Notes on Gravitational Lensing

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Contents

1 Friedmann-Lemaître cosmological models

1.1 The coordinate system and the metric

I assume that the Universe can be described on large scales by a homogeneous and isotropic Friedmann-Lemaître universe (Friedmann 1922, 1924; Lemaître 1927).[1] In order to label spacetime points, I choose a coordinate system $(t, \beta_1, \beta_2, \chi)$ based on physical time t, two angular coordinates $\boldsymbol{\beta} = (\beta_1, \beta_2)$, and line-of-sight comoving distance χ (see Fig. 1.1). The spacetime metric of the model is then given by a Friedmann-Lemaître-Robertson-Walker (FLRW) metric (Friedmann 1922, 1924; Lemaître 1927; Robertson 1933, 1935; Walker 1935):

$$g = -\mathrm{c}^2 \mathrm{d}t^2 + a^2 \left\{ \mathrm{d}\chi^2 + f_K^2(\chi) \left[\mathrm{d}\beta_1^2 + \cos^2(\beta_1) \mathrm{d}\beta_2^2 \right] \right\}. \tag{1.1}$$

Here, c denotes the speed of light, and $a = a(t)$ the scale factor. The comoving angular diameter distance

$$f_K(\chi) = \begin{cases} 1/\sqrt{K} \sin\left(\sqrt{K}\chi\right) & \text{for } K > 0, \\ \chi & \text{for } K = 0, \text{ and} \\ 1/\sqrt{-K} \sinh\left(\sqrt{-K}\chi\right) & \text{for } K < 0, \end{cases} \tag{1.2}$$

where K is the external curvature of space. Note that the choice for the angular coordinates $\boldsymbol{\beta} = (\beta_1, \beta_2)$ differs from the usual one for spherical coordinates. The choice will be more convenient for the small-angle approximation.

1.2 The scale factor and the Hubble parameter

The scale factor a is chosen such that $a(t_0) = 1$ at the present time t_0. The relative time variation of the scale factor defines the Hubble parameter

$$H(t) = \frac{1}{a(t)} \frac{\mathrm{d}a(t)}{\mathrm{d}t}. \tag{1.3}$$

Its present-time value $H_0 = H(t_0)$ is called the Hubble constant. The present-time value is often quantified by the dimensionless value

$$h = \frac{H_0}{H_{100}}, \tag{1.4}$$

with $H_{100} = 100\,\mathrm{km}s^{-1}\mathrm{Mpc}^{-1}$.

The Hubble parameter determines the critical density

$$\rho_{\mathrm{crit}}^{\mathrm{ph}}(t) = \frac{3H(t)^2}{8\pi\mathrm{G}}, \tag{1.5}$$

[1] See Bartelmann and Schneider (2001) or Schneider et al. (2006) for a review of cosmology for lensing.

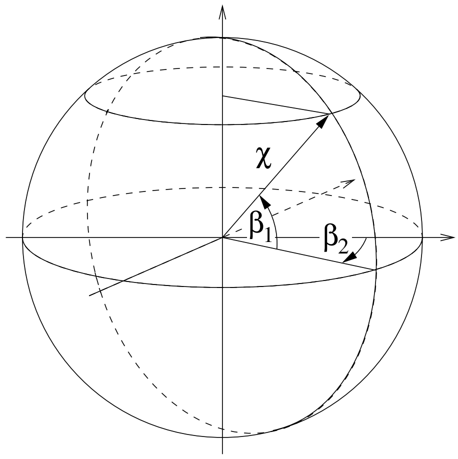

Figure 1.1: Illustration of the spatial part of the used spacetime coordinate system: a spherical coordinate system (β_1, β_2, χ) based on two angular coordinates $\boldsymbol{\beta} = (\beta_1, \beta_2)$ and line-of-sight comoving distance χ.

where G denotes Newton's gravitational constant. Its present-day value is denoted by $\rho^{\rm ph}_{\rm crit,0} = \rho^{\rm ph}_{\rm crit}(t_0)$.

Here, I assume that the average stress-energy density of the universe determines the time evolution of the metric (1.1) and the scale factor $a(t)$ via Einstein's field equations. I only consider contributions from a pressureless matter fluid with mean physical density $\bar{\rho}^{\rm ph}_{\rm m} = a^{-3}(t)\bar{\rho}^{\rm ph}_{\rm m,0}$ and a cosmological constant Λ.[2] Einstein's GR field equations for the metric (1.1) then reduce to the following equations for the scale factor:

$$\frac{1}{a(t)^2}\left(\frac{{\rm d}a(t)}{{\rm d}t}\right)^2 = \frac{8\pi{\rm G}}{3}\bar{\rho}^{\rm ph}_{\rm m}(t) + \frac{K{\rm c}^2}{a(t)^2} + \frac{\Lambda}{3}, \tag{1.6}$$

$$\frac{1}{a(t)}\frac{{\rm d}^2a(t)}{{\rm d}t^2} = -\frac{4\pi{\rm G}}{3}\bar{\rho}^{\rm ph}_{\rm m}(t) + \frac{K{\rm c}^2}{a(t)^2} + \frac{\Lambda}{3}. \tag{1.7}$$

As is commonly done, I define the density parameters

$$\Omega_{\rm m} = \frac{\bar{\rho}^{\rm ph}_{\rm m,0}}{\rho^{\rm ph}_{\rm crit,0}}, \qquad \Omega_K = -\frac{K{\rm c}^2}{H_0^2}, \text{ and } \qquad \Omega_\Lambda = \frac{\Lambda}{3H_0^2}. \tag{1.8}$$

The time evolution of the scale factor is then given by the ordinary differential equation:

$$H(t) = H_0\sqrt{a(t)^{-3}\Omega_{\rm m} + a(t)^{-2}\Omega_K + \Omega_\Lambda}. \tag{1.9}$$

The equation implies that

$$1 = \Omega_{\rm m} + \Omega_K + \Omega_\Lambda. \tag{1.10}$$

In particular, $1 = \Omega_{\rm m} + \Omega_\Lambda$ for flat universes (i.e. universes with $K = 0$). Furthermore,

$$\begin{aligned}\rho^{\rm ph}_{\rm crit}(t) &= \frac{3h^2H_{100}^2}{8\pi{\rm G}}\left[a(t)^{-3}\Omega_{\rm m} + a(t)^{-2}\Omega_K + \Omega_\Lambda\right] \\ &= \rho_{\rm crit}\left[a(t)^{-3}\Omega_{\rm m} + a(t)^{-2}\Omega_K + \Omega_\Lambda\right],\end{aligned} \tag{1.11}$$

[2]Other contributions are (most likely) not relevant for the evolution epoch of interest for gravitational-lensing studies.

with the comoving critical density

$$\begin{aligned}\rho_{\mathrm{crit}} &= \frac{3h^2 H_{100}^2}{8\pi\mathrm{G}} \\ &= 1.83137 \times 10^{-26} h^2 \,\frac{\mathrm{kg}}{\mathrm{m}^3} \\ &= 2.70509 \times 10^{11} h^2 \,\frac{\mathrm{M}_\odot}{\mathrm{Mpc}^3}.\end{aligned} \tag{1.12}$$

If, as assumed here, matter is not created nor destroyed over time, the comoving mean matter density $\bar{\rho}_{\mathrm{m}} = \Omega_{\mathrm{m}}\rho_{\mathrm{crit},0}$ stays constant. A combination of constants involving $\bar{\rho}_{\mathrm{m}}$ frequently encountered later is

$$\frac{4\pi\mathrm{G}\bar{\rho}_{\mathrm{m}}}{\mathrm{c}^2} = \frac{3H_0^2\Omega_{\mathrm{m}}}{2\mathrm{c}^2} = \frac{3H_{100}^2}{2\mathrm{c}^2} h^2 \Omega_{\mathrm{m}}. \tag{1.13}$$

1.3 Cosmological redshift

The definitions of the cosmological redshift and the various cosmological distances are based on the light propagation in FLRW cosmological models. Here, I will give a brief introduction to this, and postpone a more detailed discussion to Chapter 2, which deals with the light propagation in both plain and perturbed FLRW models.

Consider a source at fixed comoving coordinates $(\boldsymbol{\beta}, \chi)$. Consider photons emitted from the source at times t_{S}, i.e. at spacetime positions $(t_{\mathrm{S}}, \boldsymbol{\beta}, \chi)$, reaching a comoving observer at spacetime positions $\big(t_{\mathrm{O}}, \boldsymbol{\beta}, 0\big)$. Since the path of such a photon, $q^\alpha(t) = (t, \boldsymbol{\beta}, \chi(t))$, is a null geodesic,

$$0 = g_{\mu\nu}\big(q^\alpha(t)\big) \frac{\mathrm{d}q^\mu(t)}{\mathrm{d}t} \frac{\mathrm{d}q^\nu(t)}{\mathrm{d}t}. \tag{1.14}$$

Hence,

$$\frac{\mathrm{d}\chi(t)}{\mathrm{d}t} = -\mathrm{c}a(t)^{-1}. \tag{1.15}$$

For fixed comoving distance χ, the emission time t_{S} and observation time t_{O} are related by:

$$\begin{aligned} \chi = \int_0^\chi \mathrm{d}\chi = \int_{t_{\mathrm{O}}(t_{\mathrm{S}})}^{t_{\mathrm{S}}} \frac{\mathrm{d}\chi(t)}{\mathrm{d}t}\mathrm{d}t = \mathrm{c}\int_{t_{\mathrm{S}}}^{t_{\mathrm{O}}(t_{\mathrm{S}})} a(t)^{-1}\mathrm{d}t \\ \Rightarrow \qquad 0 = \frac{1}{\mathrm{c}}\frac{\mathrm{d}\chi}{\mathrm{d}t_{\mathrm{S}}} = a\big(t_{\mathrm{O}}(t_{\mathrm{S}})\big)^{-1}\frac{\mathrm{d}t_{\mathrm{O}}(t_{\mathrm{S}})}{\mathrm{d}t_{\mathrm{S}}} - a(t_{\mathrm{S}})^{-1} \\ \Rightarrow \qquad \frac{\mathrm{d}t_{\mathrm{O}}(t_{\mathrm{S}})}{\mathrm{d}t_{\mathrm{S}}} = \frac{a\big(t_{\mathrm{O}}(t_{\mathrm{S}})\big)}{a(t_{\mathrm{S}})}. \end{aligned} \tag{1.16}$$

For emission received today, i.e. $t_{\mathrm{O}}(t_{\mathrm{S}}) = t_0$ and $a\big(t_{\mathrm{O}}(t_{\mathrm{S}})\big) = 1$,

$$\frac{\mathrm{d}t_{\mathrm{O}}(t_{\mathrm{S}})}{\mathrm{d}t_{\mathrm{S}}} = a(t_{\mathrm{S}})^{-1}. \tag{1.17}$$

This implies that the emitted and observed period of the photons differ by a factor $a(t_{\mathrm{S}})$, which results in a redshift of the observed wavelength with respect to the emitted wavelength of photon by

$$z(t_{\mathrm{S}}) = a(t_{\mathrm{S}})^{-1} - 1. \tag{1.18}$$

1.4 Cosmological distances

Consider a comoving source emitting photons from spacetime position $\left(t, \boldsymbol{\beta}, \chi\right)$ that reach the observer at position $(t_0, \boldsymbol{\beta}, 0)$. According to Eq. 1.18, the scale factor a at the time of emission t is then related to the redshift z the photon experiences between source and observer by:

$$a = (z+1)^{-1}. \tag{1.19}$$

Furthermore, the emission time t of the photon, its redshift z, and the line-of-sight comoving coordinate χ of the source are related by (also see Hogg 1999, for a brief discussion of distances):

$$\begin{aligned} \chi &= \int_0^{\chi} \mathrm{d}\chi' = \int_{t_0}^{t} a(t')^{-1} \mathrm{c}\mathrm{d}t' = \mathrm{c} \int_0^{z} a(z')^{-1} \left|\frac{\mathrm{d}t}{\mathrm{d}z}\right| \mathrm{d}z' \\ &= \mathrm{c} \int_0^{z} a(z')^{-1} \frac{a(z')}{H(z')} \mathrm{d}z' = \mathrm{c} \int_0^{z} H(z')^{-1} \mathrm{d}z'. \end{aligned} \tag{1.20}$$

Hence, the line-of-sight comoving distance $\chi(z)$ of a source at redshift z is given by:

$$\chi = \chi(z) = \chi_{\mathrm{H}_0} \int_0^{z} \left[(1+z')^3 \Omega_{\mathrm{m}} + (1+z')^2 \Omega_K + \Omega_\Lambda\right]^{-1/2} \mathrm{d}z' \tag{1.21}$$

whith the Hubble distance

$$\chi_{\mathrm{H}_0} = \frac{\mathrm{c}}{H_0}. \tag{1.22}$$

For universes with a monotonous expansion between $t = 0$ and $t = t_0$, there is a one-to-one correspondence between the time t, the scale factor $a(t)$, the redshift z, and line-of-sight comoving distance χ for spacetime points in the backward light cone of an observer at spacetime position $(t_0, \mathbf{0}, 0)$. Hence, either of t, a, z, or χ can be used as temporal coordinate.

From Eq. (1.21) follows that

$$\frac{\mathrm{d}\chi}{\mathrm{d}z} = \frac{\mathrm{c}}{hH_{100}} \left[(1+z)^3 \Omega_{\mathrm{m}} + (1+z)^2 \Omega_K + \Omega_\Lambda\right]^{-1/2}. \tag{1.23}$$

The comoving line-of-sight distance and comoving angular-diameter distance (1.2) can be related to a number of other distances. For example, the physical angular-diameter distance

$$D_{\mathrm{A}}(z) = a(z) f_K\big(\chi(z)\big), \tag{1.24}$$

the luminosity distance

$$D_{\mathrm{L}}(z) = a(z)^{-1} f_K\big(\chi(z)\big), \tag{1.25}$$

and the distance modulus

$$D_{\mathrm{M}}(z) = 5 \log_{10}\left[\frac{D_{\mathrm{L}}(z)}{10\,\mathrm{pc}}\right]. \tag{1.26}$$

2 Light propagation in an inhomogeneous universe

2.1 The weakly perturbed FLRW metric

As stated above I assume that the Universe can be approximated on large scales by a Friedmann-Lemaître universe. That means that I approximate the scale factor a, redshifts z, cosmological distances etc. by their expressions in the *homogeneous* Friedmann-Lemaître universe. On somewhat smaller scales relevant for gravitational light deflection, I assume that the Universe can be described by a *weakly perturbed* Friedmann-Lemaître universe. As in the homogeneous universe model, I label spacetime points using a coordinate system $(t, \beta_1, \beta_2, \chi)$ based on physical time t, two angular coordinates $\boldsymbol{\beta} = (\beta_1, \beta_2)$, and line-of-sight comoving distance χ (see Fig. 1.1). I furthermore assume that the spacetime metric of the model is given by a weakly perturbed FLRW metric:

$$g = -\left(1 + \frac{2\Phi}{\mathrm{c}^2}\right)\mathrm{c}^2\mathrm{d}t^2 + \left(1 - \frac{2\Phi}{\mathrm{c}^2}\right)a^2\left\{\mathrm{d}\chi^2 + f_K^2(\chi)\left[\mathrm{d}\beta_1^2 + \cos^2(\beta_1)\mathrm{d}\beta_2^2\right]\right\}. \tag{2.1}$$

Here, $\Phi = \Phi(t, \boldsymbol{\beta}, \chi)$ denotes the peculiar gravitational potential. Whereas the cosmological constant Λ, the curvature K, and the mean matter density $\bar{\rho}_{\mathrm{m}}^{\mathrm{ph}}$ determine the evolution of the scale a factor [via Eq.(1.9)], local deviations from the mean density determine the peculiar gravitational potential Φ. The relative matter density contrast δ_{m} is defined by:

$$\delta_{\mathrm{m}}(t, \boldsymbol{\beta}, \chi) = \frac{\rho_{\mathrm{m}}(t, \boldsymbol{\beta}, \chi) - \bar{\rho}_{\mathrm{m}}(t)}{\bar{\rho}_{\mathrm{m}}(t)}, \tag{2.2}$$

where $\rho_{\mathrm{m}}(t, \boldsymbol{\beta}, \chi)$ denotes the comoving matter density at spacetime point $(t, \boldsymbol{\beta}, \chi)$, and $\bar{\rho}_{\mathrm{m}}(t) = a^3(t)\bar{\rho}_{\mathrm{m}}^{\mathrm{ph}}(t)$ the mean comoving matter density at cosmic time t. The potential Φ and the density contrast δ_{m} then satisfy the Poisson equation

$$\triangle^{\mathrm{com}}_{(\boldsymbol{\beta},\chi)}\Phi(t, \boldsymbol{\beta}, \chi) = \frac{4\pi\mathrm{G}\bar{\rho}_{\mathrm{m}}(t)}{a(t)}\delta_{\mathrm{m}}(t, \boldsymbol{\beta}, \chi)\ \textit{+curvature? +gauge terms?}, \tag{2.3}$$

where

$$\begin{aligned}
\triangle^{\mathrm{com}}_{(\boldsymbol{\beta},\chi)} &= \frac{1}{f_K^2(\chi)\cos(\beta_1)}\frac{\partial}{\partial\beta_1}\cos(\beta_1)\frac{\partial}{\partial\beta_1} + \frac{1}{f_K^2(\chi)\cos(\beta_1)^2}\frac{\partial^2}{\partial\beta_2^2} \\
&\quad + \frac{1}{f_K^2(\chi)}\frac{\partial}{\partial\chi}f_K^2(\chi)\frac{\partial}{\partial\chi} \\
&= \frac{\partial^2}{\partial\beta_1^2} - \tan(\beta_1)\frac{\partial}{\partial\beta_1} + \frac{1}{\cos(\beta_1)^2}\frac{\partial^2}{\partial\beta_2^2} + \frac{\partial^2}{\partial\chi^2} + \frac{2}{f_K(\chi)}\frac{\mathrm{d}f_K(\chi)}{\mathrm{d}\chi}\frac{\partial}{\partial\chi}
\end{aligned} \tag{2.4}$$

denotes the 3D Laplace operator with respect to comoving coordinates. This operator can also be expressed as:

$$\triangle^{\mathrm{com}}_{(\boldsymbol{\beta},\chi)} = \frac{1}{f_K^2(\chi)}\left[\triangle^{\mathbb{S}^2}_{\boldsymbol{\beta}} + \frac{\partial}{\partial\chi}f_K^2(\chi)\frac{\partial}{\partial\chi}\right], \tag{2.5}$$

where

$$\triangle_{\boldsymbol{\beta}}^{\mathbb{S}^2} = \frac{1}{\cos(\beta_1)}\frac{\partial}{\partial\beta_1}\cos(\beta_1)\frac{\partial}{\partial\beta_1} + \frac{1}{\cos(\beta_1)^2}\frac{\partial^2}{\partial\beta_2^2} \tag{2.6}$$

denotes the Laplace Beltrami operator on the two-sphere $\mathbb{S}^2$ (see Chapter 3 for applications).

