1

PRIMORDIAL BLACK HOLES:

FORMATION AND ABUNDANCE

SAM YOUNG

17.10.2019

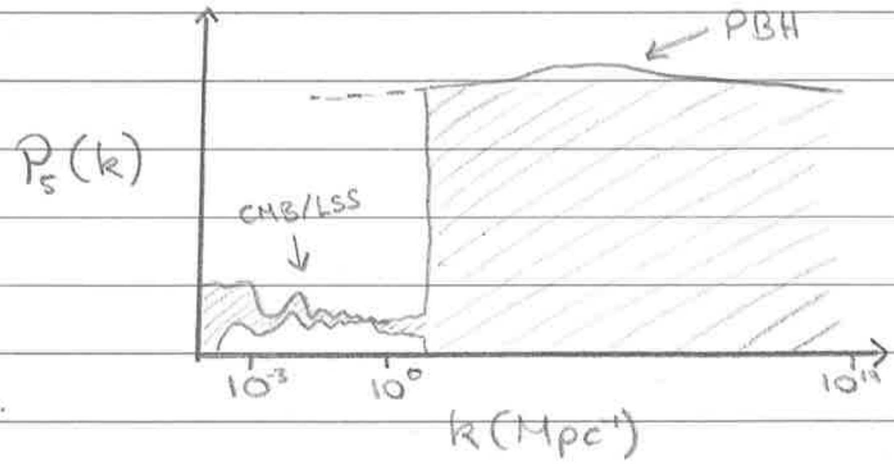
Contents

SECTION 1: WHY SHOULD I CARE ABOUT PBHs?

PBHs are interesting for several reasons:

1) They're black holes. From the dawn of the Universe. THIS IS COOL.

2) They provide unique constraints on the primordial power spectrum.



3) They're viable dark matter candidates

	CDM	PBHs
We can't see it	✓	✓
Little to no interactions with SM particles	✓	✓
Gravitational interactions	✓	✓
"Cold"	✓	✓

3

4) We *MIGHT* have already seen them
- Clesse et al (2017)
- LIGO black holes (unusual mass and low spin)
- Micro-lensing events
- Correlation between IR and X-ray background
- Early existence of SMBHs

5) They can explain a lot about the early universe, potentially responsible for reheating, or for the matter/anti-matter asymmetry

SECTION 2: WHAT ARE PBHs?

PBHs are black holes which may have formed in the early universe. There are various formation mechanisms, which we will see soon.

PBH mass is normally roughly equal to the horizon mass at the time of formation

$$M_{PBH} \sim M_H \sim 10^{15}\left(\frac{t}{10^{-23}s}\right)g$$

- depends on the number of relativistic degrees of freedom.

At such early times, the density is already much higher than in stars today, and there is no need to overcome electron or neutron degeneracy pressure
- arbitrarily small PBHs can hypothetically form (maybe with a MINIMUM of the Planck mass)

As realised by Hawking, BHs give out radiation and evaporate on a timescale $\tau(M_{PBH})$

