{2^{13}\pi^6 m^8}$$

Agreement with Causal Green Functions

- Field Theory / Stochastic agreement persists when breaking down to different diagrams in terms of causal propagators:



- The fact that we do not left multiply the CTP diagrams with an extra propagator has no deep meaning but is just calculational convenience when using the Schwinger-Dyson approach.

Back to Massless Case

- Apparently, have complete chain of arguments to show Field Theory / Stochastic agreement also for massless $\lambda\phi^4$ theory in dS.

Lorentzian Feynman Diagrams

↓ agree with

Stochastic Diagrams for Langevin Equation

↓ add up to

Stochastic Diagrams for Late Time Probability Distribution Function

↓ are identical to

Diagrammatic Expansion with Dynamical Mass

↓ can be resummed [Beneke, Moch (2012)]

Result that Agrees with the one from Stochastic Approach

Conclusion:
Agreement also in massless case
– goes a bit beyond what we state in [BG, Rigopoulos, Zhu (2013)]

Conclusions

- Leading IR predictions light/massless $\lambda\phi^4$ theory agree for Field Theory and Stochastic approaches.
- Would still be nice to work out more details, shorten above chain of arguments.
- Massless $\lambda\phi^4$ theory behaves benignely in de Sitter space.
- Lorentzian QFT calculation merits interest, because inflation & present acceleration phase are no exact de Sitter phases that can be continued to Euclidean space.
- It would be interesting to apply the techniques discussed here to address Gravitons in Euclidean/Lorentzian de Sitter space, because it is sometimes argued [Woodard et. al.] that for these, IR effects lead to a significant backreaction, which would be relevant for early & present day acceleration.