If one restricts the discussion to matter in form of a pressureless fluid of non-relativistic particles with conserved particle number, then

$$\bar{\rho}_{\rm m}(t) = \bar{\rho}_{\rm m} = \frac{3H_0^2\Omega_{\rm m}}{8\pi{\rm G}} \tag{2.7}$$

and hence,

$$\triangle^{\rm com}_{(\boldsymbol{\beta},\chi)}\Phi(t,\boldsymbol{\beta},\chi) = \frac{3H_0^2\Omega_{\rm m}}{2a(t)}\delta_{\rm m}(t,\boldsymbol{\beta},\chi). \tag{2.8}$$

A formal solution to this equation is given by:

$$\Phi(t,\boldsymbol{\beta},\chi) = -\frac{{\rm G}\bar{\rho}_{\rm m}(t)}{a(t)}\int_{\mathbb{R}^+}{\rm d}\chi' f_K^2(\chi')\int_{\mathbb{S}^2}{\rm d}^2\boldsymbol{\beta}'\frac{\delta_{\rm m}(t,\boldsymbol{\beta}',\chi')}{g_K\{d_K[(\boldsymbol{\beta},\chi),(\boldsymbol{\beta}',\chi')]\}}. \tag{2.9}$$

Here, $d_K[(\boldsymbol{\beta},\chi),(\boldsymbol{\beta}',\chi')]$ denotes the comoving distance between $(\boldsymbol{\beta},\chi)$ and $(\boldsymbol{\beta}',\chi')$, and

$$g_K(\chi) = \begin{cases} 1/\sqrt{K}\tan\left(\sqrt{K}\chi\right) & \text{for } K>0,\\ \chi & \text{for } K=0, \text{ and}\\ 1/\sqrt{-K}\tanh\left(\sqrt{-K}\chi\right) & \text{for } K<0. \end{cases} \tag{2.10}$$

2.2 Metric tensor components and Christoffel symbols

Here, I list the components of the metric (2.1) and the resulting Levi-Civita connection coefficients, a.k.a. Christoffel symbols. The only non-zero components of the metric tensor are the diagonal elements (setting c = 1):

$$\begin{aligned} g_{tt} &= [-1-2\Phi(t,\boldsymbol{\beta},\chi)]\,,\\ g_{\chi\chi} &= [1-2\Phi(t,\boldsymbol{\beta},\chi)]\,a(t)^2,\\ g_{\beta_1\beta_1} &= [1-2\Phi(t,\boldsymbol{\beta},\chi)]\,a(t)^2 f_K(\chi)^2,\\ g_{\beta_2\beta_2} &= [1-2\Phi(t,\boldsymbol{\beta},\chi)]\,a(t)^2 f_K(\chi)^2\cos(\beta_1)^2. \end{aligned} \tag{2.11}$$

The non-zero covariant components are given by:

$$\begin{aligned} g^{tt} &= [-1-2\Phi(t,\boldsymbol{\beta},\chi)]^{-1}\,,\\ g^{\chi\chi} &= [1-2\Phi(t,\boldsymbol{\beta},\chi)]^{-1}\,a(t)^{-2},\\ g^{\beta_1\beta_1} &= [1-2\Phi(t,\boldsymbol{\beta},\chi)]^{-1}\,a(t)^{-2} f_K(\chi)^{-2},\\ g^{\beta_2\beta_2} &= [1-2\Phi(t,\boldsymbol{\beta},\chi)]^{-1}\,a(t)^{-2} f_K(\chi)^{-2}\cos(\beta_1)^{-2}. \end{aligned} \tag{2.12}$$

The Christoffel symbols are calculated from a metric g by:

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}\left(g_{\nu\beta,\alpha}+g_{\alpha\nu,\beta}-g_{\alpha\beta,\nu}\right). \tag{2.13}$$

Here, indices after the comma denote (partial) derivatives, i.e.: $\circ_{,\alpha} = (\partial \circ / \partial q^\alpha)$. For a diagonal metric, the only non-zero Christoffel symbols are:

$$\Gamma^\mu_{\mu\alpha} = \frac{1}{2} g^{-1}_{\mu\mu}\, g_{\mu\mu,\alpha} \quad \forall\, \alpha, \mu, \text{ and} \tag{2.14}$$

$$\Gamma^\mu_{\alpha\alpha} = -\frac{1}{2} g^{-1}_{\mu\mu}\, g_{\alpha\alpha,\mu} \quad \forall\, \alpha, \mu : \alpha \neq \mu. \tag{2.15}$$

The non-zero Christoffel symbols of the metric (2.1) to linear order in the Newton potential Φ read:

$$\begin{aligned}
\Gamma^t_{tt} &= \Phi_{,t}\,, & \Gamma^\chi_{\chi\chi} &= -\Phi_{,\chi}\,,\\
\Gamma^t_{t\beta_1} &= \Phi_{,\beta_1}\,, & \Gamma^\chi_{\chi\beta_1} &= -\Phi_{,\beta_1}\,,\\
\Gamma^t_{t\beta_2} &= \Phi_{,\beta_2}\,, & \Gamma^\chi_{\chi\beta_2} &= -\Phi_{,\beta_2}\,,\\
\Gamma^t_{t\chi} &= \Phi_{,\chi}\,, & \Gamma^\chi_{\chi t} &= \frac{a_{,t}}{a} - \Phi_{,t}\,,\\
\Gamma^t_{\beta_1\beta_1} &= a^2 f_K^2 \left(\frac{a_{,t}}{a} - \Phi_{,t} - 4\frac{a_{,t}}{a}\Phi\right)\,, & \Gamma^\chi_{\beta_1\beta_1} &= f_K^2 \left(\Phi_{,\chi} - \frac{f_{K,\chi}}{f_K}\right)\,,\\
\Gamma^t_{\beta_2\beta_2} &= a^2 f_K^2 \cos(\beta_1)^2 \left(\frac{a_{,t}}{a} - \Phi_{,t} - 4\frac{a_{,t}}{a}\Phi\right)\,, & \Gamma^\chi_{\beta_2\beta_2} &= f_K^2 \cos(\beta_1)^2 \left(\Phi_{,\chi} - \frac{f_{K,\chi}}{f_K}\right)\,,\\
\Gamma^t_{\chi\chi} &= a^2 \left(\frac{a_{,t}}{a} - \Phi_{,t} - 4\frac{a_{,t}}{a}\Phi\right)\,, & \Gamma^\chi_{tt} &= \frac{\Phi_{,\chi}}{a^2}\,,\\
\Gamma^{\beta_1}_{\beta_1 t} &= \frac{a_{,t}}{a} - \Phi_{,t}\,, & \Gamma^{\beta_2}_{\beta_2 t} &= \frac{a_{,t}}{a} - \Phi_{,t}\,,\\
\Gamma^{\beta_1}_{\beta_1\beta_1} &= -\Phi_{,\beta_1}\,, & \Gamma^{\beta_2}_{\beta_2\beta_2} &= -\Phi_{,\beta_2}\,,\\
\Gamma^{\beta_1}_{\beta_1\beta_2} &= -\Phi_{,\beta_2}\,, & \Gamma^{\beta_2}_{\beta_2\beta_1} &= -\tan(\beta_1) - \Phi_{,\beta_1}\,,\\
\Gamma^{\beta_1}_{\beta_1\chi} &= \frac{f_{K,\chi}}{f_K} - \Phi_{,\chi}\,, & \Gamma^{\beta_2}_{\beta_2\chi} &= \frac{f_{K,\chi}}{f_K} - \Phi_{,\chi}\,,\\
\Gamma^{\beta_1}_{tt} &= \frac{\Phi_{,\beta_1}}{f_K^2 a^2}\,, & \Gamma^{\beta_2}_{tt} &= \frac{\Phi_{,\beta_2}}{f_K^2 a^2 \cos(\beta_1)^2}\,,\\
\Gamma^{\beta_1}_{\beta_2\beta_2} &= \cos(\beta_1)^2 \left(\Phi_{,\beta_1} + \tan(\beta_1)\right)\,, & \Gamma^{\beta_2}_{\beta_1\beta_1} &= \frac{\Phi_{,\beta_2}}{\cos(\beta_1)^2}\,,\\
\Gamma^{\beta_1}_{\chi\chi} &= \frac{\Phi_{,\beta_1}}{f_K^2}\,, & \Gamma^{\beta_2}_{\chi\chi} &= \frac{\Phi_{,\beta_2}}{f_K^2 \cos(\beta_1)^2}\,.
\end{aligned}$$

2.3 The geodesic equations

In Einstein's General Relativity, photon paths are null geodesics of the spacetime metric. Using an affine parameter λ to parametrise the light path $\{q^\mu\}$,

$$q^\mu(\lambda) = \big(t(\lambda), \beta_1(\lambda), \beta_2(\lambda), \chi(\lambda)\big), \tag{2.16}$$

the null equation reads (e.g. Weinberg 1972):

$$0 = \sum_{\alpha\beta} g_{\alpha\beta} \frac{\mathrm{d}q^\alpha}{\mathrm{d}\lambda} \frac{\mathrm{d}q^\beta}{\mathrm{d}\lambda}. \tag{2.17}$$

The geodesic equation for the light path reads:

$$\frac{\mathrm{d}^2 q^\mu}{\mathrm{d}\lambda^2} = -\sum_{\alpha\beta} \Gamma^\mu_{\alpha\beta} \frac{\mathrm{d}q^\alpha}{\mathrm{d}\lambda} \frac{\mathrm{d}q^\beta}{\mathrm{d}\lambda}, \tag{2.18}$$

If the light path is parametrised by a non-affine parameter $\sigma = \sigma(\lambda)$, the geodesic equation becomes:

$$\begin{aligned} \frac{\mathrm{d}^2 q^\mu}{\mathrm{d}\sigma^2} &= \frac{\mathrm{d}^2 q^\mu}{\mathrm{d}\lambda^2} \left(\frac{\mathrm{d}\lambda}{\mathrm{d}\sigma}\right)^2 + \frac{\mathrm{d}q^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}^2\lambda}{\mathrm{d}\sigma^2} \\ &= -\sum_{\alpha\beta} \Gamma^\mu_{\alpha\beta} \frac{\mathrm{d}q^\alpha}{\mathrm{d}\lambda} \frac{\mathrm{d}q^\beta}{\mathrm{d}\lambda} \left(\frac{\mathrm{d}\lambda}{\mathrm{d}\sigma}\right)^2 + \frac{\mathrm{d}q^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}\lambda}{\mathrm{d}\sigma} \left(\frac{\mathrm{d}\lambda}{\mathrm{d}\sigma}\right)^{-1} \frac{\mathrm{d}^2\lambda}{\mathrm{d}\sigma^2} \\ &= -\sum_{\alpha\beta} \Gamma^\mu_{\alpha\beta} \frac{\mathrm{d}q^\alpha}{\mathrm{d}\sigma} \frac{\mathrm{d}q^\beta}{\mathrm{d}\sigma} + h(\sigma) \frac{\mathrm{d}q^\mu}{\mathrm{d}\sigma} \end{aligned} \tag{2.19}$$

with

$$h(\sigma) = \left(\frac{\mathrm{d}\lambda}{\mathrm{d}\sigma}\right)^{-1} \frac{\mathrm{d}^2\lambda}{\mathrm{d}\sigma^2}. \tag{2.20}$$

Often it is (possible and) convenient to choose a coordinate q^σ as parameter. Then, the geodesic equation for q^σ,

$$0 = \frac{\mathrm{d}^2 q^\sigma}{\mathrm{d}(q^\sigma)^2} = -\sum_{\alpha\beta} \Gamma^\sigma_{\alpha\beta} \frac{\mathrm{d}q^\alpha}{\mathrm{d}q^\sigma} \frac{\mathrm{d}q^\beta}{\mathrm{d}q^\sigma} + h(\sigma), \tag{2.21}$$

can be used to calculate the function $f(q^\sigma)$:

$$h(q^\sigma) = \sum_{\alpha\beta} \Gamma^\sigma_{\alpha\beta} \frac{\mathrm{d}q^\alpha}{\mathrm{d}q^\sigma} \frac{\mathrm{d}q^\beta}{\mathrm{d}q^\sigma}. \tag{2.22}$$

This yields:

$$\frac{\mathrm{d}^2 q^\mu}{\mathrm{d}(q^\sigma)^2} = -\sum_{\alpha\beta} \left(\Gamma^\mu_{\alpha\beta} - \Gamma^\sigma_{\alpha\beta} \frac{\mathrm{d}q^\mu}{\mathrm{d}q^\sigma}\right) \frac{\mathrm{d}q^\alpha}{\mathrm{d}q^\sigma} \frac{\mathrm{d}q^\beta}{\mathrm{d}q^\sigma}. \tag{2.23}$$

2.4 Ordinary differential equations

Calculating the photon path by solving the geodesic equation (2.18) or (2.23) requires solving a *nonlinear* system of ordinary differential equations (ODEs). For a homogeneous universe, the problem is sufficiently symmetric to allow one to solve the nonlinear system exactly, but there is little hope to find an explicit and exact solution in a general inhomogeneous universe. However, one can formally transform the problem into an inhomogeneous *linear* system of ordinary differential equations with all non-linearities hidden in the inhomogeneity or source term. Approximations like those discussed in Sec. 2.5 can then be used to justify this approach.

The general task is to find a sufficiently smooth function

$$\boldsymbol{x} : \mathbb{R} \to \mathbb{R}^d : t \mapsto \boldsymbol{x}(t) \tag{2.24}$$

that satisfies the 1st-order ODE (higher-order ODEs can be recast into a 1st-order ODE by a simple trick):

$$\mathbf{0} = \boldsymbol{F}\left(t, \boldsymbol{x}(t), \frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t}\right). \tag{2.25}$$

Here,

$$\boldsymbol{F} : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d : (t, \boldsymbol{x},) \mapsto \boldsymbol{F}(t, \boldsymbol{x}, \boldsymbol{x}') \tag{2.26}$$

is a possibly nonlinear function of the parameter t, the function $\boldsymbol{x}(t)$, and its derivative $\dot{\boldsymbol{x}}(t) = \mathrm{d}\boldsymbol{x}/\mathrm{d}t$. Furthermore, the function $\boldsymbol{x}(t)$ is required to satisfy the initial conditions

$$\left.\frac{\mathrm{d}^i \boldsymbol{x}(t)}{\mathrm{d}t}\right|_{t=t_0} = \boldsymbol{x}_{0,i} \qquad i = 0, 1. \tag{2.27}$$

The approach taken here treats the non-linear parts of the function $\boldsymbol{F}$ as a perturbation to a linear function. The function $\boldsymbol{F}$ is split into parts that are linear in $\boldsymbol{x}$ and $\dot{\boldsymbol{x}}$, and the rest:

$$\mathbf{0} = \mathbf{c}_1(t)\dot{\boldsymbol{x}}(t) + \mathbf{c}_0(t)\boldsymbol{x}(t) + \boldsymbol{F}^{(\neq 1)}\big(t, \boldsymbol{x}(t), \dot{\boldsymbol{x}}(t)\big) \tag{2.28}$$

If $\mathbf{c}_1(t)$ is invertible, the differential equation can be rearranged to read:

$$\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} = \mathbf{A}(t)\boldsymbol{x}(t) + \boldsymbol{\Phi}\big(t, \boldsymbol{x}(t), \dot{\boldsymbol{x}}(t)\big), \tag{2.29}$$

where $\mathbf{A} = -\mathbf{c}_1^{-1}\mathbf{c}_0$, and $\boldsymbol{\Phi} = -\mathbf{c}_1^{-1}\boldsymbol{F}^{(\neq 1)}$.

The standard way to obtain a solution is to first find $\mathbf{U}(t, t_0)$ with

$$\frac{\mathrm{d}\mathbf{U}(t, t_0)}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{U}(t, t_0) \qquad \text{and} \qquad \mathbf{U}(t_0, t_0) = \mathbf{1}. \tag{2.30}$$

If, for example, $\mathbf{A}(t) = \mathbf{A} = \text{const.}$,

$$\mathbf{U}(t, t_0) = \mathbf{U}(t - t_0) = \exp\left[(t - t_0)\mathbf{A}\right]. \tag{2.31}$$

Then, the solution to the ODE can be expressed as:

$$\boldsymbol{x}(t) = \mathbf{U}(t, t_0)\boldsymbol{x}(t_0) + \mathbf{U}(t, t_0)\int_{t_0}^{t} \mathbf{U}(t', t_0)^{-1}\boldsymbol{\Phi}\big(t', \boldsymbol{x}(t'), \dot{\boldsymbol{x}}(t')\big)\mathrm{d}t'. \tag{2.32}$$

This is indeed a solution, since

$$\begin{aligned}
\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{U}(t, t_0)\boldsymbol{x}(t_0) + \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{U}(t, t_0)\int_{t_0}^{t} \mathbf{U}(t', t_0)^{-1}\boldsymbol{\Phi}\big(t', \boldsymbol{x}(t'), \dot{\boldsymbol{x}}(t')\big)\mathrm{d}t' \\
&= \mathbf{A}(t)\mathbf{U}(t, t_0)\boldsymbol{x}(t_0) + \mathbf{A}(t)\mathbf{U}(t, t_0)\int_{t_0}^{t} \mathbf{U}(t', t_0)^{-1}\boldsymbol{\Phi}\big(t', \boldsymbol{x}(t'), \dot{\boldsymbol{x}}(t')\big)\mathrm{d}t' \\
&\quad + \mathbf{U}(t, t_0)\mathbf{U}(t, t_0)^{-1}\boldsymbol{\Phi}\big(t, \boldsymbol{x}(t), \dot{\boldsymbol{x}}(t)\big) \\
&= \mathbf{A}(t)\boldsymbol{x}(t) + \boldsymbol{\Phi}\big(t, \boldsymbol{x}(t), \dot{\boldsymbol{x}}(t)\big).
\end{aligned} \tag{2.33}$$

The first term of the full solution (2.32) is provided by the solution to the homogeneous linear ODE with the suitable initial conditions. The second term is a solution to the full inhomogeneous ODE with vanishing initial conditions.

The general solution (2.32) can also be written as

$$\boldsymbol{y}(t) = \mathbf{U}(t,t_0)\boldsymbol{y}(t_0) + \int_{-\infty}^{\infty} \mathbf{G}(t,t',t_0)\boldsymbol{\Phi}\big(t', \boldsymbol{y}(t'), \dot{\boldsymbol{y}}(t')\big)\mathrm{d}t', \tag{2.34}$$

with the Green's function

$$\begin{aligned}\mathbf{G}(t,t',t_0) = &\quad \Theta(t-t')\Theta(t'-t_0)\mathbf{U}(t,t_0)\mathbf{U}(t',t_0)^{-1} \\ &- \Theta(t_0-t')\Theta(t'-t)\mathbf{U}(t,t_0)\mathbf{U}(t',t_0)^{-1}.\end{aligned} \tag{2.35}$$

In the special case of constant $\mathbf{A}$,

$$\begin{aligned}\mathbf{G}(t,t',t_0) = &\quad \Theta(t-t')\Theta(t'-t_0)\exp\big[(t-t')\mathbf{A}\big] \\ &- \Theta(t_0-t')\Theta(t'-t)\exp\big[(t-t')\mathbf{A}\big].\end{aligned} \tag{2.36}$$

2.4.1 Example: Comoving transverse distance

As shown later, the comoving transverse distance $\boldsymbol{x} = (x_1, x_2)$ as a function of comoving line-of-sight distance χ obeys

$$\frac{\mathrm{d}^2 x_i(\chi)}{\mathrm{d}\chi^2} + K x_i(\chi) = -\frac{2}{\mathrm{c}}\Phi_{,x_i}\big(\boldsymbol{x}(\chi),\chi\big), \tag{2.37}$$

where K denotes the global curvature of space and Φ denotes the peculiar Newtownian potential of the weakly perturbed FLRW metric.

Consider the initial value problem with ODE (2.37) and

$$x_i(0) = 0 \text{ and } \left.\frac{\mathrm{d}x_i(\chi)}{\mathrm{d}\chi}\right|_0 = \theta_i. \tag{2.38}$$

Let us write Eq. (2.37) in the vector form (2.29),

$$\frac{\mathrm{d}}{\mathrm{d}\chi}\begin{pmatrix} x_i(\chi) \\ x_{i,\chi}(\chi)\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -K & 0\end{pmatrix}\begin{pmatrix} x_i(\chi) \\ x_{i,\chi}(\chi)\end{pmatrix} - \frac{2}{\mathrm{c}}\begin{pmatrix} 0 \\ \Phi_{,x_i}\big(\boldsymbol{x}(\chi),\chi\big)\end{pmatrix}, \tag{2.39}$$

and identify terms in Eq. (2.37) with terms in Eq. (2.29). Then,

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -K & 0\end{pmatrix}, \tag{2.40}$$

$$\mathbf{U}(\chi,\chi_0 0) = \exp\left[(\chi-\chi_0)\mathbf{A}\right] = \begin{pmatrix} \cos\big[\sqrt{K}(\chi-\chi_0)\big] & f_K(\chi-\chi_0) \\ K f_K(\chi-\chi_0) & \cos\big[\sqrt{K}(\chi-\chi_0)\big]\end{pmatrix}, \text{ and} \tag{2.41}$$

$$\mathbf{U}(\chi,\chi_0)\mathbf{U}(\chi',\chi_0)^{-1} = \mathbf{U}(\chi,\chi') = \begin{pmatrix} \cos\big[\sqrt{K}(\chi-\chi')\big] & f_K(\chi-\chi') \\ K f_K(\chi-\chi') & \cos\big[\sqrt{K}(\chi-\chi')\big]\end{pmatrix}. \tag{2.42}$$

Hence,

$$\begin{aligned}\begin{pmatrix} x_i(\chi) \\ x_{i,\chi}(\chi)\end{pmatrix} = &\begin{pmatrix} \cos\big[\sqrt{K}(\chi-\chi_0)\big] & f_K(\chi-\chi_0) \\ K f_K(\chi-\chi_0) & \cos\big[\sqrt{K}(\chi-\chi_0)\big]\end{pmatrix}\begin{pmatrix} x_i(\chi_0) \\ x_{i,\chi}(\chi_0)\end{pmatrix} \\ &- \frac{2}{\mathrm{c}}\int_{\chi_0}^{\chi}\mathrm{d}\chi' \begin{pmatrix} \cos\big[\sqrt{K}(\chi-\chi')\big] & f_K(\chi-\chi') \\ K f_K(\chi-\chi') & \cos\big[\sqrt{K}(\chi-\chi')\big]\end{pmatrix}\begin{pmatrix} 0 \\ \Phi_{,x_i}\big(\boldsymbol{x}(\chi'),\chi'\big)\end{pmatrix}.\end{aligned} \tag{2.43}$$

Using the initial values $\chi_0 = 0$, $x_i(0) = 0$, and $x_{i,\chi}(0) = \theta_i$,

$$x_i(\boldsymbol{\theta},\chi) = f_K(\chi)\theta_i - \frac{2}{\mathrm{c}}\int_0^{\chi} f_K(\chi-\chi')\Phi_{,x_i}\big(\boldsymbol{x}(\boldsymbol{\theta},\chi'),\chi'\big)\mathrm{d}\chi'. \tag{2.44}$$

2.5 The weak-deflection approximation

Now let us return to discuss the light propagation in an inhomogeneous universe. If a light ray reaching the observer at $(t_0, 0, 0, 0)$ in the global coordinate system $q^\mu = (t, \beta_1, \beta_2, \chi)$ is never strongly deflected, its direction $(\mathrm{d}q^\mu/\mathrm{d}\lambda)$ is always almost 'radial'. This means that the comoving radial coordinate χ can be used to parametrise the light ray. In the following, This is equivalent to the "small deflection angle" approximation (see, e.g., Seljak 1994) for light paths reaching the observer. To be precise, I neglect

- all terms of second and higher order in the peculiar potential Φ,
- all terms of second and higher order in the angular velocities $(\mathrm{d}\boldsymbol{\beta}/\mathrm{d}\chi)$, and
- all terms of first and higher order in both the peculiar potential Φ and angular velocities $(\mathrm{d}\boldsymbol{\beta}/\mathrm{d}\chi)$

in the calculation of the angular coordinates of the light path (see Dodelson et al. 2005, for a discussion of the importance of additional contributions neglected by these approximations). The first approximation accounts for the fact that the metric (2.1) is valid only in the weak-field limit $\Phi/\mathrm{c}^2 \ll 1$. The second approximation ignores that even in absence of a varying gravitational potential Φ , global curvature K, or time-dependent scale factor a, the equations of motion for photons with non-radial direction look very complicated in spherical coordinates. The third approximation ignores that for photons travelling not exactly in the radial direction, there is a non-zero χ-component contributing to the deflection of the photon in the angular direction. The first approximation is justified if the peculiar gravitational field strength is small. The second and third approximations are justified if the photon direction has only very small angular components.