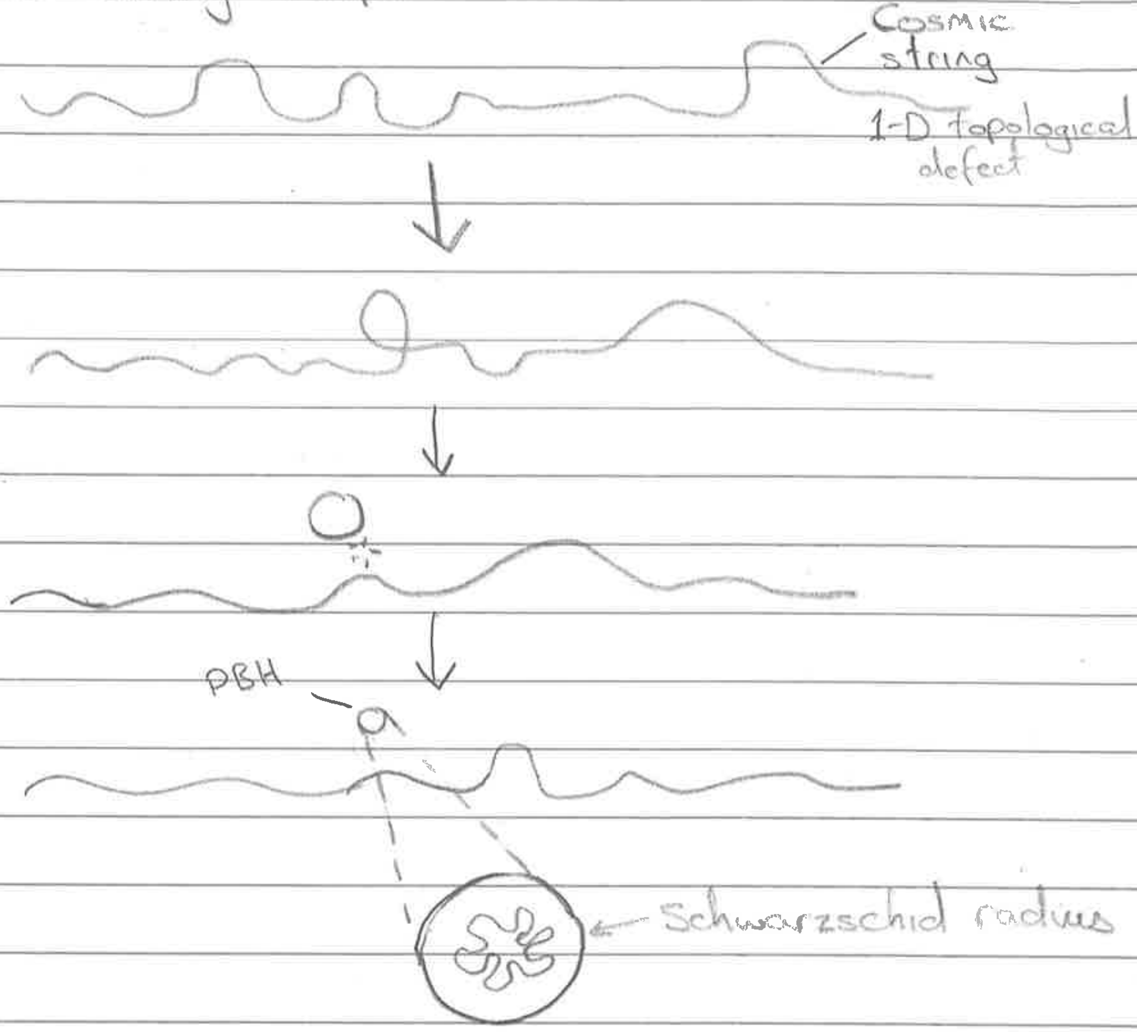
$$\tau(M_{PBH}) \sim \frac{\hbar c^4}{G^2 M_{PBH}^3} \sim 10^{10}\left(\frac{M_{PBH}}{10^{15}g}\right)^3 yr$$

→ PBHs of $\sim 10^{15}g$ are evaporating today
Lighter PBHs have already evaporated.
As BHs get smaller, they also get hotter - resulting in PBH "bombs"

⑤

2.1 : FORMATION MECHANISMS

1) Cosmic string loops

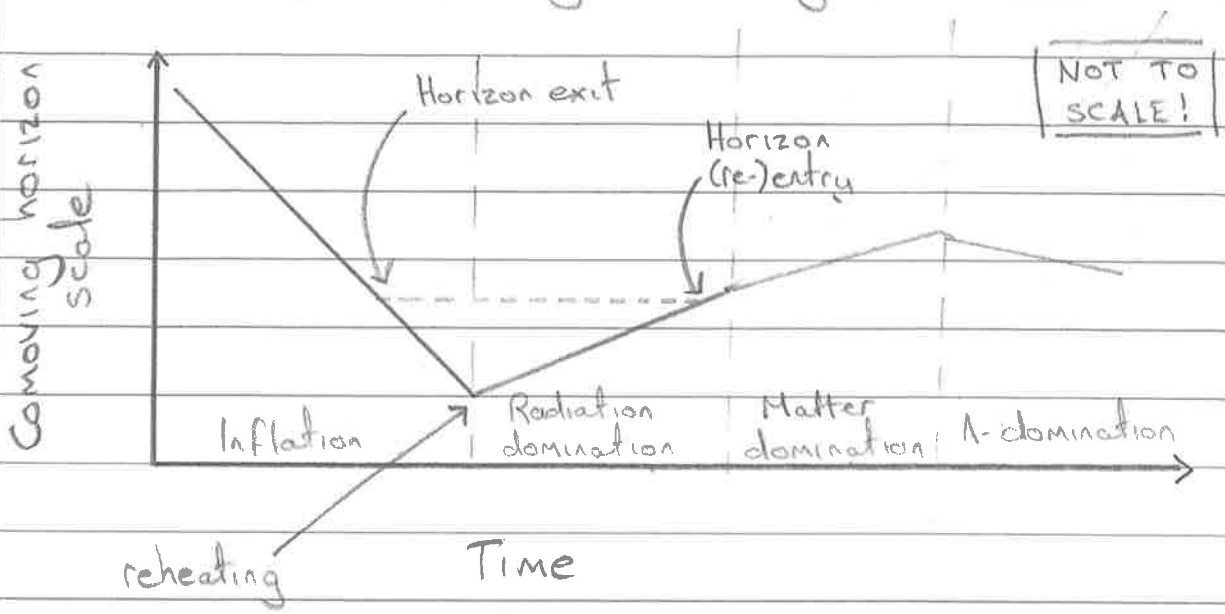


2) Bubble collisions

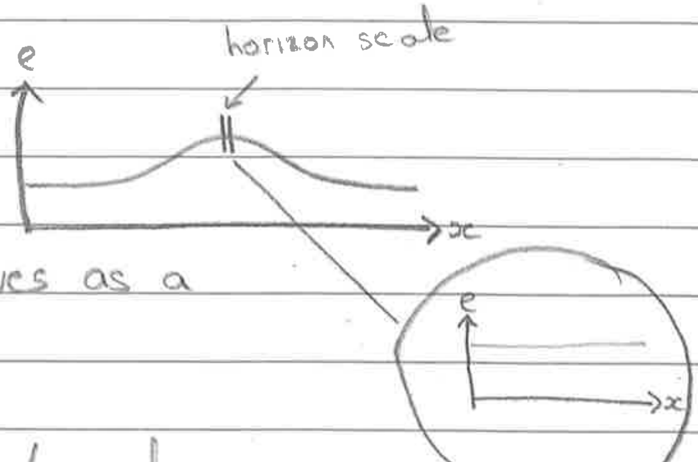
Phase transitions can occur through 'bubble nucleations' of the new phase, which then expand and collide. The collisions can form PBHs with mass approximately the horizon mass.

6

3) The collapse of large density fluctuations



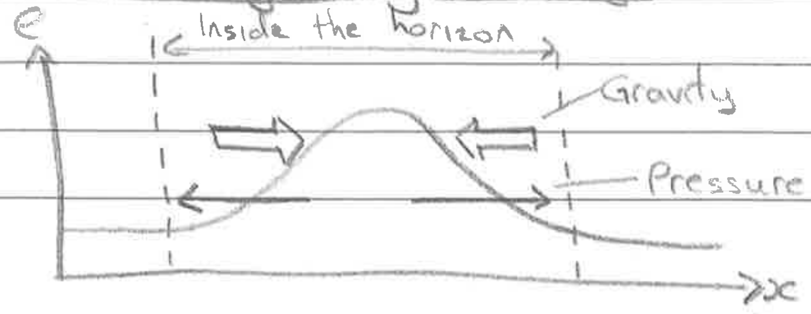
During inflation, the comoving horizon scale shrinks. Upon horizon exit, classical density perturbations are formed from quantum fluctuations.
After inflation ends, the Hubble grows, and modes re-enter the horizon.
If a perturbation has a large enough amplitude, gravity can overcome pressure forces, and collapse to form a PBH.



Super-horizon regime

Each Hubble patch evolves as a separate flat universe

Horizon crossing/entry/re-entry



2.2: HOW BIG ARE PBHs?

In most mechanisms, PBHs form with approximately the horizon mass, but what is the Schwarzschild radius?

Hubble radius: $r_H = H^{-1}$

Density of a flat Friedmann universe:

$$\rho_c = \frac{3H^2}{8\pi G}$$

Horizon mass:

$$M = \frac{4}{3}\pi R_H^3 \rho_c$$

$$= \frac{4}{3}\pi H^{-3} \times \frac{3H^2}{8\pi G}$$

$$= \frac{H^{-1}}{2G}$$

Schwarzschild radius

$$R_s = 2GM$$

$$= 2G \times \frac{H^{-1}}{2G}$$

$$= H^{-1} = R_H$$

$$\boxed{R_s = R_H}$$

SECTION 3: CONSTRAINTS ON THE ABUNDANCE OF PBHs

DISCLAIMER: old and new constraints are constantly being debated and updated. Constraints come from a huge range of sources, and both observations and calculations are becoming more sophisticated.
See Carr, Kühnel & Sandstad (2016), 1607.06077 for a recent (ish) compilation of constraints

How is the abundance stated?