2.5.1 The t-component

The null-equation for the light path reads:

$$\begin{aligned}
0 &= g_{\alpha\beta} \frac{\mathrm{d}q^\alpha}{\mathrm{d}\chi} \frac{\mathrm{d}q^\beta}{\mathrm{d}\chi} \\
&= \left[-1 - 2\Phi(t, \boldsymbol{\beta}, \chi)\right] \left(\frac{\mathrm{d}t}{\mathrm{d}\chi}\right)^2 \\
&\quad + \left[1 - 2\Phi(t, \boldsymbol{\beta}, \chi)\right] a(t)^2 \left[1 + f_K(\chi)^2 \left(\frac{\mathrm{d}\beta_1}{\mathrm{d}\chi}\right)^2 + f_K(\chi)^2 \cos(\beta_1)^2 \left(\frac{\mathrm{d}\beta_2}{\mathrm{d}\chi}\right)^2\right].
\end{aligned} \tag{2.45}$$

Hence,

$$\begin{aligned}
\left(\frac{\mathrm{d}t}{\mathrm{d}\chi}\right) &= \pm \left[1 - 2\Phi(t, \boldsymbol{\beta}, \chi)\right] a(t) \\
&\quad \times \sqrt{1 + f_K(\chi)^2 \left[\left(\frac{\mathrm{d}\beta_1}{\mathrm{d}\chi}\right)^2 + \cos(\beta_1)^2 \left(\frac{\mathrm{d}\beta_2}{\mathrm{d}\chi}\right)^2\right]} + \mathcal{O}\left[\Phi^2\right].
\end{aligned} \tag{2.46}$$

To zeroth order in Φ, the coordinate $t = t(\chi)$ along the path of a photon reaching the observer is given by:

$$\frac{\mathrm{d}t}{\mathrm{d}\chi} = -a(t). \tag{2.47}$$

Hence,

$$t(\chi) = t_0 - \int_0^\chi a(t(\chi'))\mathrm{d}\chi'. \tag{2.48}$$

This approximation for $t = t(\chi)$ is sufficiently accurate for calculating the cosmological emission time of a photon received by an observer. However, within this approximation, all photons that are simultaneously emitted by a source reach the observer simultaneously. Hence, there is no time delay between multiple images of a strongly lensed source (contrary to observations). To calculate the time delay with the required accuracy, one not only has to include terms to first order in the potential Φ (the 'potential time delay'), but also terms of second order in the angular velocities (the 'geometric time delay'). The result is an almost exact travel-time equation (terms of higher order in Φ are neglected):

$$t(\chi) = t_0 - \int_0^\chi a\big(t(\chi')\big)n(\chi')\mathrm{d}\chi' \tag{2.49a}$$

with radial refractive index

$$n(\chi) = \big[1 - 2\Phi\big(t(\chi), \boldsymbol{\beta}(\chi), \chi\big)\big]\sqrt{1 + f_K^2(\chi)\left[\left(\frac{\mathrm{d}\beta_1}{\mathrm{d}\chi}\right)^2 + \cos^2(\beta_1)\left(\frac{\mathrm{d}\beta_2}{\mathrm{d}\chi}\right)^2\right]}, \tag{2.49b}$$

which can be expanded as:

$$n(\chi) = 1 - 2\Phi\big(t(\chi), \boldsymbol{\beta}(\chi), \chi\big) + \frac{f_K^2(\chi)}{2}\left[\left(\frac{\mathrm{d}\beta_1}{\mathrm{d}\chi}\right)^2 + \cos^2(\beta_1)\left(\frac{\mathrm{d}\beta_2}{\mathrm{d}\chi}\right)^2\right] + \dots \tag{2.49c}$$

Time delays are discussed in Chapter 6.

2.5.2 The β_1-component (short derivation)

The geodesic equation for the coordinate $\beta_1 = \beta_1(\chi)$ reads:

$$\begin{aligned}
\frac{\mathrm{d}^2\beta_1}{\mathrm{d}\chi^2} &= \frac{\mathrm{d}\beta_1}{\mathrm{d}\chi}\sum_{\alpha\beta}\Gamma^\chi_{\alpha\beta}\frac{\mathrm{d}q^\alpha}{\mathrm{d}\chi}\frac{\mathrm{d}q^\beta}{\mathrm{d}\chi} - \sum_{\alpha\beta}\Gamma^{\beta_1}_{\alpha\beta}\frac{\mathrm{d}q^\alpha}{\mathrm{d}\chi}\frac{\mathrm{d}q^\beta}{\mathrm{d}\chi} \\
&= \frac{\mathrm{d}\beta_1}{\mathrm{d}\chi}\Big[-2\Gamma^\chi_{wt}a + \Gamma^\chi_{\chi\chi} + \Gamma^\chi_{tt}a^2 + 2\Gamma^{\beta_1}_{\beta_1 t}a - 2\Gamma^{\beta_1}_{\beta_1\chi}\Big] \\
&\quad - \Gamma^{\beta_1}_{tt}a^2 - \Gamma^{\beta_1}_{\chi\chi} + \mathcal{O}\big[\Phi^2\big] + \mathcal{O}\left[\left(\frac{\mathrm{d}\boldsymbol{\beta}}{\mathrm{d}\chi}\right)^2\right] \\
&= \frac{\mathrm{d}\beta_1}{\mathrm{d}\chi}\left[-2\left(\frac{a_{,t}}{a} - \Phi_{,t}\right)a - \Phi_{,\chi} + \frac{\Phi_{,\chi}}{a^2}a^2\right. \\
&\qquad\left. +2\left(\frac{a_{,t}}{a} - \Phi_{,t}\right)a - 2\left(\frac{f_{K,\chi}}{f_K} - \Phi_{,\chi}\right)\right] \\
&\quad - \frac{\Phi_{,\beta_1}}{f_K^2 a^2}a^2 - \frac{\Phi_{,\beta_1}}{f_K^2} + \mathcal{O}\big[\Phi^2\big] + \mathcal{O}\left[\left(\frac{\mathrm{d}\boldsymbol{\beta}}{\mathrm{d}\chi}\right)^2\right] \\
&= -2\frac{\mathrm{d}\beta_1}{\mathrm{d}\chi}\frac{f_{K,\chi}}{f_K} - 2\left[\frac{\Phi_{,\beta_1}}{f_K^2} + \Phi_{,\chi}\frac{\mathrm{d}\beta_1}{\mathrm{d}\chi}\right] + \mathcal{O}\big[\Phi^2\big] + \mathcal{O}\left[\left(\frac{\mathrm{d}\boldsymbol{\beta}}{\mathrm{d}\chi}\right)^2\right] \\
&= -2\frac{\mathrm{d}\beta_1}{\mathrm{d}\chi}\frac{f_{K,\chi}}{f_K} - 2\frac{\Phi_{,\beta_1}}{f_K^2} + \mathcal{O}\big[\Phi^2\big] + \mathcal{O}\left[\Phi\frac{\mathrm{d}\boldsymbol{\beta}}{\mathrm{d}\chi}\right] + \mathcal{O}\left[\left(\frac{\mathrm{d}\boldsymbol{\beta}}{\mathrm{d}\chi}\right)^2\right].
\end{aligned} \tag{2.50}$$

Now it's time for a change of variables:

$$\beta_1 \mapsto x_1 = f_K(\chi)\beta_1. \tag{2.51}$$

The geodesic equation for the new coordinate x_1 is then given by:

$$\begin{aligned}
\frac{\mathrm{d}^2 x_1}{\mathrm{d}\chi^2} + K x_1 &= K x_1 + \frac{\mathrm{d}^2[f_K(\chi)\beta_1(\chi)]}{\mathrm{d}\chi^2} \\
&= K x_1 + f_{K,\chi\chi}\beta_1 + 2 f_{K,\chi}\frac{\mathrm{d}\beta_1}{\mathrm{d}\chi} + f_K \frac{\mathrm{d}^2\beta_1}{\mathrm{d}\chi^2} \\
&= K x_1 - K f_K \beta_1 + 2 f_{K,\chi}\frac{\mathrm{d}\beta_1}{\mathrm{d}\chi} - 2\frac{\mathrm{d}\beta_1}{\mathrm{d}\chi} f_{K,\chi} \\
&\quad - 2\frac{\Phi_{,\beta_1}}{f_K} + \mathcal{O}\left[\Phi^2\right] + \mathcal{O}\left[\Phi \frac{\mathrm{d}\boldsymbol{\beta}}{\mathrm{d}\chi}\right] + \mathcal{O}\left[\left(\frac{\mathrm{d}\boldsymbol{\beta}}{\mathrm{d}\chi}\right)^2\right] \\
&= -2\Phi_{,x_1} + \mathcal{O}\left[\Phi^2\right] + \mathcal{O}\left[\Phi \frac{\mathrm{d}\boldsymbol{\beta}}{\mathrm{d}\chi}\right] + \mathcal{O}\left[\left(\frac{\mathrm{d}\boldsymbol{\beta}}{\mathrm{d}\chi}\right)^2\right].
\end{aligned} \tag{2.52}$$

Neglecting the terms of second and higher order, one obtains the (familiar) linear differential equation (with c put back in):

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\chi^2} + K\right) x_1(\chi) = -\frac{2}{\mathrm{c}^2}\Phi_{,x_1}\big(t(\chi), \boldsymbol{x}(\chi), \chi\big). \tag{2.53}$$

The solution of Eq.(2.53) with initial conditions

$$x_1(\chi)\big|_{\chi=0} = 0 \text{ and} \tag{2.54a}$$

$$\left.\frac{\mathrm{d}x_1(\chi)}{\mathrm{d}\chi}\right|_{\chi=0} = \theta_1 \tag{2.54b}$$

reads:

$$x_1(\chi) = f_K(\chi)\theta_1 - \frac{2}{\mathrm{c}^2}\int_0^\chi f_K(\chi - \chi')\Phi_{,x_1}\big(t(\chi), \boldsymbol{x}(\chi), \chi\big)\mathrm{d}\chi'. \tag{2.55}$$

A transformation back to angular coordinates yields:

$$\beta_1(\chi) = \theta_1 - \frac{2}{\mathrm{c}^2}\int_0^\chi \frac{f_K(\chi - \chi')}{f_K(\chi) f_K(\chi')}\Phi_{,\beta_1}\big(t(\chi), \boldsymbol{\beta}(\chi), \chi\big)\mathrm{d}\chi'. \tag{2.56}$$

2.5.3 The β_2-component

In the radial approximation, the geodesic equation for the coordinate $\beta_2 = \beta_2(\chi)$ has exactly the same form as the geodesic equation for β_1. Its solution is thus:

$$\beta_2(\chi) = \theta_2 - \frac{2}{\mathrm{c}^2}\int_0^\chi \frac{f_K(\chi - \chi')}{f_K(\chi) f_K(\chi')}\Phi_{,\beta_2}\big(t(\chi), \boldsymbol{\beta}(\chi), \chi\big)\mathrm{d}\chi'. \tag{2.57}$$

2.6 The lens equation

Combining the results of the preceding sections, one obtains the following: Assume an observer at spacetime point $(t_0, \mathbf{0}, 0)$ in a universe with a weakly perturbed FLRW metric (2.1) receiving a photon with incident direction $\boldsymbol{\beta}$ from a source at redshift z. The position $(t, \boldsymbol{\beta}, \chi)$ of the source is then found by tracing back the photon along the light path $(t(\chi), \boldsymbol{\beta}(\boldsymbol{\beta}, \chi), \chi)$, where

$$\beta_i(\boldsymbol{\theta}, \chi) = \theta_i - \frac{2}{\mathrm{c}^2} \int_0^\chi \frac{f_K(\chi - \chi')}{f_K(\chi) f_K(\chi')} \Phi_{,\beta_i}(t(\chi'), \boldsymbol{\beta}(\boldsymbol{\theta}, \chi'), \chi') \mathrm{d}\chi'. \tag{2.58}$$

For a given mass distribution $\rho(t, \boldsymbol{\beta}, \chi)$ generating a gravitational potential $\Phi(t, \boldsymbol{\beta}, \chi)$, this equation can be integrated numerically.

The relative position of nearby light rays is quantified by:

$$\begin{aligned} A_{ij}(\boldsymbol{\theta}, \chi) &= \frac{\partial \beta_i(\boldsymbol{\theta}, \chi)}{\partial \theta_j} \\ &= \delta_{ij} - \frac{2}{\mathrm{c}^2} \int_0^\chi \frac{f_K(\chi - \chi')}{f_K(\chi) f_K(\chi')} \Phi_{,\beta_i \beta_k}(t(\chi'), \boldsymbol{\beta}(\boldsymbol{\theta}, \chi'), \chi') A_{kj}(\boldsymbol{\theta}, \chi') \, \mathrm{d}\chi'. \end{aligned} \tag{2.59}$$

The image distortions of small light sources can be described by the *distortion matrix* $\mathsf{A}(\boldsymbol{\theta}, \chi) = \big(A_{ij}(\boldsymbol{\theta}, \chi)\big)$, which will be discussed in more detail in Chap. 4.

For the transverse comoving coordinates

$$\boldsymbol{x}(\boldsymbol{\theta}, \chi) = f_K(\chi) \boldsymbol{\beta}(\boldsymbol{\theta}, \chi), \tag{2.60}$$

the lens equation yields:

$$x_i(\boldsymbol{\theta}, \chi) = f_K(\chi) \theta_i - \frac{2}{\mathrm{c}^2} \int_0^\chi f_K(\chi - \chi') \Phi_{,x_i}(t(\chi'), \boldsymbol{x}(\boldsymbol{\theta}, \chi'), \chi') \mathrm{d}\chi', \tag{2.61}$$

and

$$\begin{aligned} \frac{\partial x_i(\boldsymbol{\theta}, \chi)}{\partial \theta_j} &= f_K(\chi) \delta_{ij} \\ &\quad - \frac{2}{\mathrm{c}^2} \int_0^\chi f_K(\chi - \chi') \Phi_{,x_i x_k}(t(\chi'), \boldsymbol{x}(\boldsymbol{\theta}, \chi'), \chi') \frac{\partial x_k(\boldsymbol{\theta}, \chi')}{\partial \theta_j} \mathrm{d}\chi'. \end{aligned} \tag{2.62}$$

2.7 The lens map

2.7.1 Definition

The lens equation (2.58) can be interpreted as a map:

$$\mathcal{L} : \mathbb{S}^2 \times \mathbb{I} \to \mathbb{S}^2 : (\boldsymbol{\theta}, \chi) \mapsto \boldsymbol{\beta} = \boldsymbol{\beta}(\boldsymbol{\theta}, \chi), \tag{2.63}$$

where $\mathbb{S}^2$ denotes the 2-sphere, and $\mathbb{I} = [0, \chi_{\max}) \in \mathbb{R}$ the half-open interval between zero and the largest comoving distance $\chi_{\max}$ from which light has reached the observer so far from all directions.

2.7.2 Deflection angle

The lens map (2.63) can be written as:

$$\mathcal{L} : \mathbb{S}^2 \times \mathbb{I} \to \mathbb{S}^2 : (\boldsymbol{\theta}, \chi) \mapsto \boldsymbol{\beta} = \boldsymbol{\theta} + \boldsymbol{\alpha}(\boldsymbol{\theta}, \chi). \tag{2.64}$$

The (scaled) deflection angle

$$\alpha_i(\boldsymbol{\theta}, \chi) = -\frac{2}{\mathrm{c}^2} \int_0^{\chi} \frac{f_K(\chi - \chi')}{f_K(\chi) f_K(\chi')} \Phi_{,\beta_i}(t(\chi'), \boldsymbol{\beta}(\boldsymbol{\theta}, \chi'), \chi') \mathrm{d}\chi' \tag{2.65}$$

encodes the deviation of the angular photon positions caused by a given gravitational field Φ.

2.7.3 Cosmological constant and light deflection

If Eq. (2.8) holds, the potential Φ depends only on the matter distribution and the scale factor, but not on the cosmological constant Λ. As a consequence, there is no effect of the cosmological constant Λ on the light deflection. Only the involved distances along the l.o.s. may be affected by Λ.

2.7.4 Mass-sheet degeneracy

Consider an affine linear transformation $\mathcal{D}^{(\boldsymbol{\beta}_0, \mathbf{D})}$, parametrized by a shift vector $\boldsymbol{\beta}_0$ and a distortion matrix $\mathbf{D}$, of the source position $\boldsymbol{\beta}$ in a region $\mathbb{U} \subset \mathbb{S}^2$ to a new source position $\boldsymbol{\beta}^{(\boldsymbol{\beta}_0, \mathbf{D})}$:

$$\mathcal{D}^{(\boldsymbol{\beta}_0, \mathbf{D})} : \mathbb{U} \to \mathbb{S}^2 : \boldsymbol{\beta} \mapsto \boldsymbol{\beta}^{(\boldsymbol{\beta}_0, \mathbf{D})} = \mathbf{D}\boldsymbol{\beta} + \boldsymbol{\beta}_0. \tag{2.66}$$

Define a new lens map

$$\mathcal{L}^{(\boldsymbol{\beta}_0, \mathbf{D})} = \mathcal{D}^{(\boldsymbol{\beta}_0, \mathbf{D})} \circ \mathcal{L} : \mathbb{U} \to \mathbb{S}^2 : (\boldsymbol{\theta}, \chi) \mapsto \boldsymbol{\beta}^{(\boldsymbol{\beta}_0, \mathbf{D})} = \mathbf{D}\boldsymbol{\beta}(\boldsymbol{\theta}, \chi) + \boldsymbol{\beta}_0. \tag{2.67}$$

Then

$$\begin{aligned} \boldsymbol{\beta}^{(\boldsymbol{\beta}_0, \mathbf{D})} &= \mathbf{D}\big[\boldsymbol{\theta} + \boldsymbol{\alpha}(\boldsymbol{\theta}, \chi)\big] + \boldsymbol{\beta}_0 \\ &= \boldsymbol{\theta} + \mathbf{D}\boldsymbol{\alpha}(\boldsymbol{\theta}, \chi) + (\mathbf{D} - \mathbf{1})\,\boldsymbol{\theta} + \boldsymbol{\beta}_0 \\ &= \boldsymbol{\theta} + \boldsymbol{\alpha}^{(\boldsymbol{\beta}_0, \mathbf{D})}(\boldsymbol{\theta}, \chi), \end{aligned} \tag{2.68}$$

with

$$\boldsymbol{\alpha}^{(\boldsymbol{\beta}_0, \mathbf{D})}(\boldsymbol{\theta}, \chi) = \mathbf{D}\boldsymbol{\alpha}(\boldsymbol{\theta}, \chi) + (\mathbf{D} - \mathbf{1})\,\boldsymbol{\theta} + \boldsymbol{\beta}_0. \tag{2.69}$$

Complete information on the lens map and hence the underlying matter distribution can be obtained from known image and source positions. However, often only the positions and shapes of objects in the image plane can be inferred from the data but not their source positions, shapes, or sizes. Thus, for every lens map satisfying the observational constraints on image positions and shapes, there is a whole family of lens maps that all yield the same observed images, but for different source positions and shapes. If one restricts attention to simple scale transformations $\mathcal{D}^{(\boldsymbol{\beta}_0, \mathbf{D})}$ with $\boldsymbol{\beta}_0 = \mathbf{0}$, and $\mathbf{D} = (1 - \lambda)\mathbf{1}$, this is known as *mass sheet degeneracy*.

3 Approximations to the lens equation

3.1 Plane-sky approximation

The plane-sky approximation approximates the angular part of the geometry on a small part on the sphere by the geometry of a plane. For the part on the sphere with $|\boldsymbol{\beta}| \ll 1$, this implies:

$$g \approx -\mathrm{c}^2 \mathrm{d}t^2 + a^2 \left\{ \mathrm{d}\chi^2 + f_K^2(\chi) \left[\mathrm{d}\beta_1^2 + \mathrm{d}\beta_2^2 \right] \right\}. \tag{3.1}$$

The plane-sky approximation is not useful by itself, but often simplifies calculations involving approximations to the lens equation discussed in the following.