1) The mass fraction of the universe collapsing to form PBHs at the time of formation*

$$\beta \equiv \frac{\rho_{PBH}}{\rho_c}$$

PBH energy density

Background density of the universe

$$\rho_c = \frac{3H^2}{8\pi G}$$

2) The fraction of dark matter composed of PBHs

$$f \equiv \frac{\rho_{PBH}}{\rho_{CDM}} = \frac{\Omega_{PBH}}{\Omega_{CDM}}$$

Constraints on the abundance can be grouped into several categories:

1) Evaporation
2) Lensing
3) Dynamical effects
4) Miscellaneous

*I prefer to take the time of formation as the horizon crossing time. The actual formation time is more complicated.

3.1: Evaporation constraints

PBHs with mass $\lesssim 10^{15}$g have already evaporated, We can constrain their abundance by looking for the effects of their evaporation

Relic particles

Evaporating PBHs may leave behind a stable Planck mass relic, The abundance of these relics today cannot exceed the dark matter content.
$M_{PBH} \lesssim 10^{15}$g

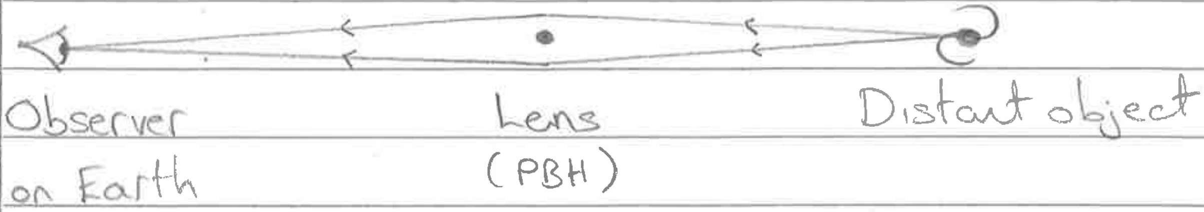
Nucleosynthesis

PBHs evaporating during nucleosynthesis would affect the abundance of different elements forming in the universe. $10^{9}g < M_{PBH} < 10^{10}g$

γ-rays

PBHs evaporating now (or in the recent past) would emit a lot of γ-rays that we should observe. $10^{13}g < M_{PBH} < 10^{17}g$
It has been suggested this is where GRBs come from

3.2: Lensing constraints



A PBH passing in front of a distant object can cause a change in the apparent magnitude of that object.

GRB Femtolensing
Lensing around a PBH can cause an interference pattern in GRBs, if the time delay is close to the period of the γ-rays. $10^{17}g < M_{PBH} < 10^{20}g$

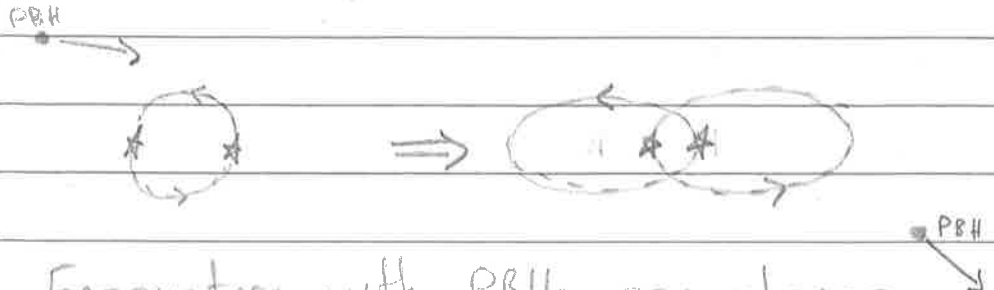
Galactic microlensing
EROS & MACHO surveys looked for MAssive Compact Halo Objects by looking for temporary increase in the brightness of stars in the large and Small Magellanic Clouds. $10^{26}g < M_{PBH} < 10^{34}g$.

Quasar microlensing
PBH lensing could increase the continuum emission spectrum without affecting the line emission.
$10^{30}g < M_{PBH} < 10^{35}g$

Radio source millilensing
Multiple images of radio sources could be resolved with Very Long Base Interferometry.
$10^{39}g < M_{PBH} < 10^{41}g$