3.2 The sudden-deflection approximation

A crucial step in some of the following approximations to the lens equation involves the approximation of the continuous deflection along the light path by a discrete deflection at the l.o.s. distance of the deflector, which I call *sudden-deflection approximation*. In the plane-sky approximation, the integral form of the sudden-deflection approximation reads:

$$\begin{aligned}
\psi(\boldsymbol{\beta}) &\equiv \int_{\chi_0}^{\chi_1} \frac{\mathrm{d}\chi}{g_K\{d_K[(\boldsymbol{\beta},\chi),(\boldsymbol{\beta}',\chi')]\}} \approx \int_{\chi_0}^{\chi_1} \frac{\mathrm{d}\chi}{\sqrt{f_K^2(\chi')|\boldsymbol{\beta}-\boldsymbol{\beta}'|^2 + |\chi-\chi'|^2}} \\
&= \operatorname{arsinh}\left[\frac{\chi'-\chi_0}{f_K(\chi')\,|\boldsymbol{\beta}-\boldsymbol{\beta}'|}\right] + \operatorname{arsinh}\left[\frac{\chi_1-\chi'}{f_K(\chi')\,|\boldsymbol{\beta}-\boldsymbol{\beta}'|}\right] \\
&\approx \operatorname{sign}(\chi'-\chi_0)\log\left[\frac{2|\chi'-\chi_0|}{f_K(\chi')|\boldsymbol{\beta}-\boldsymbol{\beta}'|}\right] + \operatorname{sign}(\chi_1-\chi')\log\left[\frac{2|\chi_1-\chi'|}{f_K(\chi')|\boldsymbol{\beta}-\boldsymbol{\beta}'|}\right] \\
&= \operatorname{sign}(\chi'-\chi_0)\log\left[\frac{2|\chi'-\chi_0|}{f_K(\chi')}\right] + \operatorname{sign}(\chi_1-\chi')\log\left[\frac{2|\chi_1-\chi'|}{f_K(\chi')}\right] \\
&\quad - 2\left[\Theta(\chi'-\chi_0)\Theta(\chi_1-\chi') - \Theta(\chi_0-\chi')\Theta(\chi'-\chi_1)\right]\log|\boldsymbol{\beta}-\boldsymbol{\beta}'|.
\end{aligned} \tag{3.2}$$

For $\chi_0 < \chi' < \chi_1$, this simplifies to

$$\begin{aligned}
\psi(\boldsymbol{\beta}) &= \int_{\chi_0}^{\chi_1} \frac{\mathrm{d}\chi}{g_K\{d_K[(\boldsymbol{\beta},\chi),(\boldsymbol{\beta}',\chi')]\}} \approx \int_{\chi_0}^{\chi_1} \frac{\mathrm{d}\chi}{\sqrt{f_K^2(\chi')|\boldsymbol{\beta}-\boldsymbol{\beta}'|^2 + |\chi-\chi'|^2}} \\
&= \log\left[\frac{\chi_1-\chi'+\sqrt{|\chi_1-\chi'|^2 + f_K^2(\chi')|\boldsymbol{\beta}-\boldsymbol{\beta}'|^2}}{\chi_0-\chi'+\sqrt{|\chi_0-\chi'|^2 + f_K^2(\chi')|\boldsymbol{\beta}-\boldsymbol{\beta}'|^2}}\right] \\
&\approx \log\left[\frac{2|\chi_1-\chi'|}{f_K(\chi')}\right] + \log\left[\frac{2|\chi'-\chi_0|}{f_K(\chi')}\right] - 2\log|\boldsymbol{\beta}-\boldsymbol{\beta}'|.
\end{aligned} \tag{3.3}$$

The last approximation becomes exact in the limit of $\chi_1 \to +\infty$ and $\chi_1 \to -\infty$.

The 'differential' form of the sudden-deflection approximation, in the plane-sky approximation, is practically nothing but:

$$
\begin{aligned}
\alpha_i(\boldsymbol{\beta},\chi) &\equiv \frac{\partial}{\partial\beta_i}\frac{1}{g_K\big\{d_K[(\boldsymbol{\beta},\chi),(\boldsymbol{\beta}',\chi')]\big\}} \approx \frac{\partial}{\partial\beta_i}\frac{1}{\sqrt{f_K^2(\chi')|\boldsymbol{\beta}-\boldsymbol{\beta}'|^2+|\chi-\chi'|^2}} \\
&= -\frac{f_K^2(\chi')(\beta_i-\beta_i')}{\big[f_K^2(\chi')|\boldsymbol{\beta}-\boldsymbol{\beta}'|^2+|\chi-\chi'|^2\big]^{3/2}} \\
&\approx -2\delta_{\mathrm{D}}(\chi-\chi')\frac{\beta_i-\beta_i'}{|\boldsymbol{\beta}-\boldsymbol{\beta}'|^2} \\
&= -2\delta_{\mathrm{D}}(\chi-\chi')\frac{\partial}{\partial\beta_i}\log\left|\boldsymbol{\beta}-\boldsymbol{\beta}'\right| .
\end{aligned}
\tag{3.4}
$$

The smooth but very "peaky" function $\alpha_i(\chi)$ is approximated by a Dirac delta "function" with the same integral.

3.3 The first-order approximation

From the theory of ordinary differential equations follows that the solution $\boldsymbol{\beta}(\boldsymbol{\theta},\chi)$ of the lens equation (2.58) is an attractive fixed point of the mapping

$$\zeta_i(\boldsymbol{\theta},\chi) \mapsto \theta_i - \frac{2}{\mathrm{c}^2}\int_0^{\chi}\frac{f_K(\chi-\chi')}{f_K(\chi)f_K(\chi')}\Phi_{,\zeta_i}(t(\chi'),\boldsymbol{\zeta}(\boldsymbol{\theta},\chi'),\chi')\mathrm{d}\chi' \tag{3.5}$$

(at least in some interval $\mathbb{I}=[0,w_{\max})\in\mathbb{R}$ whose extent depends on the properties of Φ, f_K, and a). Thus, one can take a good guess for $\boldsymbol{\beta}(\boldsymbol{\theta},\chi)$ and obtain an even better one by applying the mapping (3.5). The approximate solution

$$\beta_i(\boldsymbol{\theta},\chi) = \theta_i - \frac{2}{\mathrm{c}^2}\int_0^{\chi}\frac{f_K(\chi-\chi')}{f_K(\chi)f_K(\chi')}\Phi_{,\theta_i}(t(\chi'),\boldsymbol{\theta},\chi')\mathrm{d}\chi' \tag{3.6}$$

obtained by taking the 'unperturbed' path $\big(t(\chi),\boldsymbol{\theta},\chi'\big)$ is input is called the *first-order approximation* to the lens equation. Using the first-order lens potential

$$\psi(\boldsymbol{\theta},\chi) = \frac{2}{\mathrm{c}^2}\int_0^{\chi}\frac{f_K(\chi-\chi')}{f_K(\chi)f_K(\chi')}\Phi(t(\chi'),\boldsymbol{\theta},\chi')\,\mathrm{d}\chi' \tag{3.7}$$

and the (scaled) deflection angle

$$\alpha_i(\boldsymbol{\theta},\chi) = -\psi_{,\theta_i}(\boldsymbol{\theta},\chi), \tag{3.8}$$

this can be written as:

$$\beta_i(\boldsymbol{\theta},\chi) = \theta_i + \alpha_i(\boldsymbol{\theta},\chi). \tag{3.9}$$

The resulting approximation to the distortion reads:

$$
\begin{aligned}
A_{ij}(\boldsymbol{\theta},\chi) &= \delta_{ij} - \frac{2}{\mathrm{c}^2}\int_0^{\chi}\frac{f_K(\chi-\chi')}{f_K(\chi)f_K(\chi')}\Phi_{,\theta_i\theta_j}(t(\chi'),\boldsymbol{\theta},\chi')\,\mathrm{d}\chi' \\
&= \delta_{ij} + U_{ij}(\boldsymbol{\theta},\chi),
\end{aligned}
\tag{3.10}
$$

where the shear matrix

$$U_{ij}(\boldsymbol{\theta},\chi) = -\psi_{,\theta_i\theta_j}(\boldsymbol{\theta},\chi). \tag{3.11}$$

The shear matrix **U**, and hence the distortion matrix **A** are manifestly symmetric, and thus free of *B-modes* .

Applying the Laplace operator on the 2-sphere

$$\triangle^{\mathbb{S}^2}_{\boldsymbol{\beta}} = \frac{1}{\cos(\beta_1)}\frac{\partial}{\partial\beta_1}\cos(\beta_1)\frac{\partial}{\partial\beta_1} + \frac{1}{\cos(\beta_1)^2}\frac{\partial^2}{\partial\beta_2^2}. \tag{3.12}$$

to the first-order lens potential $\psi(\boldsymbol{\theta},\chi)$ yields:

$$\begin{aligned}\triangle^{\mathbb{S}^2}_{\boldsymbol{\theta}}\psi(\boldsymbol{\theta},\chi) &= \frac{2}{\mathrm{c}^2}\int_0^\chi \frac{f_K(\chi-\chi')}{f_K(\chi)f_K(\chi')}\triangle^{\mathbb{S}^2}_{\boldsymbol{\theta}}\Phi(t(\chi'),\boldsymbol{\theta},\chi')\,\mathrm{d}\chi' \\ &= \frac{2}{\mathrm{c}^2}\int_0^\chi \frac{f_K(\chi-\chi')f_K(\chi')}{f_K(\chi)}\triangle^{\mathrm{com}}_{(\boldsymbol{\theta},\chi')}\Phi(t(\chi'),\boldsymbol{\theta},\chi')\,\mathrm{d}\chi' \\ &\quad - \frac{2}{\mathrm{c}^2}\int_0^\chi \frac{f_K(\chi-\chi')}{f_K(\chi)f_K(\chi')}\frac{\partial\left[f_k^2(\chi')\Phi_{,\chi'}(t(\chi'),\boldsymbol{\theta},\chi')\right]}{\partial\chi'}\mathrm{d}\chi'.\end{aligned} \tag{3.13}$$

Neglecting the second integral (see, e.g., Jain et al. 2000, for a discussion) and using Eq. (2.3), one obtains for the first-order lens potential:

$$\begin{aligned}\triangle^{\mathbb{S}^2}_{\boldsymbol{\theta}}\psi(\boldsymbol{\theta},\chi) &= \frac{2}{\mathrm{c}^2}\int_0^\chi \frac{f_K(\chi-\chi')f_K(\chi')}{f_K(\chi)}\triangle^{\mathrm{com}}_{(\boldsymbol{\theta},\chi')}\Phi(t(\chi'),\boldsymbol{\theta},\chi')\,\mathrm{d}\chi' \\ &= \frac{8\pi\mathrm{G}}{\mathrm{c}^2}\int_0^\chi \frac{f_K(\chi-\chi')f_K(\chi')}{f_K(\chi)}\frac{\bar{\rho}_\mathrm{m}}{a\big(t(\chi')\big)}\delta_\mathrm{m}\big(t(\chi'),\boldsymbol{\theta},\chi'\big)\mathrm{d}\chi' \\ &= 2\kappa(\boldsymbol{\theta},\chi),\end{aligned} \tag{3.14}$$

where $\kappa(\boldsymbol{\theta},\chi)$ denotes the first-order lensing convergence:

$$\begin{aligned}\kappa(\boldsymbol{\theta},\chi) &= \frac{4\pi\mathrm{G}\bar{\rho}_\mathrm{m}}{\mathrm{c}^2}\int_0^\chi \frac{f_K(\chi-\chi')f_K(\chi')}{f_K(\chi)a\big(t(\chi')\big)}\delta_\mathrm{m}\big(t(\chi'),\boldsymbol{\theta},\chi'\big)\mathrm{d}\chi' \\ &= \frac{3H_0^2\Omega_\mathrm{m}}{2\mathrm{c}^2}\int_0^\chi \frac{f_K(\chi-\chi')f_K(\chi')}{f_K(\chi)a\big(t(\chi')\big)}\delta_\mathrm{m}\big(t(\chi'),\boldsymbol{\theta},\chi'\big)\mathrm{d}\chi'.\end{aligned} \tag{3.15}$$

The lens potential thus satisfies the Poisson equation:

$$\triangle^{\mathbb{S}^2}_{\boldsymbol{\theta}}\psi(\boldsymbol{\theta},\chi) = 2\kappa(\boldsymbol{\theta},\chi). \tag{3.16}$$

In the plane-sky approximation, the solution is thus given by:

$$\psi(\boldsymbol{\theta},\chi) \approx \frac{1}{\pi}\int_{\mathbb{S}^2}\mathrm{d}^2\boldsymbol{\theta'}\kappa(\boldsymbol{\theta'},\chi)\log\left(|\boldsymbol{\theta}-\boldsymbol{\theta'}|\right), \tag{3.17}$$

and the source position is given by:

$$\boldsymbol{\beta}(\boldsymbol{\theta},\chi) \approx \boldsymbol{\theta} + \boldsymbol{\alpha}(\boldsymbol{\theta},\chi) \tag{3.18}$$

with the (scaled) deflection angle

$$\begin{aligned}\boldsymbol{\alpha}(\boldsymbol{\theta},\chi) &= -\nabla_{\boldsymbol{\theta}}\psi(\boldsymbol{\theta},\chi) \\ &= -\frac{1}{\pi}\int_{\mathbb{S}^2}\mathrm{d}^2\boldsymbol{\theta'}\kappa(\boldsymbol{\theta'},\chi)\frac{\boldsymbol{\theta}-\boldsymbol{\theta'}}{|\boldsymbol{\theta}-\boldsymbol{\theta'}|^2} \\ &= -\frac{4\mathrm{G}\bar{\rho}_\mathrm{m}}{\mathrm{c}^2}\int_{\mathbb{S}^2}\mathrm{d}^2\boldsymbol{\theta'}\int_0^\chi\mathrm{d}\chi'\frac{f_K(\chi-\chi')f_K(\chi')}{f_K(\chi)a(t')}\frac{\boldsymbol{\theta}-\boldsymbol{\theta'}}{|\boldsymbol{\theta}-\boldsymbol{\theta'}|^2}\delta_\mathrm{m}(t',\boldsymbol{\theta'},\chi') \\ &= -\frac{3H_0^2\Omega_\mathrm{m}}{2\pi\mathrm{c}^2}\int_{\mathbb{S}^2}\mathrm{d}^2\boldsymbol{\theta'}\int_0^\chi\mathrm{d}\chi'\frac{f_K(\chi-\chi')f_K(\chi')}{f_K(\chi)a(t')}\frac{\boldsymbol{\theta}-\boldsymbol{\theta'}}{|\boldsymbol{\theta}-\boldsymbol{\theta'}|^2}\delta_\mathrm{m}(t',\boldsymbol{\theta'},\chi').\end{aligned} \tag{3.19}$$

Here, the abbreviation $t' = t(\chi')$ has been used.

3.4 The thin-lens approximation

One is often interested in the lens effect of an isolated and well-confined matter inhomogeneity at a given redshift z^{L}, corresponding to a distance χ^{L} and scale factor a^{L}, which can be approximated by:

$$\delta_{\mathrm{m}}\big(t,\boldsymbol{\beta},\chi\big) = \frac{\Sigma^{\mathrm{ang}}(\boldsymbol{\beta})\delta_{\mathrm{D}}(\chi-\chi^{\mathrm{L}})}{f_K^2(\chi^{\mathrm{L}})\bar{\rho}_{\mathrm{m}}}, \tag{3.20}$$

where $\Sigma^{\mathrm{ang}}(\boldsymbol{\beta})$ denotes the angular surface mass density. The gravitational potential of such a matter distribution is given by:

$$\begin{aligned}\Phi(t,\boldsymbol{\beta},\chi) &= -\frac{\mathrm{G}}{a(t)}\int_{\mathbb{S}^2}\mathrm{d}^2\boldsymbol{\beta}'\frac{\Sigma^{\mathrm{ang}}(\boldsymbol{\beta}')}{g_K\big\{d_K\big[(\boldsymbol{\beta},\chi),(\boldsymbol{\beta}',\chi^{\mathrm{L}})\big]\big\}}\\ &\approx -\frac{\mathrm{G}}{a(t)}\int_{\mathbb{S}^2}\mathrm{d}^2\boldsymbol{\beta}'\frac{\Sigma^{\mathrm{ang}}(\boldsymbol{\beta}')}{\sqrt{f_K^2(\chi^{\mathrm{L}})\,|\boldsymbol{\beta}-\boldsymbol{\beta}'|^2+|\chi'-\chi^{\mathrm{L}}|^2}}.\end{aligned} \tag{3.21}$$

The latter expression is the potential for $|\boldsymbol{\beta}| \ll 1$ in the plane-sky approximation. Now this expression can be put into the lens equation and the results subjected to the sudden-deflection approximation:

$$\begin{aligned}\beta_i(\boldsymbol{\theta},\chi) &= \theta_i - \frac{2}{\mathrm{c}^2}\int_0^{\chi}\mathrm{d}\chi'\frac{f_K(\chi-\chi')}{f_K(\chi)f_K(\chi')}\frac{\partial}{\partial\beta_i}\Phi(t(\chi'),\boldsymbol{\beta}(\boldsymbol{\theta},\chi'),\chi')\\ &\approx \theta_i + \frac{2\mathrm{G}}{\mathrm{c}^2}\int_0^{\chi}\mathrm{d}\chi'\frac{f_K(\chi-\chi')}{f_K(\chi)f_K(\chi')a\big(t(\chi')\big)}\times\\ &\qquad\times\int_{\mathbb{S}^2}\mathrm{d}^2\boldsymbol{\beta}'\frac{\partial}{\partial\beta_i}\frac{\Sigma^{\mathrm{ang}}(\boldsymbol{\beta}')}{\sqrt{f_K^2(\chi^{\mathrm{L}})|\boldsymbol{\beta}(\boldsymbol{\theta},\chi')-\boldsymbol{\beta}'|^2+|\chi'-\chi^{\mathrm{L}}|^2}}\\ &\approx \theta_i - \frac{4\mathrm{G}}{\mathrm{c}^2}\int_0^{\chi}\mathrm{d}\chi'\frac{f_K(\chi-\chi')}{f_K(\chi)f_K(\chi')a\big(t(\chi')\big)}\times\\ &\qquad\times\int_{\mathbb{S}^2}\mathrm{d}^2\boldsymbol{\beta}'\Sigma^{\mathrm{ang}}(\boldsymbol{\beta}')\frac{\beta_i(\boldsymbol{\theta},\chi')-\beta_i'}{|\boldsymbol{\beta}(\boldsymbol{\theta},\chi')-\boldsymbol{\beta}'|^2}\delta_{\mathrm{D}}(\chi'-\chi^{\mathrm{L}})\\ &= \theta_i - \frac{4\mathrm{G}}{\mathrm{c}^2}\frac{f_K(\chi-\chi^{\mathrm{L}})}{f_K(\chi)f_K(\chi^{\mathrm{L}})a^{\mathrm{L}}}\Theta(\chi-\chi^{\mathrm{L}})\int_{\mathbb{S}^2}\mathrm{d}^2\boldsymbol{\beta}'\Sigma^{\mathrm{ang}}(\boldsymbol{\beta}')\frac{\theta_i-\beta_i'}{|\boldsymbol{\theta}-\boldsymbol{\beta}'|^2}\\ &= \theta_i - \psi_{,\theta_i}(\boldsymbol{\theta},\chi).\end{aligned} \tag{3.22}$$

Here, the lensing potential

$$\psi(\boldsymbol{\theta},\chi) = \frac{4\mathrm{G}}{\mathrm{c}^2}\frac{f_K(\chi-\chi^{\mathrm{L}})}{f_K(\chi)f_K(\chi^{\mathrm{L}})a^{\mathrm{L}}}\Theta(\chi-\chi^{\mathrm{L}})\int_{\mathbb{S}^2}\mathrm{d}^2\boldsymbol{\beta}'\,\Sigma^{\mathrm{ang}}(\boldsymbol{\beta}')\log\big(|\boldsymbol{\theta}-\boldsymbol{\beta}'|\big)\,. \tag{3.23}$$

The lensing potential satisfies the Poisson equation:

$$\triangle_{\boldsymbol{\theta}}^{\mathbb{S}^2}\psi(\boldsymbol{\theta},\chi) = 2\kappa(\boldsymbol{\theta},\chi)\ \text{with} \tag{3.24}$$

$$\kappa(\boldsymbol{\theta},\chi) = \frac{\Sigma^{\mathrm{ang}}(\boldsymbol{\theta})}{\Sigma^{\mathrm{ang}}_{\mathrm{crit}}(\chi)}\ \text{and} \tag{3.25}$$

$$\Sigma^{\mathrm{ang}}_{\mathrm{crit}}(\chi) = \left[\frac{4\pi\mathrm{G}}{\mathrm{c}^2}\frac{f_K(\chi-\chi^{\mathrm{L}})}{f_K(\chi)f_K(\chi^{\mathrm{L}})a^{\mathrm{L}}}\Theta(\chi-\chi^{\mathrm{L}})^{-1}\right]^{-1}. \tag{3.26}$$

An alternative derivation leading to the this result employs the first-order approximation.
From Eq. (3.22), one can read off for the source position:

$$\beta_i(\boldsymbol{\theta}, \chi) = \boldsymbol{\theta} + \boldsymbol{\alpha}(\boldsymbol{\theta}, \chi) \tag{3.27}$$

with the scaled deflection angle

$$\begin{aligned}
\boldsymbol{\alpha}(\boldsymbol{\theta}, \chi) &= -\nabla_{\boldsymbol{\theta}} \psi(\boldsymbol{\theta}, \chi) \\
&= -\frac{1}{\pi} \int_{\mathbb{S}^2} \mathrm{d}^2\boldsymbol{\beta}' \frac{\Sigma^{\mathrm{ang}}(\boldsymbol{\beta}')}{\Sigma^{\mathrm{ang}}_{\mathrm{crit}}(\chi)} \frac{\boldsymbol{\theta} - \boldsymbol{\beta}'}{|\boldsymbol{\theta} - \boldsymbol{\beta}'|^2} \\
&= -\frac{4\mathrm{G}}{\mathrm{c}^2} \frac{f_K(\chi - \chi^{\mathrm{L}})}{f_K(\chi) f_K(\chi^{\mathrm{L}}) a^{\mathrm{L}}} \Theta(\chi - \chi^{\mathrm{L}}) \int_{\mathbb{S}^2} \mathrm{d}^2\boldsymbol{\beta}' \Sigma^{\mathrm{ang}}(\boldsymbol{\beta}') \frac{\boldsymbol{\theta} - \boldsymbol{\beta}'}{|\boldsymbol{\theta} - \boldsymbol{\beta}'|^2}.
\end{aligned} \tag{3.28}$$

The distortion matrix reads:

$$A_{ij}(\boldsymbol{\theta}, \chi) = \delta_{ij} + U_{ij}(\boldsymbol{\theta}, \chi), \text{ with} \tag{3.29}$$

$$U_{ij}(\boldsymbol{\theta}, \chi) = -\psi_{,\theta_i\theta_j}(\boldsymbol{\theta}, \chi). \tag{3.30}$$

As in the first-order approximation, the distortion matrix $\mathbf{A}$ is manifestly symmetric.