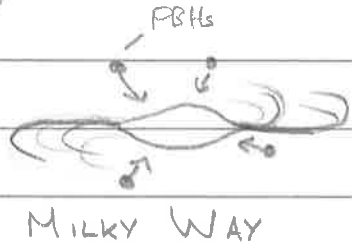
3.3: Dynamical effects

Disruption of wide binaries



Encounters with PBHs can change the orbital parameters of wide binary systems

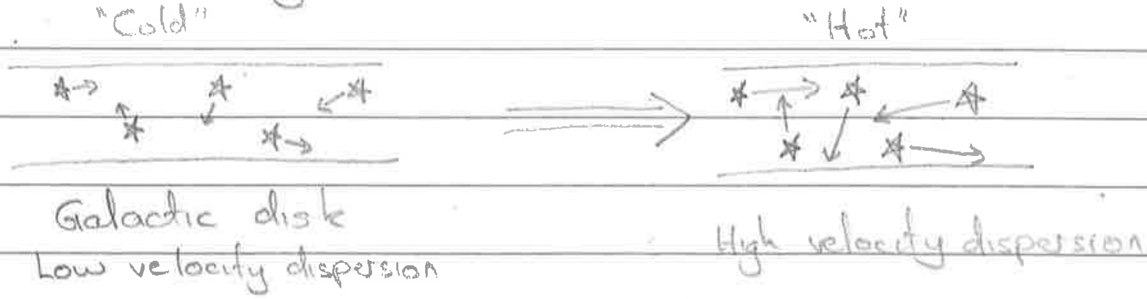
Dynamic friction



Compact objects will be dragged into the centre of the Milky Way. Constraints on the central mass constrain PBH abundance.

$10^{37}g < M_{PBH} < 10^{45}g$

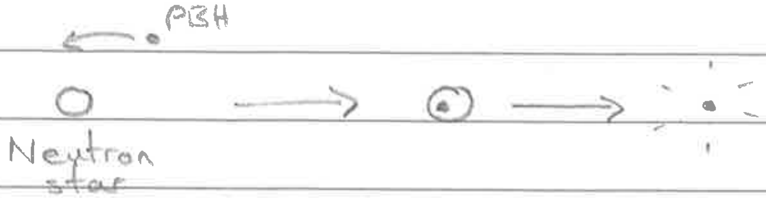
Disk heating



PBHs traversing the galactic disk can increase the velocity dispersion. $10^{40}g < M_{PBH} < 10^{45}g$

3.4: Other constraints

Stars



Neutron stars can capture PBHs and be quickly destroyed. Similarly, accretion of PBHs during formation can prevent stars forming.

$10^{16}g < M_{PBH} < 10^{24}g$

Gravitational waves

The large perturbations necessary for PBH formation also generate second order tensor perturbations, which is constrained by pulsar timing.

$10^{35}g < M_{PBH} < 10^{37}g$

CMB

Accretion onto PBHs at the time of decoupling would produce X-rays, and produce a measurable effect on the CMB spectrum and anisotropies

$10^{34}g < M_{PBH} < 10^{43}g$

LSS

Poisson fluctuations in the PBH number density enhance the DM power spectrum.

$10^{37}g < M_{PBH} < 10^{43}g$

13

3.5: Dark matter content

The PBH density cannot exceed the dark matter density of the universe.

→ lets assume radiation domination until the time of matter-radiation equality, at which point $\Omega_{PBH}\big|_{eq} \leq \Omega_{CDM}\big|_{eq} = 0.5$

Lets look at how the initial PBH abundance is related to the density parameter at matter radiation equality

$$a(t) \propto t^{1/2}$$

$$r_H = H^{-1} = \frac{a}{\dot{a}} \propto \frac{t^{1/2}}{t^{-1/2}} \propto a^2$$

$$\rho_{rad} \propto a^{-4}$$

$$\rho_{mat} \propto a^{-3}$$

Horizon mass: $M_H \propto r_H^3 \rho_{rad} \propto a^2$

During radiation domination, the matter density parameter increases $\propto a$

$$\Omega_{Mat} = \frac{\rho_{Mat}}{\rho_{rad}} \propto \frac{a^{-3}}{a^{-4}} \propto a^1$$

$$\Longrightarrow \Omega_{PBH} \propto a \propto M_H^{1/2}$$

$$\Longrightarrow \Omega_{PBH}\Big|_{eq} = \underbrace{\left(\frac{M_{PBH}}{M_{eq}}\right)^{-1/2}}_{\text{Red-shift of matter density parameter}} \beta < \Omega_{CDM}\Big|_{eq} = 0.5$$

$M_{PBH} \ll M_{eq}$ so β is typically very small

SECTION 4: FORMATION OF PBHs FROM DENSITY FLUCTUATIONS

4.1: Carr and Hawking (1974)

Bernard Carr (the PBH Godfather) and Stephen Hawking's pioneering work provided an order of magnitude estimate for how large the amplitude of a perturbation must be to collapse.