3.5 The infinitely-many-lens-planes approximation

The sudden-deflection approximation (3.4) can also be used for general mass distributions:

$$\begin{aligned}
\beta_i(\boldsymbol{\theta}, \chi) &= \theta_i - \frac{2}{\mathrm{c}^2} \int_0^{\chi} \mathrm{d}\chi' \frac{f_K(\chi - \chi')}{f_K(\chi) f_K(\chi')} \frac{\partial}{\partial \beta_i} \Phi(t(\chi'), \boldsymbol{\beta}(\boldsymbol{\theta}, \chi'), \chi') \\
&= \theta_i + \frac{2\mathrm{G}}{\mathrm{c}^2} \int_0^{\chi} \mathrm{d}\chi' \frac{f_K(\chi - \chi')}{f_K(\chi) f_K(\chi')} \frac{\bar{\rho}_{\mathrm{m}}}{a\big(t(\chi')\big)} \times \\
&\quad \times \int_{\mathbb{R}} \mathrm{d}\chi'' f_K^2(\chi'') \int_{\mathbb{S}^2} \mathrm{d}^2\boldsymbol{\beta}' \frac{\partial}{\partial \beta_i} \frac{\delta_{\mathrm{m}}(t, \boldsymbol{\beta}', \chi'')}{g_K\big\{d_K\big[\big(\boldsymbol{\beta}(\boldsymbol{\theta}, \chi'), \chi'\big), \big(\boldsymbol{\beta}', \chi''\big)\big]\big\}} \\
&\approx \theta_i - \frac{4\mathrm{G}}{\mathrm{c}^2} \int_0^{\chi} \mathrm{d}\chi' \frac{f_K(\chi - \chi')}{f_K(\chi) f_K(\chi')} \frac{\bar{\rho}_{\mathrm{m}}\big(t(\chi')\big)}{a\big(t(\chi')\big)} \int_{\mathbb{R}} \mathrm{d}\chi'' f_K^2(\chi'') \times \\
&\quad \times \int_{\mathbb{S}^2} \mathrm{d}^2\boldsymbol{\beta}' \delta_{\mathrm{m}}\big(t(\chi'), \boldsymbol{\beta}', \chi''\big) \delta_{\mathrm{D}}(\chi' - \chi'') \frac{\partial}{\partial \beta_i} \log \big|\boldsymbol{\beta}(\boldsymbol{\theta}, \chi') - \boldsymbol{\beta}'\big| \\
&= \theta_i - \frac{4\mathrm{G}\bar{\rho}_{\mathrm{m}}}{\mathrm{c}^2} \int_0^{\chi} \mathrm{d}\chi' \frac{f_K(\chi - \chi') f_K(\chi')}{f_K(\chi) a\big(t(\chi')\big)} \times \\
&\quad \times \int_{\mathbb{S}^2} \mathrm{d}^2\boldsymbol{\beta}' \delta_{\mathrm{m}}\big(t(\chi'), \boldsymbol{\beta}', \chi'\big) \frac{\partial}{\partial \beta_i} \log \big|\boldsymbol{\beta}(\boldsymbol{\theta}, \chi') - \boldsymbol{\beta}'\big| \\
&= \theta_i - \frac{4\mathrm{G}\bar{\rho}_{\mathrm{m}}}{\mathrm{c}^2} \int_0^{\chi} \mathrm{d}\chi' \frac{f_K(\chi - \chi') f_K(\chi')}{f_K(\chi) a\big(t(\chi')\big)} \times \\
&\quad \times \int_{\mathbb{S}^2} \mathrm{d}^2\boldsymbol{\beta}' \delta_{\mathrm{m}}\big(t(\chi'), \boldsymbol{\beta}', \chi'\big) \frac{\beta_i(\boldsymbol{\theta}, \chi') - \beta_i'}{|\boldsymbol{\beta}(\boldsymbol{\theta}, \chi') - \boldsymbol{\beta}'|^2}.
\end{aligned} \tag{3.31}$$

This can be written as:

$$\beta_i(\boldsymbol{\theta}, \chi) = \theta_i - \int_0^{\chi} \mathrm{d}\chi' \frac{f_K(\chi - \chi')}{f_K(\chi)} \tilde{\psi}_{,\beta_i}\big(\boldsymbol{\beta}(\boldsymbol{\theta}, \chi'), \chi'\big), \tag{3.32}$$

where the lensing potential

$$\tilde{\psi}(\boldsymbol{\beta},\chi)=\frac{4\mathrm{G}\bar{\rho}_{\mathrm{m}}}{\mathrm{c}^2}\frac{f_K(\chi)}{a\big(t(\chi)\big)}\int_{\mathbb{S}^2}\mathrm{d}^2\boldsymbol{\beta}'\delta_{\mathrm{m}}\big(t(\chi),\boldsymbol{\beta}',\chi\big)\log\left|\boldsymbol{\beta}-\boldsymbol{\beta}'\right| \tag{3.33}$$

satisfies the Poisson equation (again I make sloppy use of the plane-sky approximation):

$$\begin{aligned}\triangle^{\mathbb{S}^2}_{\boldsymbol{\beta}}\tilde{\psi}(\boldsymbol{\beta},\chi)&=\frac{8\pi\mathrm{G}\bar{\rho}_{\mathrm{m}}}{\mathrm{c}^2}\frac{f_K(\chi)}{a\big(t(\chi)\big)}\delta_{\mathrm{m}}\big(t(\chi),\boldsymbol{\beta},\chi\big)\\&=2\tilde{\kappa}(\boldsymbol{\beta},\chi),\ \text{with}\end{aligned} \tag{3.34}$$

$$\tilde{\kappa}(\boldsymbol{\beta},\chi)=\frac{4\pi\mathrm{G}\bar{\rho}_{\mathrm{m}}}{\mathrm{c}^2}\frac{f_K(\chi)}{a\big(t(\chi)\big)}\delta_{\mathrm{m}}\big(t(\chi),\boldsymbol{\beta},\chi\big). \tag{3.35}$$

The lensing potential $\tilde{\psi}(\boldsymbol{\beta},\chi)$ is related to the Newtonian potential

$$\Phi\big(t,\boldsymbol{\beta},\chi,\chi''\big)=-\frac{\mathrm{G}\bar{\rho}_{\mathrm{m}}}{a(t)}f_K^2(\chi'')\int_{\mathbb{S}^2}\mathrm{d}^2\boldsymbol{\beta}''\frac{\delta_{\mathrm{m}}(t,\boldsymbol{\beta}'',\chi'')}{g_K\big\{d_K\big[(\boldsymbol{\beta},\chi),(\boldsymbol{\beta}'',\chi'')\big]\big\}} \tag{3.36}$$

generated by the matter at comoving distance χ'' via:

$$\begin{aligned}&\frac{2}{\mathrm{c}^2}\int_0^{\chi}\mathrm{d}\chi'\ \Phi\big(t(\chi'),\boldsymbol{\beta}(\boldsymbol{\theta},\chi'),\chi',\chi''\big)\approx\frac{2}{\mathrm{c}^2}\int_0^{\chi}\mathrm{d}\chi'\ \Phi\big(t(\chi''),\boldsymbol{\beta}(\boldsymbol{\theta},\chi''),\chi',\chi''\big)\\&\approx-\frac{2\mathrm{G}\bar{\rho}_{\mathrm{m}}}{\mathrm{c}^2}\frac{f_K^2(\chi'')}{a\big(t(\chi'')\big)}\int_{\mathbb{S}^2}\mathrm{d}^2\boldsymbol{\beta}''\int_0^{\chi}\mathrm{d}\chi'\ \frac{\delta_{\mathrm{m}}\big(t(\chi''),\boldsymbol{\beta}'',\chi''\big)}{g_K\big\{d_K\big[\big(\boldsymbol{\beta}(\boldsymbol{\theta},\chi''),\chi'\big),(\boldsymbol{\beta}'',\chi'')\big]\big\}}\\&\approx-\frac{2\mathrm{G}\bar{\rho}_{\mathrm{m}}}{\mathrm{c}^2}\frac{f_K^2(\chi'')}{a\big(t(\chi'')\big)}\int_{\mathbb{S}^2}\mathrm{d}^2\boldsymbol{\beta}''\delta_{\mathrm{m}}\big(t(\chi''),\boldsymbol{\beta}'',\chi''\big)\Bigg\{\log\left[2\frac{\chi''}{f_K(\chi'')}\right]\\&\quad+\operatorname{sign}(\chi-\chi'')\log\left[2\frac{|\chi-\chi''|}{f_K(\chi'')}\right]-2\Theta(\chi-\chi'')\log\left|\boldsymbol{\beta}(\boldsymbol{\theta},\chi'')-\boldsymbol{\beta}''\right|\Bigg\}\\&=\frac{4\mathrm{G}\bar{\rho}_{\mathrm{m}}}{\mathrm{c}^2}\Theta(\chi-\chi'')\frac{f_K^2(\chi'')}{a\big(t(\chi'')\big)}\int_{\mathbb{S}^2}\mathrm{d}^2\boldsymbol{\beta}''\delta_{\mathrm{m}}\big(t(\chi''),\boldsymbol{\beta}'',\chi''\big)\log\left|\boldsymbol{\beta}(\boldsymbol{\theta},\chi'')-\boldsymbol{\beta}''\right|\\&=\Theta(\chi-\chi'')f_K(\chi'')\tilde{\psi}(\boldsymbol{\theta},\chi'').\end{aligned} \tag{3.37}$$

Equation (3.32) can also be written as

$$\beta_i(\boldsymbol{\theta},\chi)=\theta_i-\int_0^{\chi}\mathrm{d}\chi'\frac{f_K(\chi-\chi')}{f_K(\chi)}\tilde{\Psi}_{,x_i}\big(\boldsymbol{x}(\boldsymbol{\theta},\chi'),\chi'\big), \tag{3.38}$$

with the potential

$$\tilde{\Psi}(\boldsymbol{x},\chi)=\frac{4\mathrm{G}\bar{\rho}_{\mathrm{m}}}{a\big(t(\chi)\big)\mathrm{c}^2}\int\mathrm{d}^2\boldsymbol{x}'\delta_{\mathrm{m}}\big(t(\chi),\boldsymbol{\beta}(\boldsymbol{x}',\chi),\chi\big)\log\left|\boldsymbol{x}-\boldsymbol{x}'\right|. \tag{3.39}$$

3.6 The multiple-lens-plane approximation

A feasible computational scheme to calculate the photon path is based on a discretization of Eq. (3.32). Take an ordered partition $\big\{\mathbb{I}^{(k)}\mid\mathbb{I}^{(k)}=[\chi_{\mathrm{L}}^{(k)},\chi_{\mathrm{U}}^{(k)})\subset\mathbb{R},\,k=1,\dots,N_{\max}\big\}$ of the interval $[0, w_{\max})$ into $N_{\max}$ subintervals ($w_{\max}$ being the largest source distance of any interest). Within each $\mathbb{I}^{(k)}$, select a point $\chi^{(k)} \in \mathbb{I}^{(k)}$, e.g. by choosing $\chi^{(k)} = (\chi_{\rm L}^{(k)} + \chi_{\rm U}^{(k)})/2$. Then,

$$
\begin{aligned}
\beta_i(\boldsymbol{\theta}, \chi) &= \theta_i - \int_0^{\chi} \mathrm{d}\chi' \frac{f_K(\chi - \chi')}{f_K(\chi)} \tilde{\psi}_{,\beta_i}\big(\boldsymbol{\beta}(\boldsymbol{\theta}, \chi'), \chi'\big) \\
&= \theta_i - \sum_{k=1}^{N_{\max}} \int_{\chi_{\rm L}^{(k)}}^{\chi_{\rm U}^{(k)}} \mathrm{d}\chi'\, \Theta(\chi - \chi') \frac{f_K(\chi - \chi')}{f_K(\chi)} \tilde{\psi}_{,\beta_i}\big(\boldsymbol{\beta}(\boldsymbol{\theta}, \chi'), \chi'\big) \\
&\approx \theta_i - \sum_{k=1}^{N_{\max}} \int_{\chi_{\rm L}^{(k)}}^{\chi_{\rm U}^{(k)}} \mathrm{d}\chi'\, \Theta(\chi - \chi^{(k)}) \frac{f_K(\chi - \chi^{(k)})}{f_K(\chi)} \tilde{\psi}_{,\beta_i}\big(\boldsymbol{\beta}(\boldsymbol{\theta}, \chi^{(k)}), \chi'\big) \\
&= \theta_i - \sum_{k=1}^{N(\chi)} \frac{f_K(\chi - \chi^{(k)})}{f_K(\chi)} \tilde{\psi}^{(k)}_{,\beta_i}\big(\boldsymbol{\beta}^{(k)}(\boldsymbol{\theta})\big) \\
&= \theta_i + \sum_{k=1}^{N(\chi)} \frac{f_K(\chi - \chi^{(k)})}{f_K(\chi)} \tilde{\alpha}^{(k)}_i\big(\boldsymbol{\beta}^{(k)}(\boldsymbol{\theta})\big),
\end{aligned}
\tag{3.40}
$$

where $N(\chi) = \sup\{i \in \mathbb{N}, \chi^{(i)} < \chi\}$,

$$\tilde{\boldsymbol{\alpha}}^{(k)}(\boldsymbol{\beta}) = -\nabla_{\boldsymbol{\beta}} \tilde{\psi}^{(k)}\big(\boldsymbol{\beta}\big), \tag{3.41}$$

$$\tilde{\psi}^{(k)}(\boldsymbol{\beta}) = \int_{\chi_{\rm L}^{(k)}}^{\chi_{\rm U}^{(k)}} \mathrm{d}\chi\, \tilde{\psi}\big(\boldsymbol{\beta}, \chi\big) \quad \text{and} \tag{3.42}$$

$$\boldsymbol{\beta}^{(k)}(\boldsymbol{\theta}) = \boldsymbol{\beta}\big(\boldsymbol{\theta}, \chi^{(k)}\big) \tag{3.43}$$

$$= \boldsymbol{\theta} + \sum_{i=1}^{k-1} \frac{f_K^{(k,i)}}{f_K^{(k)}} \tilde{\boldsymbol{\alpha}}^{(i)}\big(\boldsymbol{\beta}^{(i)}(\boldsymbol{\theta})\big) \tag{3.44}$$

with $f_K^{(k)} = f_K(\chi^{(k)})$ and $f_K^{(k,i)} = f_K(\chi^{(k)} - \chi^{(i)})$. The result of the derivation (3.40) is called *multiple-lens-plane approximation*. The derivation exploits the fact that $f_K(\chi)$ and $\boldsymbol{\beta}(\boldsymbol{\theta}, \chi)$ are continuous, slowly varying functions of χ. In contrast, the lens potential $\tilde{\psi}(t, \boldsymbol{\beta}, \chi)$ (unlike the gravitational potential Φ) can vary with χ as rapidly as the (possibly very clumpy) matter distribution $\delta_{\rm m}(t, \boldsymbol{\beta}, \chi)$.

The lensing potential

$$
\begin{aligned}
\tilde{\psi}^{(k)}(\boldsymbol{\beta}) &= \int_{\chi_{\rm L}^{(k)}}^{\chi_{\rm U}^{(k)}} \mathrm{d}\chi\, \tilde{\psi}\big(\boldsymbol{\beta}, \chi\big) \\
&= \int_{\chi_{\rm L}^{(k)}}^{\chi_{\rm U}^{(k)}} \mathrm{d}\chi\, \frac{4\mathrm{G}\bar{\rho}_{\rm m}}{\mathrm{c}^2} \frac{f_K(\chi)}{a\big(t(\chi)\big)} \int_{\mathbb{S}^2} \mathrm{d}^2\boldsymbol{\beta}' \log\big|\boldsymbol{\beta} - \boldsymbol{\beta}'\big|\, \delta_{\rm m}\big(t(\chi), \boldsymbol{\beta}', \chi\big) \\
&\approx \frac{4\mathrm{G}\bar{\rho}_{\rm m}}{\mathrm{c}^2} \frac{1}{f_K^{(k)} a^{(k)}} \int_{\mathbb{S}^2} \mathrm{d}^2\boldsymbol{\beta}' \log\big|\boldsymbol{\beta} - \boldsymbol{\beta}'\big|\, \Sigma^{\mathrm{ang}(k)}(\boldsymbol{\beta}') \\
&= \frac{4\mathrm{G}\bar{\rho}_{\rm m}}{\mathrm{c}^2} \frac{f_K^{(k)}}{a^{(k)}} \int_{\mathbb{S}^2} \mathrm{d}^2\boldsymbol{\beta}' \log\big|\boldsymbol{\beta} - \boldsymbol{\beta}'\big|\, \Sigma^{(k)}(f_K^{(k)} \boldsymbol{\beta}') \\
&= \frac{1}{\pi} \int_{\mathbb{S}^2} \mathrm{d}^2\boldsymbol{\beta}' \log\big|\boldsymbol{\beta} - \boldsymbol{\beta}'\big|\, \tilde{\kappa}^{(k)}(\boldsymbol{\beta}')
\end{aligned}
\tag{3.45}
$$

with the abbreviations $t^{(k)} = t(\chi^{(k)})$ and $a^{(k)} = a\big(t(\chi^{(k)})\big)$, the angular surface mass density

$$\Sigma^{\mathrm{ang}(k)}(\boldsymbol{\beta}) = \big(f_K^{(k)}\big)^2 \int_{\chi_{\mathrm{L}}^{(k)}}^{\chi_{\mathrm{U}}^{(k)}} \mathrm{d}\chi\, \delta_{\mathrm{m}}\big(t(\chi), \boldsymbol{\beta}, \chi\big) = \big(f_K^{(k)}\big)^2 \Sigma^{(k)}\big(f_K^{(k)}\boldsymbol{\beta}\big), \tag{3.46}$$

the comoving surface mass density

$$\Sigma^{(k)}(\boldsymbol{x}) = \int_{\chi_{\mathrm{L}}^{(k)}}^{\chi_{\mathrm{U}}^{(k)}} \mathrm{d}\chi\, \delta_{\mathrm{m}}\big(t(\chi), \boldsymbol{x}/f_K^{(k)}, \chi\big), \tag{3.47}$$

and the convergence

$$\begin{aligned}\tilde{\kappa}^{(k)}(\boldsymbol{\beta}) &= \frac{4\pi \mathrm{G}\bar{\rho}_{\mathrm{m}}}{\mathrm{c}^2} \int_{\chi_{\mathrm{L}}^{(k)}}^{\chi_{\mathrm{U}}^{(k)}} \mathrm{d}\chi\, \frac{f_K(\chi)}{a\big(t(\chi)\big)} \delta_{\mathrm{m}}\big(t(\chi), \boldsymbol{\beta}, \chi\big) \\ &\approx \frac{4\pi \mathrm{G}\bar{\rho}_{\mathrm{m}}}{\mathrm{c}^2} \frac{f_K^{(k)}}{a^{(k)}} \int_{\chi_{\mathrm{L}}^{(k)}}^{\chi_{\mathrm{U}}^{(k)}} \mathrm{d}\chi\, \delta_{\mathrm{m}}\big(t^{(k)}, \boldsymbol{\beta}, \chi\big) \\ &= \frac{3H_0^2\Omega_{\mathrm{m}}}{2\mathrm{c}^2} \frac{f_K^{(k)}}{a^{(k)}} \int_{\chi_{\mathrm{L}}^{(k)}}^{\chi_{\mathrm{U}}^{(k)}} \mathrm{d}\chi\, \delta_{\mathrm{m}}\big(t^{(k)}, \boldsymbol{\beta}, \chi\big) \\ &= \frac{4\pi \mathrm{G}\bar{\rho}_{\mathrm{m}}}{\mathrm{c}^2} \frac{1}{f_K^{(k)} a^{(k)}} \Sigma^{\mathrm{ang}(k)}(\boldsymbol{\beta}) \\ &= \frac{4\pi \mathrm{G}\bar{\rho}_{\mathrm{m}}}{\mathrm{c}^2} \frac{f_K^{(k)}}{a^{(k)}} \Sigma^{(k)}\big(f_K^{(k)}\boldsymbol{\beta}\big).\end{aligned} \tag{3.48}$$

The lens potentials (3.42) satisfy the Poisson equation [see Eq. (3.34)]:

$$\triangle_{\boldsymbol{\beta}}^{\mathbb{S}^2} \tilde{\psi}^{(k)}(\boldsymbol{\beta}) = 2\tilde{\kappa}^{(k)}(\boldsymbol{\beta}). \tag{3.49}$$

The lensing potential $\tilde{\psi}^{(k)}(\boldsymbol{\beta})$ is related to the Newtonian potential

$$\Phi^{(k)}\big(t, \boldsymbol{\beta}, \chi\big) = -\frac{\mathrm{G}\bar{\rho}_{\mathrm{m}}}{a(t)} \int_{\chi_{\mathrm{L}}^{(k)}}^{\chi_{\mathrm{U}}^{(k)}} \mathrm{d}\chi'' f_K^2(\chi'') \int_{\mathbb{S}^2} \mathrm{d}^2\boldsymbol{\beta}'' \frac{\delta_{\mathrm{m}}(t, \boldsymbol{\beta}'', \chi'')}{g_K\big\{d_K\big[(\boldsymbol{\beta}, \chi), (\boldsymbol{\beta}'', \chi'')\big]\big\}} \tag{3.50}$$

generated by the matter in the k-th slice via:

$$\frac{2}{\mathrm{c}^2} \int_0^{\chi} \mathrm{d}\chi'\; \Phi^{(k)}\big(t(\chi'), \boldsymbol{\beta}(\boldsymbol{\theta}, \chi'), \chi'\big) \approx \Theta\big(\chi - \chi^{(k)}\big) f_K^{(k)} \tilde{\psi}^{(k)}(\boldsymbol{\theta}). \tag{3.51}$$

When working with comoving transverse coordinates on the lens planes, deflection angle can be computed from

$$\tilde{\boldsymbol{\alpha}}^{(k)}(\boldsymbol{x}) = -\nabla_{\boldsymbol{x}} \tilde{\Psi}^{(k)}(\boldsymbol{x}) \tag{3.52}$$

with the potential

$$\tilde{\Psi}^{(k)}(\boldsymbol{x}) = f_K^{(k)} \tilde{\psi}^{(k)}\big(\boldsymbol{x}/f_K^{(k)}\big) = \frac{4\mathrm{G}\bar{\rho}_{\mathrm{m}}}{a^{(k)}\mathrm{c}^2} \int \mathrm{d}^2\boldsymbol{x}' \log\big|\boldsymbol{x} - \boldsymbol{x}'\big|\, \Sigma^{(k)}(\boldsymbol{x}'). \tag{3.53}$$

The multiple-lens-plane approximation appears as if the continuous matter distribution has been approximated by a number of thin lens planes located at distances $\chi^{(k)}$ from the observer, and the

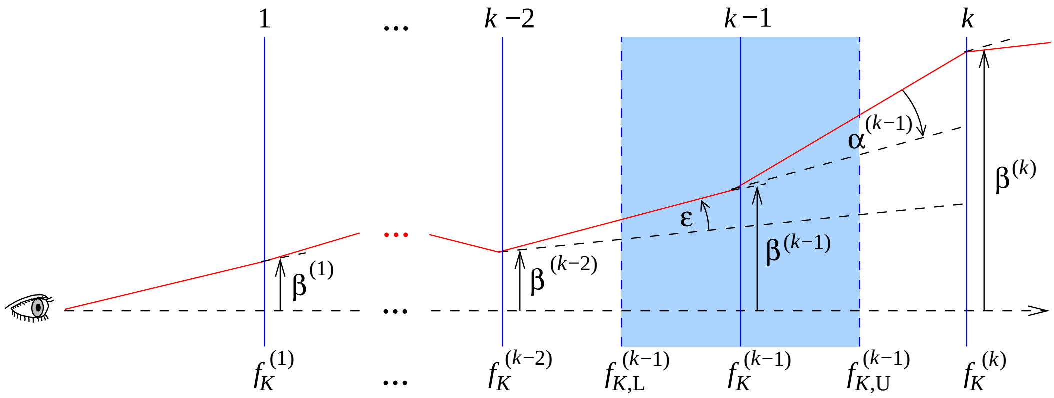

Figure 3.1: Schematic view of the observer's backward light cone in the multiple-lens-plane approximation. A light ray (red line) experiences a deflection only when passing through a lens plane (solid blue lines). The deflection angle $\tilde{\boldsymbol{\alpha}}^{(k-1)}$ of a ray passing through the lens plane at distance $f_K^{(k-1)}$ from the observer is obtained from the matter distribution between $f_{K,\mathrm{U}}^{(k-1)}$ and $f_{K,\mathrm{L}}^{(k-1)}$ projected onto the plane. Using the deflection angle $\tilde{\boldsymbol{\alpha}}^{(k-1)}$ of the light ray at the previous lens plane and the ray's angular positions $\boldsymbol{\beta}^{(k-1)}$ and $\boldsymbol{\beta}^{(k-2)}$ on the two previous planes, the angular position $\boldsymbol{\beta}^{(k)}$ on the current plane can be computed.

light deflection by these lens planes is approximated using the sudden-deflection approximation. Such a procedure obviously leads to the same result.

The errors introduced by the approximations in Eqs. (3.42) and (3.48) decrease with decreasing size of the intervals $\mathbb{I}^{(k)}$ and may be made arbitrarily small by using enough lens planes. However, the error introduced by the sudden-deflection approximation (3.4) cannot be made arbitrarily small by increasing the number of lens planes.

Equation (3.43) is not practical for tracing rays through many lens planes. An alternative expression is obtained as follows (see, e.g., Seitz et al. 1994, for a different derivation): The angular position $\boldsymbol{\beta}^{(k)}$ of a light ray on the lens plane k is related to its positions $\boldsymbol{\beta}^{(k-2)}$ and $\boldsymbol{\beta}^{(k-1)}$ on the two previous lens planes by (see Fig. 3.1):

$$f_K^{(k)}\boldsymbol{\beta}^{(k)} = f_K^{(k)}\boldsymbol{\beta}^{(k-2)} + f_K^{(k,k-2)}\boldsymbol{\epsilon}^{(k-2)} + f_K^{(k,k-1)}\tilde{\boldsymbol{\alpha}}^{(k-1)}\big(\boldsymbol{\beta}^{(k-1)}\big), \tag{3.54}$$

where

$$\boldsymbol{\epsilon}^{(k-2)} = \frac{f_K^{(k-1)}}{f_K^{(k-1,k-2)}}\left(\boldsymbol{\beta}^{(k-1)} - \boldsymbol{\beta}^{(k-2)}\right). \tag{3.55}$$

Hence,

$$\begin{aligned}\boldsymbol{\beta}^{(k)}(\boldsymbol{\theta}) &= \left(1 - \frac{f_K^{(k-1)}}{f_K^{(k)}} \frac{f_K^{(k,k-2)}}{f_K^{(k-1,k-2)}}\right) \boldsymbol{\beta}^{(k-2)}(\boldsymbol{\theta}) + \frac{f_K^{(k-1)}}{f_K^{(k)}} \frac{f_K^{(k,k-2)}}{f_K^{(k-1,k-2)}} \boldsymbol{\beta}^{(k-1)}(\boldsymbol{\theta}) \\ &\quad + \frac{f_K^{(k,k-1)}}{f_K^{(k)}} \tilde{\boldsymbol{\alpha}}^{(k-1)}\big(\boldsymbol{\beta}^{(k-1)}(\boldsymbol{\theta})\big) \\ &= \boldsymbol{\beta}^{(k-1)}(\boldsymbol{\theta}) + \frac{f_K^{(k-2)} f_K^{(k,k-1)}}{f_K^{(k)} f_K^{(k-1,k-2)}} \left(\boldsymbol{\beta}^{(k-1)}(\boldsymbol{\theta}) - \boldsymbol{\beta}^{(k-2)}(\boldsymbol{\theta})\right) \\ &\quad + \frac{f_K^{(k,k-1)}}{f_K^{(k)}} \tilde{\boldsymbol{\alpha}}^{(k-1)}\big(\boldsymbol{\beta}^{(k-1)}(\boldsymbol{\theta})\big).\end{aligned} \tag{3.56}$$

For a light ray reaching the observer from angular position $\boldsymbol{\beta}$ on the first lens plane, one can compute its angular position on the other lens planes by iterating (3.56) with initial values $\boldsymbol{\beta}^{(0)} = \boldsymbol{\beta}^{(1)} = \boldsymbol{\theta}$.

Differentiating (3.56) with respect to $\boldsymbol{\theta}$, one obtains a recurrence relation for the distortion matrix $\mathbf{A}^{(k)}(\boldsymbol{\theta}) = \mathbf{A}(\boldsymbol{\theta}, \chi^{(k)})$:

$$\begin{aligned}A_{ij}^{(k)}(\boldsymbol{\theta}) &= \left(1 - \frac{f_K^{(k-1)}}{f_K^{(k)}} \frac{f_K^{(k,k-2)}}{f_K^{(k-1,k-2)}}\right) A_{ij}^{(k-2)}(\boldsymbol{\theta}) + \frac{f_K^{(k-1)}}{f_K^{(k)}} \frac{f_K^{(k,k-2)}}{f_K^{(k-1,k-2)}} A_{ij}^{(k-1)}(\boldsymbol{\theta}) \\ &\quad + \frac{f_K^{(k,k-1)}}{f_K^{(k)}} \tilde{U}_{im}^{(k-1)}\big(\boldsymbol{\beta}^{(k-1)}(\boldsymbol{\theta})\big) A_{mj}^{(k-1)}(\boldsymbol{\theta})\end{aligned} \tag{3.57}$$

$$\text{with} \quad \tilde{U}_{ij}^{(k)}(\boldsymbol{\beta}) = -\frac{\partial^2 \tilde{\psi}^{(k)}\big(\boldsymbol{\beta}\big)}{\partial \beta_i \partial \beta_j}. \tag{3.58}$$

Unlike in the first-order lensing approximation or the single-plane approximation, the distortion matrix (3.57) is not symmetric in general.

For the reduced distortion matrix $\mathbf{U}^{(k)} = \mathbf{A}^{(k)} - \mathbf{1}$, the recurrence relation reads:

$$\begin{aligned}U_{ij}^{(k)}(\boldsymbol{\theta}) &= \left(1 - \frac{f_K^{(k-1)}}{f_K^{(k)}} \frac{f_K^{(k,k-2)}}{f_K^{(k-1,k-2)}}\right) U_{ij}^{(k-2)}(\boldsymbol{\theta}) + \frac{f_K^{(k-1)}}{f_K^{(k)}} \frac{f_K^{(k,k-2)}}{f_K^{(k-1,k-2)}} U_{ij}^{(k-1)}(\boldsymbol{\theta}) \\ &\quad + \frac{f_K^{(k,k-1)}}{f_K^{(k)}} \tilde{U}_{im}^{(k-1)}\big(\boldsymbol{\beta}^{(k-1)}(\boldsymbol{\theta})\big) \left[U_{mj}^{(k-1)}(\boldsymbol{\theta}) + \delta_{mj}\right].\end{aligned} \tag{3.59}$$

4 Image distortions

4.1 Standard decomposition of the distortion field

The position of the light ray with respect to the global coordinate system is given by the lens equation (2.58), the relative position of nearby light rays is quantified by the distortion matrix (2.59). The distortion matrix is usually decomposed into a rotation matrix and a symmetric matrix:

$$\mathbf{A}(\boldsymbol{\theta}, \chi) = \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix} \begin{pmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ -\gamma_2 & 1-\kappa+\gamma_1 \end{pmatrix} . \tag{4.1}$$

The decomposition defines the rotation angle $\omega = \omega(\boldsymbol{\theta}, \chi)$, the convergence $\kappa = \kappa(\boldsymbol{\theta}, \chi)$, and the two components $\gamma_1 = \gamma_1(\boldsymbol{\theta}, \chi)$ and $\gamma_2 = \gamma_2(\boldsymbol{\theta}, \chi)$ of the shear, which may be combined into the complex shear $\gamma = \gamma_1 + \mathrm{i}\gamma_2$.

The (signed) magnification $\mu(\boldsymbol{\theta}, \chi)$ of an image is given by the inverse determinant of the distortion matrix:

$$\mu = (\det \mathbf{A})^{-1} . \tag{4.2}$$

The reduced shear $g = \gamma/(1-\kappa)$ determines the major-to-minor axis ratio

$$r = \left| \frac{1+|g|}{1-|g|} \right| \tag{4.3}$$

of the elliptical images of sufficiently small circular sources.

The determinant and trace of the distortion matrix,

$$\det \mathbf{A} = A_{11}A_{22} - A_{12}A_{21} \text{ and} \tag{4.4}$$

$$\operatorname{tr} \mathbf{A} = A_{11} + A_{22}, \text{ resp.} \tag{4.5}$$

may be used to categorise images (Schneider et al. 1992):

- type I: $\det \mathbf{A} > 0$ and $\operatorname{tr} \mathbf{A} > 0$,
- type II: $\det \mathbf{A} < 0$,
- type III: $\det \mathbf{A} > 0$ and $\operatorname{tr} \mathbf{A} < 0$.

In all situations relevant for gravitational lensing, images of type II and type III belong to sources that have multiple images.

Magnification, convergence, shear, etc. can be calculated by:

$$\mu = (\det \mathbf{A})^{-1} = (A_{11}A_{22} - A_{12}A_{21})^{-1}, \tag{4.6}$$

$$\omega = -\arctan\left(\frac{A_{12} - A_{21}}{A_{11} + A_{22}}\right), \tag{4.7}$$

$$\kappa = 1 - \frac{A_{11} + A_{22}}{2\cos\omega} = 1 - \frac{1}{2}(A_{11} + A_{22})\sec\omega, \tag{4.8}$$

$$\begin{aligned} \gamma_1 &= -\frac{1}{2}\left[(A_{11} - A_{22})\cos\omega + (A_{12} + A_{21})\sin\omega\right], \\ \gamma_2 &= -\frac{1}{2}\left[(A_{12} + A_{21})\cos\omega + (A_{22} - A_{11})\sin\omega\right]. \end{aligned} \tag{4.9}$$

Using some algebra, one can express the modulus square of the reduced shear by:

$$|g|^2 = 1 - 4\frac{A_{11}A_{22} - A_{12}A_{21}}{(A_{11} + A_{22})^2 + (A_{12} - A_{21})^2} = \frac{(A_{11} - A_{22})^2 + (A_{12} + A_{21})^2}{(A_{11} + A_{22})^2 + (A_{12} - A_{21})^2}. \tag{4.10}$$

4.2 Standard decomposition for weak distortion fields

If one considers only weak lensing, one can expect κ, γ_1, γ_2, and ω to be small compared to unity. Then, one can expand the distortion matrix (4.1) up to linear order in these quantities:

$$\mathbf{A}(\boldsymbol{\theta}, \chi) = \begin{pmatrix} 1 - \kappa - \gamma_1 & -\gamma_2 - \omega \\ -\gamma_2 + \omega & 1 - \kappa + \gamma_1 \end{pmatrix}. \tag{4.11}$$

One can use this decomposition even in the case when κ, γ_1, γ_2, and ω are *not* small, but then the relations between the quantities defined by the decomposition (4.11) and the lensing observables become quite complicated. However, the decomposition (4.11) is easier to use in certain calculations than the decomposition (4.1).

In the case of weak rotation, magnification, convergence, etc. can be calculated by:

$$\mu = (\det \mathbf{A})^{-1} = (A_{11}A_{22} - A_{12}A_{21})^{-1}, \tag{4.12}$$

$$\kappa = 1 - \frac{1}{2}(A_{11} + A_{22}), \tag{4.13}$$

$$\omega = -\frac{1}{2}(A_{12} - A_{21}), \tag{4.14}$$

$$\begin{aligned} \gamma_1 &= -\frac{1}{2}(A_{11} - A_{22}), \\ \gamma_2 &= -\frac{1}{2}(A_{12} + A_{21}), \end{aligned} \tag{4.15}$$

$$|g|^2 = \frac{(A_{11} - A_{22})^2 + (A_{12} + A_{21})^2}{(A_{11} + A_{22})^2}. \tag{4.16}$$

Using the reduced distortion matrix

$$\mathbf{U} = \mathbf{A} - \mathbf{1} = \left(\frac{\partial\boldsymbol{\alpha}}{\partial\boldsymbol{\theta}}\right) = \begin{pmatrix} -\kappa - \gamma_1 & -\gamma_2 - \omega \\ -\gamma_2 + \omega & -\kappa + \gamma_1 \end{pmatrix}, \tag{4.17}$$

one obtains:

$$\mu = [\det(\mathbf{1} + \mathbf{U})]^{-1} = (U_{11}U_{22} - U_{12}U_{21} + U_{11} + U_{22} + 1)^{-1}\,, \tag{4.18}$$

$$\kappa = -\frac{1}{2}\,(U_{11} + U_{22})\,, \tag{4.19}$$

$$\omega = -\frac{1}{2}\,(U_{12} - U_{21})\,, \tag{4.20}$$

$$\begin{aligned} \gamma_1 &= -\frac{1}{2}\,(U_{11} - U_{22})\,, \\ \gamma_2 &= -\frac{1}{2}\,(U_{12} + U_{21})\,, \end{aligned} \tag{4.21}$$

$$|g|^2 = \frac{(U_{11} - U_{22})^2 + (U_{12} + U_{21})^2}{(U_{11} + U_{22} + 2)^2}. \tag{4.22}$$

4.3 Flexion

Higher-order image distortions can be quantified by higher derivatives of the lens map. Of particular interest are the spin-1 flexion

$$\begin{aligned} \mathcal{F}(\boldsymbol{\theta}, z) &= \left(\partial_{\theta_1} - \mathrm{i}\partial_{\theta_2}\right)\gamma(\boldsymbol{\theta}, z) \\ &= \left(\partial_{\theta_1}\gamma_1 + \partial_{\theta_2}\gamma_2\right) + \mathrm{i}\left(\partial_{\theta_1}\gamma_2 - \partial_{\theta_2}\gamma_1\right)\,, \end{aligned} \tag{4.23}$$

and the spin-3 flexion

$$\begin{aligned} \mathcal{G}(\boldsymbol{\theta}, z) &= \left(\partial_{\theta_1} + \mathrm{i}\partial_{\theta_2}\right)\gamma(\boldsymbol{\theta}, z) \\ &= \left(\partial_{\theta_1}\gamma_1 - \partial_{\theta_2}\gamma_2\right) + \mathrm{i}\left(\partial_{\theta_1}\gamma_2 + \partial_{\theta_2}\gamma_1\right)\,. \end{aligned} \tag{4.24}$$

Analogous to the reduced shear, one can define a reduced spin-1 and spin-3 flexion (Schneider and Er 2008):

$$F(\boldsymbol{\theta}, z) = \left(\partial_{\theta_1} + \mathrm{i}\partial_{\theta_2}\right) g(\boldsymbol{\theta}, z) = \frac{\mathcal{F}(\boldsymbol{\theta}, z) + g(\boldsymbol{\theta}, z)\mathcal{F}^*(\boldsymbol{\theta}, z)}{1 - \kappa(\boldsymbol{\theta}, z)}, \tag{4.25}$$

$$G(\boldsymbol{\theta}, z) = \left(\partial_{\theta_1} - \mathrm{i}\partial_{\theta_2}\right) g(\boldsymbol{\theta}, z) = \frac{\mathcal{G}(\boldsymbol{\theta}, z) + g(\boldsymbol{\theta}, z)\mathcal{F}(\boldsymbol{\theta}, z)}{1 - \kappa(\boldsymbol{\theta}, z)}. \tag{4.26}$$

5 Simple models for isolated lenses

In the following, the lensing effects of isolated mass distributions with analytical profiles are discussed. It is assumed, that the lens mass distributions can be represented by a thin matter sheet at redshift z^{L} with an analytic surface mass density profile. Any sources are assumed at redshift z^{S}. Furthermore, I define the following abbreviations:

$$\begin{aligned} a^{\mathrm{L}} &= a(z^{\mathrm{L}}), && (5.1\mathrm{a})\\ \chi^{\mathrm{S}} &= \chi(z^{\mathrm{S}}), && (5.1\mathrm{b})\\ \chi^{\mathrm{L}} &= \chi(z^{\mathrm{L}}), && (5.1\mathrm{c})\\ f^{\mathrm{S}} &= f_K(\chi^{\mathrm{S}}), && (5.1\mathrm{d})\\ f^{\mathrm{L}} &= f_K(\chi^{\mathrm{L}}), \text{ and} && (5.1\mathrm{e})\\ f^{\mathrm{SL}} &= f_K(\chi^{\mathrm{S}} - \chi^{\mathrm{L}}). && (5.1\mathrm{f}) \end{aligned}$$

Some of these enter the definition of the physical critical surface mass density

$$\Sigma_{\mathrm{crit}}^{\mathrm{ph}}(z^{\mathrm{L}}, z^{\mathrm{S}}) = \left[\frac{4\pi\mathrm{G}}{\mathrm{c}^2}\frac{f^{\mathrm{SL}} f^{\mathrm{L}} a^{\mathrm{L}}}{f^{\mathrm{S}}}\right]^{-1}, \tag{5.2}$$

the comoving critical surface-mass density

$$\Sigma_{\mathrm{crit}}(z^{\mathrm{L}}, z^{\mathrm{S}}) = \left[\frac{4\pi\mathrm{G}}{\mathrm{c}^2}\frac{f^{\mathrm{L}} f^{\mathrm{SL}}}{a^{\mathrm{L}} f^{\mathrm{S}}}\right]^{-1}, \tag{5.3}$$

or the angular critical surface-mass density

$$\Sigma_{\mathrm{crit}}^{\mathrm{ang}}(z^{\mathrm{L}}, z^{\mathrm{S}}) = \left[\frac{4\pi\mathrm{G}}{\mathrm{c}^2}\frac{f^{\mathrm{SL}}}{f^{\mathrm{S}} f^{\mathrm{L}} a^{\mathrm{L}}}\right]^{-1} \tag{5.4}$$

for lenses at redshift z^{L} and sources at redshift z^{S}. In physical units, the prefactor

$$\begin{aligned} \frac{4\pi\mathrm{G}}{\mathrm{c}^2} &= 9.33196 \times 10^{-27}\,\frac{\mathrm{m}}{\mathrm{kg}} \\ &= 6.01556 \times 10^{-19}\,\frac{\mathrm{Mpc}}{\mathrm{M}_\odot}. \end{aligned} \tag{5.5}$$

Distances along the line-of-sight will be usually quantified by χ. Positions and distances transverse to the line-of-sight will be usually quantified by transverse comoving coordinates $\boldsymbol{x} = f_K(\chi)\boldsymbol{\beta}(\boldsymbol{\theta}, \chi)$ and distances (e.g. to describe the projected mass distribution of the lens).