They assumed an overdense region was spherically symmetric and part of a closed Friedmann universe. Using a Jeans length argument, for gravity to overcome pressure, this leads to a requirement at horizon entry

$$\delta = \frac{\delta e}{e} \gtrsim \omega = \frac{1}{3} \Rightarrow \delta_c \approx \frac{1}{3}.$$

The PBHs form with $M_{PBH} \approx \omega^{3/2} M_H$

4.2: Refinements

Shibata and Sasaki (1999) studied PBH formation by numerically evolving a metric perturbation and looking for the formation of a perturbation.

This result was used by Green, Liddle, Malik and Sasaki (2004) to define a critical value $0.7 < \zeta_c < 1.2$

→ we'll see later why this doesn't work well

Niemeyer and Jedamzik (1999) numerically investigated PBH formation, finding $\delta_c \sim 0.7$. They found a critical scaling law

$$M_{PBH} = K M_H (\delta - \delta_c)^{\gamma}$$

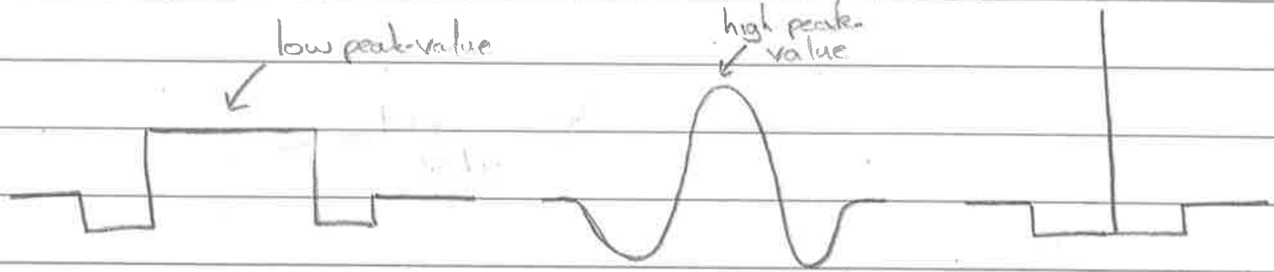
Since 2000 a great deal of work has been completed, notably by Musco, Miller, Harada and Nakama.

→ Most of the work focusses on spherically symmetric perturbations (expected of large, rare perturbations, see BBKS (1986)) during radiation domination.

→ I'll (mostly) stick to these approximations

General results

A range of perturbation shapes is considered, which can affect the threshold value and mass



"Top-hat"
High average-value

"Mexican-hat"
Low average-value

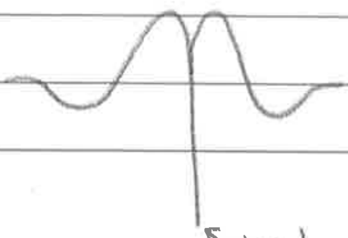
"Delta function"
(~~some analytic~~ estimates correspond to this - no pressure gradient)

"Flat" ⟷ "Spiky"

Volume-average: $\underset{\text{spiky}}{0.4} \lesssim \delta_c \lesssim \underset{\text{flat}}{2/3}$

Peak-value*: $\underset{\text{flat}}{2/3} < \delta_c < \underset{\text{spiky}}{\infty}$

* For off-centred peaks, this can theoretically have any value



"peak central value"

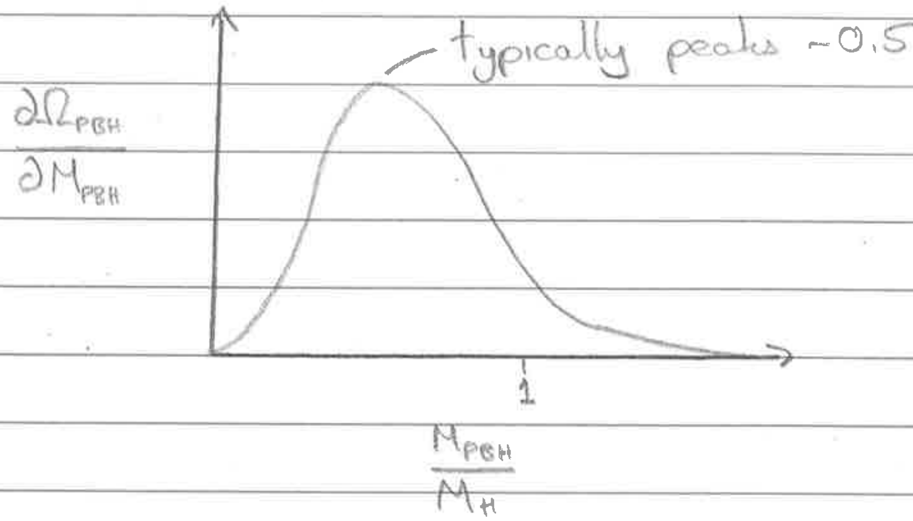
$$M_{PBH} = k M_H (\delta - \delta_c)^{\gamma}$$