Deflections will be quantified by the physical deflection angle $\tilde{\boldsymbol{\alpha}}$. In case of a single deflector, $\tilde{\boldsymbol{\alpha}}$ is related to the scaled deflection angle $\boldsymbol{\alpha}$ by:

$$\boldsymbol{\alpha}(\boldsymbol{\theta}, \chi^{\mathrm{S}}) = \frac{f^{\mathrm{SL}}}{f^{\mathrm{S}}}\tilde{\boldsymbol{\alpha}}\big(\boldsymbol{x}(\boldsymbol{\theta}, \chi^{\mathrm{L}})\big). \tag{5.6}$$

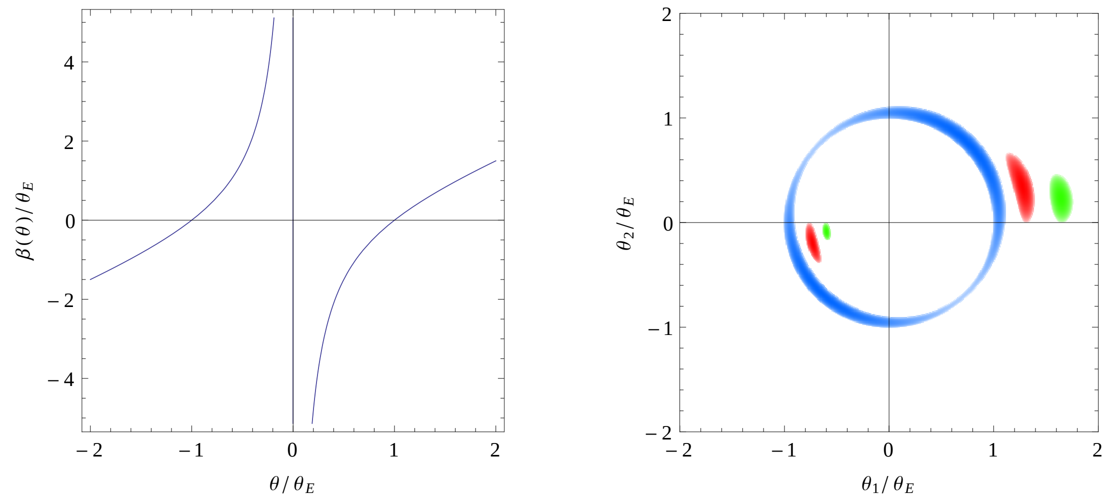

Figure 5.1: Point-mass lens: source position β as a function of image position θ (left panel), and images for circular sources (blue, red, and green) at three different source positions (right panel).

Differential deflections are quantified by $(\partial\tilde{\boldsymbol{\alpha}}/\partial\boldsymbol{x})$. The reduced distortion matrix $(\partial\boldsymbol{\alpha}/\partial\boldsymbol{\theta})$ for the case of a single lens can then be computed by:

$$\frac{\partial\boldsymbol{\alpha}(\boldsymbol{\theta},\chi^{\mathrm{S}})}{\partial\boldsymbol{\theta}} = \frac{f^{\mathrm{L}} f^{\mathrm{SL}}}{f^{\mathrm{S}}} \frac{\partial\tilde{\boldsymbol{\alpha}}\big(\boldsymbol{x}(\boldsymbol{\theta},\chi^{\mathrm{L}})\big)}{\partial\boldsymbol{x}}. \tag{5.7}$$

Lens potentials will be quantified by the comoving lens potential $\tilde{\Psi}(\boldsymbol{x})$, which is related to the physical deflection angle $\tilde{\boldsymbol{\alpha}}$ by

$$\tilde{\boldsymbol{\alpha}}(\boldsymbol{x}) = -\nabla_{\boldsymbol{x}}\tilde{\Psi}(\boldsymbol{x}), \tag{5.8}$$

and to the differential deflection $(\partial\boldsymbol{\alpha}/\partial\boldsymbol{\theta})$ by

$$\frac{\partial\tilde{\alpha}_i\big(\boldsymbol{x}(\boldsymbol{\theta},\chi^{\mathrm{L}})\big)}{\partial x_j} = \frac{\partial^2\tilde{\Psi}\big(\boldsymbol{x}(\boldsymbol{\theta},\chi^{\mathrm{L}})\big)}{\partial x_i \partial x_j}. \tag{5.9}$$

For a single lens plane, the lens potentials $\tilde{\Psi}(\boldsymbol{x})$, $\tilde{\psi}(\boldsymbol{\theta})$, and $\psi(\boldsymbol{\theta},\chi^{\mathrm{S}})$ are related by:

$$\psi(\boldsymbol{\theta},\chi^{\mathrm{S}}) = \frac{f^{\mathrm{SL}}}{f^{\mathrm{S}}}\tilde{\psi}(\boldsymbol{\theta}) = \frac{f^{\mathrm{SL}}}{f^{\mathrm{S}} f^{\mathrm{L}}}\tilde{\Psi}\big(\boldsymbol{x}(\boldsymbol{\theta})\big). \tag{5.10}$$

5.1 The point-mass lens

The deflection angle $\tilde{\boldsymbol{\alpha}}$ for a light ray passing a point mass M at redshift z^{L} with comoving impact parameter $\boldsymbol{x} = (x_1, x_2)$ much larger than the mass's Schwarzschild radius of the mass is given by:

$$\tilde{\boldsymbol{\alpha}} = (\tilde{\alpha}_1, \tilde{\alpha}_2) = -\frac{4\mathrm{G}M}{a^{\mathrm{L}}\mathrm{c}^2}\frac{\boldsymbol{x}}{r^2}. \tag{5.11}$$

Here, $r = |\boldsymbol{x}|$. Note the sign convention, which differs from the one commonly used in gravitational lensing (but follows the one commonly used in most other fields of theoretical physics).

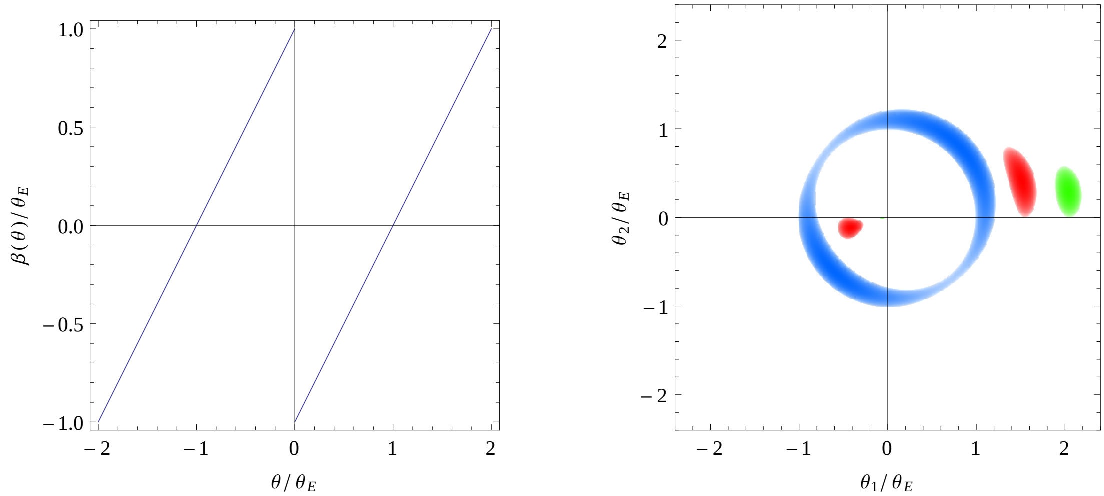

Figure 5.2: Isothermal lens: source position β as a function of image position θ (left panel), and images for circular sources (blue, red, and green) at three different source positions (right panel).

The differential deflection is given by:

$$\frac{\partial \tilde{\boldsymbol{\alpha}}}{\partial \boldsymbol{x}} = \frac{4\mathrm{G}M}{a(z^{\mathrm{L}})\mathrm{c}^2} \frac{1}{|\boldsymbol{x}|^4} \begin{pmatrix} x_0^2 - x_1^2 & 2x_0x_1 \\ 2x_0x_1 & x_1^2 - x_0^2 \end{pmatrix}. \tag{5.12}$$

All light rays of a source located at line-of-sight comoving distance $\chi^{\mathrm{S}} = \chi(z^{\mathrm{S}})$ from the observer directly behind the point-mass lens at distance $\chi^{\mathrm{L}} = \chi(z^{\mathrm{L}})$ that reach the observer impact the lens at one comoving distance r_{E} from the lens centre. The radius r_{E} is called comoving Einstein radius. Using the abbreviations (5.1), the comoving Einstein radius reads:

$$r_{\mathrm{E}} = \sqrt{\frac{4\mathrm{G}M}{a(z^{\mathrm{L}})\mathrm{c}^2} \frac{f^{\mathrm{L}} f^{\mathrm{SL}}}{f^{\mathrm{S}}}}. \tag{5.13}$$

The angular Einstein radius reads:

$$\theta_{\mathrm{E}} = \frac{r_{\mathrm{E}}}{f^{\mathrm{L}}} = \sqrt{\frac{4\mathrm{G}M}{a(z^{\mathrm{L}})\mathrm{c}^2} \frac{f^{\mathrm{SL}}}{f^{\mathrm{S}} f^{\mathrm{L}}}}. \tag{5.14}$$

The lens potential of the point mass lens reads:

$$\tilde{\Psi}(\boldsymbol{x}) = \frac{4\mathrm{G}M}{a^{\mathrm{L}}\mathrm{c}^2} \log |\boldsymbol{x}|. \tag{5.15}$$

5.2 The isothermal lens

Assume a lens at redshift z^{L} with the a *singular isothermal spherical* 3D density profile

$$\rho(r; M_{\mathrm{s}}, r_{\mathrm{s}}) = \frac{M_{\mathrm{s}}}{2r_{\mathrm{s}}^3} \frac{r_{\mathrm{s}}^2}{r^2}. \tag{5.16}$$

A projection of this density profile along the line-of-sight yields the following circularly symmetric comoving surface mass density:

$$\Sigma(r; M_\mathrm{s}, r_\mathrm{s}) = \frac{M_\mathrm{s}}{2\pi r_\mathrm{s}^2} \frac{r_\mathrm{s}}{r}. \tag{5.17}$$

The mass projected inside radius r is given by:

$$M(r; M_\mathrm{s}, r_\mathrm{s}) = M_\mathrm{s} \frac{r}{r_\mathrm{s}} \tag{5.18}$$

The deflection angle is given by:

$$\tilde{\boldsymbol{\alpha}} = -\frac{4\mathrm{G}}{a^\mathrm{L}\mathrm{c}^2} \frac{M_\mathrm{s}}{r_\mathrm{s}} \frac{\boldsymbol{x}}{r}. \tag{5.19}$$

The differential deflection is given by:

$$\frac{\partial \tilde{\boldsymbol{\alpha}}}{\partial \boldsymbol{x}} = -\frac{4\mathrm{G}}{a^\mathrm{L}\mathrm{c}^2} \frac{M_\mathrm{s}}{r_\mathrm{s}} \frac{1}{r^3} \left[\delta_{ij} r^2 - x_i x_j\right]. \tag{5.20}$$

The total mass M_tot of the lens is infinite.

6 Time delays

6.1 The light tavel time

Consider a source at spacetime position $(t^{\mathrm{S}}, \boldsymbol{\beta}^{\mathrm{S}}, \chi^{\mathrm{S}})$ emitting a photon that eventually reaches the observer. Let

$$t^{\mathrm{hom}}(\chi) = t^{\mathrm{S}} + \frac{1}{\mathrm{c}} \int_{\chi}^{\chi^{\mathrm{S}}} a\big(t^{\mathrm{hom}}(\chi')\big) \mathrm{d}\chi' \tag{6.1}$$

denote the time coordinate the photon would have in a homogenous universe. Then, the observer would receive the photon at time $t_0^{\mathrm{hom}} = t^{\mathrm{hom}}(0)$ in a completely homogenous universe.

The time $t_0 = t(0)$ at which the observer receives the photon in the inhomogeneous universe may differ from t_0^{hom}, and furthermore may differ between different paths via which the photon could reach the observer when the source is strongly lensed. Let $\big(q^{\alpha}(\chi)\big) = \big(t(\chi), \boldsymbol{\beta}(\chi), \chi\big)$ denote the photon path. According to Eq. (2.49),

$$t(\chi) = t^{\mathrm{S}} + \frac{1}{\mathrm{c}} \int_{\chi}^{\chi^{\mathrm{S}}} a\big(t(\chi')\big) n(\chi') \mathrm{d}\chi', \tag{6.2}$$

where

$$\begin{aligned} n(\chi) &= \big[1 - 2\Phi\big(t(\chi), \boldsymbol{\beta}(\chi), \chi\big)\big] \sqrt{1 + f_K^2(\chi) \left[\left(\frac{\mathrm{d}\beta_1}{\mathrm{d}\chi}\right)^2 + \cos^2(\beta_1) \left(\frac{\mathrm{d}\beta_2}{\mathrm{d}\chi}\right)^2 \right]} \\ &= 1 - 2\Phi\big(t(\chi), \boldsymbol{\beta}(\chi), \chi\big) + \frac{f_K^2(\chi)}{2} \left[\left(\frac{\mathrm{d}\beta_1}{\mathrm{d}\chi}\right)^2 + \cos^2(\beta_1) \left(\frac{\mathrm{d}\beta_2}{\mathrm{d}\chi}\right)^2 \right] + \dots \\ &= 1 - 2\Phi\big(t(\chi), \boldsymbol{\beta}(\chi), \chi\big) + \frac{f_K^2(\chi)}{2} \left| \frac{\mathrm{d}\boldsymbol{\beta}}{\mathrm{d}\chi} \right|^2 + \dots \end{aligned} \tag{6.3}$$

For a given light path $\big(t(\chi), \boldsymbol{\beta}(\chi), \chi\big)$, Eq. (6.2) can be integrated to obtain the arrival time $t(0)$. However, this approach is often impractical. Except in special situations, the integration cannot be performed analytically, but numerical integration schemes are needed. In particular, numerical errors become a problem when one wants to calculate the arrival time difference between two different light paths in a strong-lens situation, where the time difference is much smaller than the light travel times.

There are other methods to compute the travel times than direct integration. The basics of time delays between multiple images of a source lensed by a single matter inohomogeneity has be worked out by, e.g., Cooke and Kantowski (1975), Kayser and Refsdal (1983), and Borgeest (1983). The relation of the time delay to the lens equation in the single-lens-plane approximation has been discussed by, e.g., Schneider (1985). Time lelays in the multiple-plane approximation have been discussed by, e.g., Blandford and Narayan (1986) and Seitz and Schneider (1992). These works have used an eclectic approach employing analytic, geometric, and physical arguments together to derive the relevant equations for the time delay. Here, I try to provide a straight-forward, self-contained, and analytic derivation.

My approach is based on a perturbation expansion of the travel time. I define

$$t(\chi, \epsilon) = t^{\mathrm{S}} + \frac{1}{\mathrm{c}} \int_{\chi}^{\chi^{\mathrm{S}}} a\big(t(\chi', \epsilon)\big)\big(1 + \epsilon u(\chi')\big) \mathrm{d}\chi' \tag{6.4}$$

with ϵ as a formal expansion parameter for the detour

$$\begin{aligned} u(\chi) &= -1 + \left[1 - \frac{2}{\mathrm{c}^2}\Phi\big(t(\chi), \boldsymbol{\beta}(\chi), \chi\big)\right] \\ &\quad \times \sqrt{1 + f_K^2(\chi)\left[\left(\frac{\mathrm{d}\beta_1}{\mathrm{d}\chi}\right)^2 + \cos^2(\beta_1)\left(\frac{\mathrm{d}\beta_2}{\mathrm{d}\chi}\right)^2\right]} \\ &= -\frac{2}{\mathrm{c}^2}\Phi\big(t(\chi), \boldsymbol{\beta}(\chi), \chi\big) - 1 \\ &\quad + \sqrt{1 + f_K^2(\chi)\left[\left(\frac{\mathrm{d}\beta_1}{\mathrm{d}\chi}\right)^2 + \cos^2(\beta_1)\left(\frac{\mathrm{d}\beta_2}{\mathrm{d}\chi}\right)^2\right]} + \ldots \\ &= -\frac{2}{\mathrm{c}^2}\Phi\big(t(\chi), \boldsymbol{\beta}(\chi), \chi\big) + \frac{f_K^2(\chi)}{2}\left[\left(\frac{\mathrm{d}\beta_1}{\mathrm{d}\chi}\right)^2 + \cos^2(\beta_1)\left(\frac{\mathrm{d}\beta_2}{\mathrm{d}\chi}\right)^2\right] + \ldots \end{aligned} \tag{6.5}$$

The expansion of $t(\chi, \epsilon)$ around $\epsilon = 0$,

$$t(\chi, \epsilon) = \sum_{k=1}^{\infty} t^{(k)}(\chi, \epsilon) = \sum_{k=1}^{\infty} \frac{\epsilon^k}{k!} \left.\frac{\partial^k t(\chi, \epsilon)}{\partial \epsilon^k}\right|_{\epsilon=0}, \tag{6.6}$$

defines the k-th order contributions $t^{(k)}(\chi, \epsilon)$ to the time coordinate. The zeroth order reads:

$$t^{(0)}(\chi, \epsilon) = t(\chi, 0) = t^{\mathrm{hom}}(\chi). \tag{6.7}$$

For the first-order term, one obtains the following implicit equation:

$$\begin{aligned} t^{(1)}(\chi, \epsilon) &= \epsilon \left.\frac{\partial t(\chi, \epsilon)}{\partial \epsilon}\right|_{\epsilon=0} \\ &= \epsilon \frac{1}{\mathrm{c}} \int_{\chi}^{\chi^{\mathrm{S}}} \left[\dot{a}\big(t(\chi', 0)\big) \left.\frac{\partial t(\chi', \epsilon)}{\partial \epsilon}\right|_{\epsilon=0} + a\big(t(\chi', 0)\big)u(\chi')\right] \mathrm{d}\chi' \\ &= \epsilon \frac{1}{\mathrm{c}} \int_{\chi}^{\chi^{\mathrm{S}}} \left[\dot{a}\big(t^{\mathrm{hom}}(\chi')\big) \left.\frac{\partial t(\chi', \epsilon)}{\partial \epsilon}\right|_{\epsilon=0} + a\big(t^{\mathrm{hom}}(\chi')\big)u(\chi')\right] \mathrm{d}\chi' \\ &= \epsilon \frac{1}{\mathrm{c}} \int_{\chi}^{\chi^{\mathrm{S}}} \left[-\frac{\mathrm{d}\ln a\big(t^{\mathrm{hom}}(\chi')\big)}{\mathrm{d}\chi'} \left.\frac{\partial t(\chi', \epsilon)}{\partial \epsilon}\right|_{\epsilon=0} + a\big(t^{\mathrm{hom}}(\chi')\big)u(\chi')\right] \mathrm{d}\chi' \\ &= \epsilon \frac{a\big(t^{\mathrm{hom}}(\chi)\big)}{\mathrm{c}} \int_{\chi}^{\chi^{\mathrm{S}}} u(\chi') \mathrm{d}\chi'. \end{aligned} \tag{6.8}$$

One way to arrive at the last line is to define the function

$$t^{(1)}(\chi) = \left.\frac{\partial t(\chi, \epsilon)}{\partial \epsilon}\right|_{\epsilon=0}, \tag{6.9}$$

wich satisfies the integral equation:

$$t^{(1)}(\chi) = \frac{1}{\mathrm{c}} \int_{\chi}^{\chi^{\mathrm{S}}} \left[-\frac{\mathrm{d}\ln a\big(t^{\mathrm{hom}}(\chi')\big)}{\mathrm{d}\chi'} t^{(1)}(\chi') + a\big(t^{\mathrm{hom}}(\chi')\big) u(\chi') \right] \mathrm{d}\chi'. \tag{6.10}$$

This corresponds to the linear inhomogeneous ordinary differential equation:

$$\frac{\mathrm{d}t^{(1)}(\chi)}{\mathrm{d}\chi} = \frac{1}{\mathrm{c}} \frac{\mathrm{d}\ln a\big(t^{\mathrm{hom}}(\chi)\big)}{\mathrm{d}\chi} t^{(1)}(\chi) - \frac{1}{\mathrm{c}} a\big(t^{\mathrm{hom}}(\chi)\big) u(\chi). \tag{6.11}$$

The homogenous part of the solution,

$$t^{(1,\mathrm{hom})}(\chi) \propto a(t^{\mathrm{hom}}(\chi)), \tag{6.12}$$

vanishes because of the initial condition $t^{(1)}(\chi^{\mathrm{S}}) = 0$, but helps us at guessing the remaining inhomogeneous part of the solution (see Sec. 2.4):

$$t^{(1)}(\chi) = \frac{a(t^{\mathrm{hom}}(\chi))}{\mathrm{c}} \int_{\chi}^{\chi^{\mathrm{S}}} u(\chi')\mathrm{d}\chi'. \tag{6.13}$$

Differentiating the solution shows that it satisfies Eq. (6.11).

The time coordinate of the photon in the inhomogeneous universe is given by:

$$t(\chi) = t^{\mathrm{hom}}(\chi) + \frac{a(t^{\mathrm{hom}}(\chi))}{\mathrm{c}} \int_{\chi}^{\chi^{\mathrm{S}}} u(\chi')\mathrm{d}\chi' + \mathcal{O}\big[u^2\big]. \tag{6.14}$$

Thus, the arrival time of the photon can be calculated by:

$$t_0 = t(0) = t^{\mathrm{hom}}(0) + \frac{1}{\mathrm{c}} \int_0^{\chi^{\mathrm{S}}} u(\chi')\mathrm{d}\chi' + \mathcal{O}\big[u^2\big]. \tag{6.15}$$

The arrival time delay

$$\Delta = t_0 - t^{\mathrm{hom}}(0) = \frac{1}{\mathrm{c}} \int_0^{\chi^{\mathrm{S}}} u(\chi')\mathrm{d}\chi' + \mathcal{O}\big[u^2\big] \tag{6.16}$$

can be split into a potential part

$$\Delta_{\mathrm{pot}} = -\frac{2}{\mathrm{c}^3} \int_0^{\chi^{\mathrm{S}}} \Phi\big(t(\chi'), \boldsymbol{\beta}(\chi'), \chi'\big)\mathrm{d}\chi', \tag{6.17}$$

a geometric part

$$\begin{aligned} \Delta_{\mathrm{geom}} &= \frac{1}{\mathrm{c}} \int_0^{\chi^{\mathrm{S}}} \sqrt{1 + f_K^2(\chi') \left[\left(\frac{\mathrm{d}\beta_1}{\mathrm{d}\chi'}\right)^2 + \cos^2(\beta_1) \left(\frac{\mathrm{d}\beta_2}{\mathrm{d}\chi'}\right)^2 \right]} \mathrm{d}\chi' - \chi^{\mathrm{S}} \\ &= \frac{1}{2\mathrm{c}} \int_0^{\chi^{\mathrm{S}}} f_K^2(\chi') \left[\left(\frac{\mathrm{d}\beta_1}{\mathrm{d}\chi'}\right)^2 + \cos^2(\beta_1) \left(\frac{\mathrm{d}\beta_2}{\mathrm{d}\chi'}\right)^2 \right] \mathrm{d}\chi' + \dots, \end{aligned} \tag{6.18}$$

and a higher-order part, which I neglect in the following.

In the plane-sky approximation, i.e. $\cos(\beta_1) \approx 1$, the lowest-order geometric time delay reads:

$$\Delta_{\mathrm{geom}} = \frac{1}{2\mathrm{c}} \int_0^{\chi^{\mathrm{S}}} f_K^2(\chi') \left(\frac{\mathrm{d}\boldsymbol{\beta}}{\mathrm{d}\chi'}\right)^2 \mathrm{d}\chi'. \tag{6.19}$$

6.2 Time delay in the multiple-lens-plane approximation

In the multiple-plane approximation (discussed in Sec. 3.6), the comoving light path is picewise linear between successive lens planes. For $\chi^{(i)} \leq \chi \leq \chi^{(i+1)}$,

$$\begin{aligned}\boldsymbol{\beta}(\chi) &= \boldsymbol{\beta}^{(i)} + \left(\boldsymbol{\beta}^{(i+1)} - \boldsymbol{\beta}^{(i)}\right) \frac{f_K^{(i+1)}}{f_K^{(i,i+1)}} \frac{f_K\left(\chi - \chi^{(i)}\right)}{f_K(\chi)} \\ &= \boldsymbol{\beta}^{(i)} + \boldsymbol{\epsilon}^{(i)} \frac{f_K\left(\chi - \chi^{(i)}\right)}{f_K(\chi)}, \text{ and}\end{aligned} \tag{6.20}$$

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{\beta}(\chi)}{\mathrm{d}\chi} &= \left(\boldsymbol{\beta}^{(i+1)} - \boldsymbol{\beta}^{(i)}\right) \frac{f_K^{(i+1)}}{f_K^{(i,i+1)}} \\ &\quad \times \frac{f_{K,\chi}\left(\chi - \chi^{(i)}\right) f_K(\chi) - f_K\left(\chi - \chi^{(i)}\right) f_{K,\chi}(\chi)}{f_K^2(\chi)} \\ &= \boldsymbol{\epsilon}^{(i)} \frac{f_{K,\chi}\left(\chi - \chi^{(i)}\right) f_K(\chi) - f_K\left(\chi - \chi^{(i)}\right) f_{K,\chi}(\chi)}{f_K^2(\chi)},\end{aligned} \tag{6.21}$$

$$\text{where } \boldsymbol{\epsilon}^{(i)} = \left(\boldsymbol{\beta}^{(i+1)} - \boldsymbol{\beta}^{(i)}\right) \frac{f_K^{(i+1)}}{f_K^{(i,i+1)}}. \tag{6.22}$$

Thus, it is approriate to split the geometric time delay for sources at the n-th plane into contributions from (here I also use the plane-sky approximation):

$$\begin{aligned}\Delta_{\text{geom}}^{(n)} &= \frac{1}{2\mathrm{c}} \int_0^{\chi^{(n)}} f_K^2(\chi) \left(\frac{\mathrm{d}\boldsymbol{\beta}}{\mathrm{d}\chi}\right)^2 \mathrm{d}\chi \\ &= \sum_{i=1}^{n-1} \frac{1}{2\mathrm{c}} \int_{\chi^{(i)}}^{\chi^{(i+1)}} f_K^2(\chi) \left(\frac{\mathrm{d}\boldsymbol{\beta}}{\mathrm{d}\chi}\right)^2 \mathrm{d}\chi = \sum_{i=1}^{n-1} \Delta_{\text{geom}}^{(i,i+1)}.\end{aligned} \tag{6.23}$$

The geometric time delay between successive planes reads:[1]

$$\begin{aligned}\Delta_{\text{geom}}^{(i,i+1)} &= \frac{1}{2\mathrm{c}} \int_{\chi^{(i)}}^{\chi^{(i+1)}} f_K^2(\chi) \left(\frac{\mathrm{d}\boldsymbol{\beta}}{\mathrm{d}\chi}\right)^2 \mathrm{d}\chi \\ &= \frac{1}{2\mathrm{c}} \left(\boldsymbol{\beta}^{(i+1)} - \boldsymbol{\beta}^{(i)}\right)^2 \left(\frac{f_K^{(i+1)}}{f_K^{(i,i+1)}}\right)^2 \\ &\quad \times \int_{\chi^{(i)}}^{\chi^{(i+1)}} f_K^2(\chi) \left[\frac{f_{K,\chi}\left(\chi - \chi^{(i)}\right) f_K(\chi) - f_K\left(\chi - \chi^{(i)}\right) f_{K,\chi}(\chi)}{f_K^2(\chi)}\right]^2 \mathrm{d}\chi \\ &= \frac{1}{2\mathrm{c}} \left(\boldsymbol{\beta}^{(i+1)} - \boldsymbol{\beta}^{(i)}\right)^2 \left(\frac{f_K^{(i+1)}}{f_K^{(i,i+1)}}\right)^2 \frac{f_K^{(i)} f_K^{(i,i+1)}}{f_K^{(i+1)}} \\ &= \frac{1}{2\mathrm{c}} \frac{f_K^{(i)} f_K^{(i+1)}}{f_K^{(i,i+1)}} \left(\boldsymbol{\beta}^{(i+1)} - \boldsymbol{\beta}^{(i)}\right)^2 \\ &= \frac{1}{2\mathrm{c}} f_K^{(i)} \boldsymbol{\epsilon}^{(i)} \left(\boldsymbol{\beta}^{(i+1)} - \boldsymbol{\beta}^{(i)}\right).\end{aligned} \tag{6.24}$$

[1] with a little help from MATHEMATICA

Summing the contributions from all lens planes up to the source plane, one obtains the geometric time delay:

$$\Delta_{\text{geom}}^{(n)}(\boldsymbol{\theta}) = \frac{1}{\mathrm{c}} \sum_{i=1}^{n-1} \frac{1}{2} \frac{f_K^{(i)} f_K^{(i+1)}}{f_K^{(i,i+1)}} \left[\boldsymbol{\beta}^{(i+1)}(\boldsymbol{\theta}) - \boldsymbol{\beta}^{(i)}(\boldsymbol{\theta})\right]^2 . \tag{6.25}$$

Using the identities

$$\begin{aligned}
\tilde{\boldsymbol{\alpha}}^{(i)} &= \frac{f_K^{(i+1)}}{f_K^{(i+1,i)}} \left(\boldsymbol{\beta}^{(i+1)} - \boldsymbol{\beta}^{(i)}\right) - \frac{f_K^{(i-1)}}{f_K^{(i,i-1)}} \left(\boldsymbol{\beta}^{(i)} - \boldsymbol{\beta}^{(i-1)}\right), \\
f_K^{(i)} \boldsymbol{\epsilon}^{(i)} - f_K^{(i-1)} \boldsymbol{\epsilon}^{(i-1)} &= \frac{f_K^{(i+1)}}{f_K^{(i+1,i)}} \left(\boldsymbol{\beta}^{(i+1)} - \boldsymbol{\beta}^{(i)}\right) - \frac{f_K^{(i-1)}}{f_K^{(i,i-1)}} \left(\boldsymbol{\beta}^{(i)} - \boldsymbol{\beta}^{(i-1)}\right), \text{ and} \\
f_K^{(n-1)} \boldsymbol{\epsilon}^{(n-1)} &= \sum_{i=1}^{n-1} f_K^{(i)} \tilde{\boldsymbol{\alpha}}^{(i)},
\end{aligned}$$

which can be derived, e.g., from Eqs. (3.43) and (3.56), the geometric time delay can also be expressed as:

$$\begin{aligned}
\Delta_{\text{geom}}^{(n)}(\boldsymbol{\theta}) &= \frac{1}{2\mathrm{c}} \sum_{i=1}^{n-1} f_K^{(i)} \boldsymbol{\epsilon}^{(i)} \left[\boldsymbol{\beta}^{(i+1)}(\boldsymbol{\theta}) - \boldsymbol{\beta}^{(i)}(\boldsymbol{\theta})\right] \\
&= -\frac{1}{2\mathrm{c}} \sum_{i=1}^{n-1} \left(f_K^{(i)} \boldsymbol{\epsilon}^{(i)} - f_K^{(i-1)} \boldsymbol{\epsilon}^{(i-1)}\right) \boldsymbol{\beta}^{(i)}(\boldsymbol{\theta}) + \frac{1}{2\mathrm{c}} f_K^{(n-1)} \boldsymbol{\epsilon}^{(n-1)} \boldsymbol{\beta}^{(n)}(\boldsymbol{\theta}) \\
&= -\frac{1}{2\mathrm{c}} \sum_{i=1}^{n-1} f_K^{(i)} \tilde{\boldsymbol{\alpha}}^{(i)} \boldsymbol{\beta}^{(i)}(\boldsymbol{\theta}) + \frac{1}{2\mathrm{c}} f_K^{(n-1)} \boldsymbol{\epsilon}^{(n-1)} \boldsymbol{\beta}^{(n)}(\boldsymbol{\theta}) \\
&= -\frac{1}{2\mathrm{c}} \sum_{i=1}^{n-1} f_K^{(i)} \tilde{\boldsymbol{\alpha}}^{(i)} \left[\boldsymbol{\beta}^{(i)}(\boldsymbol{\theta}) - \boldsymbol{\beta}^{(n)}(\boldsymbol{\theta})\right] \\
&= -\frac{1}{2\mathrm{c}} \sum_{i=1}^{n-1} f_K^{(i)} \left[\boldsymbol{\beta}^{(n)}(\boldsymbol{\theta}) - \boldsymbol{\beta}^{(i)}(\boldsymbol{\theta})\right] \nabla_{\boldsymbol{\beta}^{(i)}} \tilde{\psi}^{(i)}(\boldsymbol{\beta}^{(i)}).
\end{aligned} \tag{6.26}$$

The potential time delay for sources on the n-th plane can be computed by a sum over contributions from each lens plane:

$$\begin{aligned}
\Delta_{\text{pot}}^{(n)}(\boldsymbol{\theta}) &= -\frac{2}{\mathrm{c}^3} \int_0^{\chi^{(n)}} \Phi\big(t(\chi), \boldsymbol{\beta}(\boldsymbol{\theta}, \chi), \chi\big) \mathrm{d}\chi \\
&= -\frac{1}{\mathrm{c}} \sum_i \frac{2}{\mathrm{c}^2} \int_0^{\chi^{(n)}} \Phi^{(i)}\big(t(\chi), \boldsymbol{\beta}(\boldsymbol{\theta}, \chi), \chi\big) \mathrm{d}\chi \\
&= -\frac{1}{\mathrm{c}} \sum_{i=1}^{n-1} f_K^{(i)} \tilde{\psi}^{(i)}(\boldsymbol{\theta}, \chi).
\end{aligned} \tag{6.27}$$

Here, $\Phi^{(i)}(t, \boldsymbol{\beta}, \chi)$ denotes the gravitational potential created by the matter projected onto lens plane i (cf. Eq. (3.50) and (3.51)), and $\tilde{\psi}^{(i)}$ denotes the lens potential on plane i (cf. Eq. (3.42)).

Adding geometric and potential contributions in the multiple-plane approximation, the time delay for light originating from the nth plane and reaching the observer from direction $\boldsymbol{\theta}$ is given by (see, e.g., Blandford and Narayan 1986; Seitz and Schneider 1992):

$$\begin{aligned}\Delta^{(n)}(\boldsymbol{\theta}) &= \frac{1}{\mathrm{c}}\sum_{i=1}^{n-1}\left\{\frac{1}{2}\frac{f_K^{(i)}f_K^{(i+1)}}{f_K^{(i,i+1)}}\left[\boldsymbol{\beta}^{(i+1)}(\boldsymbol{\theta})-\boldsymbol{\beta}^{(i)}(\boldsymbol{\theta})\right]^2-f_K^{(i)}\tilde{\psi}^{(i)}(\boldsymbol{\beta}^{(i)})\right\}\\ &= \frac{1}{\mathrm{c}}\sum_{i=1}^{n-1}f_K^{(i)}\left\{\frac{1}{2}\left[\boldsymbol{\beta}^{(n)}(\boldsymbol{\theta})-\boldsymbol{\beta}^{(i)}(\boldsymbol{\theta})\right]\left[-\nabla_{\boldsymbol{\beta}^{(i)}}\tilde{\psi}^{(i)}(\boldsymbol{\beta}^{(i)})\right]-\tilde{\psi}^{(i)}(\boldsymbol{\beta}^{(i)})\right\}\\ &= \frac{1}{\mathrm{c}}\sum_{i=1}^{n-1}f_K^{(i)}\left\{\frac{1}{2}\left[\boldsymbol{\beta}^{(n)}(\boldsymbol{\theta})-\boldsymbol{\beta}^{(i)}(\boldsymbol{\theta})\right]\tilde{\boldsymbol{\alpha}}^{(i)}(\boldsymbol{\beta}^{(i)})-\tilde{\psi}^{(i)}(\boldsymbol{\beta}^{(i)})\right\}.\end{aligned} \tag{6.28}$$

6.3 Time delay for a single strong lens

Consider a point-like source at redshift position $\boldsymbol{\beta}^{\mathrm{S}}$ and redshift z^{S} strongly and a matter structure at lens at redshift z^{L}, characterized by the lens potential $\tilde{\psi}(\boldsymbol{\theta})$. Consider an image of the source at position θ,

$$\boldsymbol{\beta}^{\mathrm{L}}(\boldsymbol{\theta}) = \boldsymbol{\theta}, \tag{6.29a}$$

$$\tilde{\boldsymbol{\alpha}}(\boldsymbol{\beta}^{\mathrm{L}}) = \nabla_{\boldsymbol{\theta}}\tilde{\psi}(\boldsymbol{\theta}), \tag{6.29b}$$

$$\boldsymbol{\beta}^{\mathrm{S}}(\boldsymbol{\theta}) = \boldsymbol{\theta} + \frac{f^{\mathrm{SL}}}{f^{\mathrm{S}}}\tilde{\boldsymbol{\alpha}}(\boldsymbol{\theta}), \tag{6.29c}$$

$$\Delta(\boldsymbol{\theta}) = \frac{1}{\mathrm{c}}\left\{\frac{1}{2}\frac{f^{\mathrm{L}}f^{\mathrm{SL}}}{f^{\mathrm{S}}}\left[\tilde{\boldsymbol{\alpha}}(\boldsymbol{\theta})\right]^2 - f^{\mathrm{L}}\tilde{\psi}(\boldsymbol{\theta})\right\}. \tag{6.29d}$$

The time delay difference $\Delta(\boldsymbol{\theta},\boldsymbol{\theta}')$ between two images at $\boldsymbol{\theta}$ and $\boldsymbol{\theta}'$ of the same source,

$$\begin{aligned}\Delta(\boldsymbol{\theta},\boldsymbol{\theta}') &= \Delta(\boldsymbol{\theta}')-\Delta(\boldsymbol{\theta})\\ &= \frac{1}{\mathrm{c}}\left\{\frac{1}{2}\frac{f^{\mathrm{L}}f^{\mathrm{SL}}}{f^{\mathrm{S}}}\left[\tilde{\boldsymbol{\alpha}}(\boldsymbol{\theta}')^2-\tilde{\boldsymbol{\alpha}}(\boldsymbol{\theta})^2\right]-f^{\mathrm{L}}\left[\tilde{\psi}(\boldsymbol{\theta}')-\tilde{\psi}(\boldsymbol{\theta})\right]\right\}\\ &= \frac{1}{\mathrm{c}}\left\{\frac{1}{2}\frac{f^{\mathrm{L}}f^{\mathrm{SL}}}{f^{\mathrm{S}}}\left[\tilde{\boldsymbol{\alpha}}(\boldsymbol{\theta}')-\tilde{\boldsymbol{\alpha}}(\boldsymbol{\theta})\right]\left[\tilde{\boldsymbol{\alpha}}(\boldsymbol{\theta}')+\tilde{\boldsymbol{\alpha}}(\boldsymbol{\theta})\right]-f^{\mathrm{L}}\left[\tilde{\psi}(\boldsymbol{\theta}')-\tilde{\psi}(\boldsymbol{\theta})\right]\right\}\\ &= -\frac{1}{\mathrm{c}}f^{\mathrm{L}}\left\{\frac{1}{2}\left[\boldsymbol{\theta}'-\boldsymbol{\theta}\right]\left[\tilde{\boldsymbol{\alpha}}(\boldsymbol{\theta}')+\tilde{\boldsymbol{\alpha}}(\boldsymbol{\theta})\right]+\left[\tilde{\psi}(\boldsymbol{\theta}')-\tilde{\psi}(\boldsymbol{\theta})\right]\right\}.\end{aligned} \tag{6.30}$$

Now write the lensing potential and the deflection angle as:

$$\tilde{\psi}(\boldsymbol{\theta}) = \tilde{\psi}_0 - \tilde{\boldsymbol{\alpha}}_0^{\mathrm{t}}\boldsymbol{\theta} - \frac{1}{2}\frac{f^{\mathrm{S}}}{f^{\mathrm{SL}}}\boldsymbol{\theta}^{\mathrm{t}}\mathbf{U}\boldsymbol{\theta} + \tilde{\psi}_{3+}(\boldsymbol{\theta}), \tag{6.31}$$

$$\tilde{\boldsymbol{\alpha}}(\boldsymbol{\theta}) = \tilde{\boldsymbol{\alpha}}_0 + \frac{f^{\mathrm{S}}}{f^{\mathrm{SL}}}\mathbf{U}\boldsymbol{\theta} + \tilde{\boldsymbol{\alpha}}_{3+}(\boldsymbol{\theta}). \tag{6.32}$$

Then the time delay

$$
\begin{aligned}
\Delta(\boldsymbol{\theta},\boldsymbol{\theta}') &= -\frac{1}{\mathrm{c}} f^{\mathrm{L}} \left\{ \frac{1}{2}\big[\boldsymbol{\theta}'-\boldsymbol{\theta}\big]\big[\tilde{\boldsymbol{\alpha}}(\boldsymbol{\theta}')+\tilde{\boldsymbol{\alpha}}(\boldsymbol{\theta})\big] + \big[\tilde{\psi}(\boldsymbol{\theta}')-\tilde{\psi}(\boldsymbol{\theta})\big] \right\} \\
&= -\frac{1}{\mathrm{c}} f^{\mathrm{L}} \left\{ \frac{1}{2}\big[\boldsymbol{\theta}'-\boldsymbol{\theta}\big]\big[\tilde{\boldsymbol{\alpha}}_{3+}(\boldsymbol{\theta}')+\tilde{\boldsymbol{\alpha}}_{3+}(\boldsymbol{\theta})\big] + \big[\tilde{\psi}_{3+}(\boldsymbol{\theta}')-\tilde{\psi}_{3+}(\boldsymbol{\theta})\big] \right\} \\
&\quad - \frac{1}{\mathrm{c}} f^{\mathrm{L}} \left\{ \frac{1}{2}\big[\boldsymbol{\theta}'-\boldsymbol{\theta}\big]\big[2\tilde{\boldsymbol{\alpha}}_0 + \frac{f^{\mathrm{S}}}{f^{\mathrm{SL}}}\mathbf{U}(\boldsymbol{\theta}'+\boldsymbol{\theta})\big] \right. \\
&\quad \left. + \big[-\tilde{\boldsymbol{\alpha}}_0(\boldsymbol{\theta}'-\boldsymbol{\theta}) - \frac{1}{2}\frac{f^{\mathrm{S}}}{f^{\mathrm{SL}}}(\boldsymbol{\theta}'\mathbf{U}\boldsymbol{\theta}'-\boldsymbol{\theta}\mathbf{U}\boldsymbol{\theta})\big] \right\} \\
&= -\frac{1}{\mathrm{c}} f^{\mathrm{L}} \left\{ \frac{1}{2}\big[\boldsymbol{\theta}'-\boldsymbol{\theta}\big]\big[\tilde{\boldsymbol{\alpha}}_{3+}(\boldsymbol{\theta}')+\tilde{\boldsymbol{\alpha}}_{3+}(\boldsymbol{\theta})\big] + \big[\tilde{\psi}_{3+}(\boldsymbol{\theta}')-\tilde{\psi}_{3+}(\boldsymbol{\theta})\big] \right\} .
\end{aligned}
\tag{6.33}
$$

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