The exact values of k and γ depend on the profile shape considered, but generally:

$$k \sim 4$$
$$\gamma \sim 0.36$$

(for a volume-averaged δ_c)

This results in an extended mass-function at PBH formation:



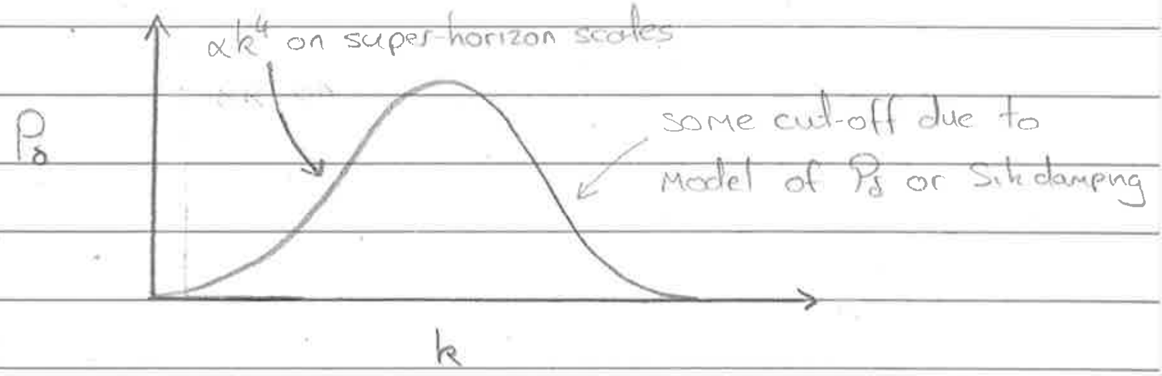
4.3: Profile shape

The average profile shape can be predicted from the shape of the 2-point correlation function, $P_\delta(k,t)$

$$F(r) = \frac{\xi(r,t)}{\xi(0,t)}$$

$$\xi(r,t) = \int_0^\infty \frac{dk}{k} \cdot \underbrace{\frac{\sin(kR)}{kR}}_{\text{comes from the Fourier transform}} P_\delta(k,t) \quad *$$

Generally, the density power spectrum is expected to look something like



For a broad range of power spectra typically considered, the central region of the profile is well described by that predicted from a δ_D power spectrum (no time dependance)

$$P_\delta = A\,\delta_D(k-k_*)$$

$$\xi(r) = \int \frac{dk}{k} \frac{\sin(kR)}{kR} A\,\delta_D(k-k_*) = \frac{A\sin(k_* r)}{k^2 R}$$

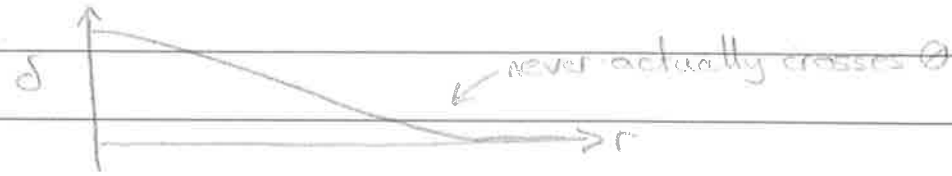
$$\Longrightarrow F(r) \sim \frac{\sin(r)}{r} = \mathrm{sinc}(r)$$

*This needs both a UV and IR cut-off

4.4: Perturbation scale and PBH mass

We've been defining the PBH mass relative to the horizon at the time a perturbation enters the horizon → to do this we need to define the characteristic scale of a perturbation

→ "Traditional" scale: r_0 defined as $\delta(r_0) = 0$. Problematic for i.e. a Gaussian profile

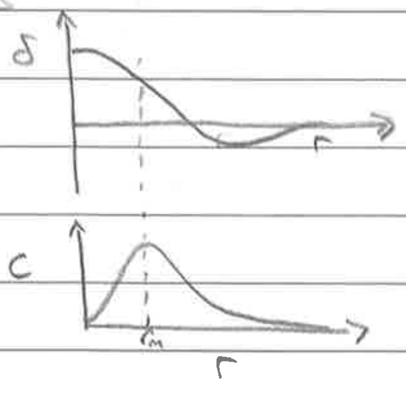


→ "Compaction" scale: based on the condition for a Schwarzschild event horizon, $\frac{2M}{R} = 1$. r_M is defined as the scale at which the compaction function peaks

$$\left.\frac{dC}{dr}\right|_{r=r_M} = 0$$

$$C(r) = \frac{2\,(M(r) - M_b(r))}{r}$$

i.e. The mass excess compared to a flat universe within areal radius r, divided by the radius



→ r_M is then used to determine the horizon mass

(19)

4.5: Comparison with LSS

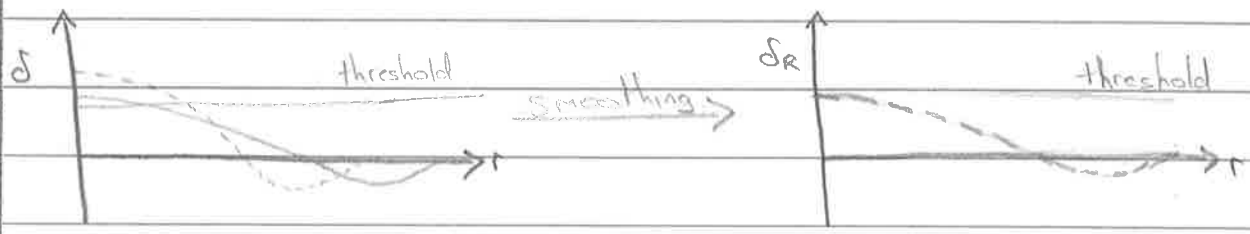
In LSS, the mass of a compact object depends on the largest smoothing scale at which the initial perturbation is above some critical value.

δ - unsmoothed

δ_R - smoothed

Perturbation A: smaller amplitude, larger scale ———

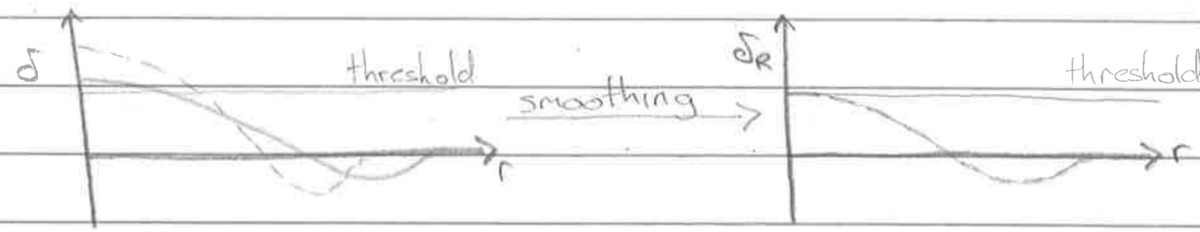
Perturbation B: larger amplitude, smaller scale — — —



Correct conclusion: both objects form with the same mass

This works because perturbations are $O(10^{-5})$, so the total mass is hardly affected.

Let's consider the same scenario for PBHs:



Incorrect conclusion: both objects form with the same mass

SECTION 5: CALCULATING THE ABUNDANCE

Let's first discuss the calculation of β (which is the tricky bit), then later the total PBH density and the mass function, and we'll assume Gaussian statistics for now.

5.1: Smoothing functions

A smoothing function removes (reduces the amplitude of) perturbations smaller than the smoothing scale, R. (You can also define functions to remove larger scales)

The smoothing essentially just finds the average of a field around a point.

Real-space: convolution (messy)

$$\delta_R(x) = \int d^3y\, W(\bar{x}-\bar{y}R)\,\delta(\bar{y})$$

Fourier-space: multiplication (easy)

$$\delta_R(k) = \tilde{W}(kR)\,\delta(k)$$

This makes the calculation of the variance $\langle \delta_R^2 \rangle$ straight forward:

$$\langle \delta^2 \rangle = \int \frac{dk}{k} P_\delta(k,t)$$

$$\langle \delta_R^2 \rangle = \int \frac{dk}{k} P_\delta(k,t)\, W(k,R)$$

I'll discuss more later the use of smoothing functions

5.2: Press-Schechter

Basic idea: pick a region at random and look at the average density. If its over δ_c, a PBH forms

$$\beta = 2\int_{\delta_c}^{\infty} d\delta_R \, P_G(\delta_R)$$

$$= 2\int_{\delta_c}^{\infty} d\delta_R \frac{1}{\sqrt{2\pi\langle\delta_R^2\rangle}} \exp\left(-\frac{\delta_R^2}{2\langle\delta_R^2\rangle}\right)$$

$$= \mathrm{erfc}\left(\frac{\delta_c}{\sqrt{2\langle\delta_R^2\rangle}}\right)$$

$$\Big\downarrow \text{asymptotic expansion}$$

$$\beta = \sqrt{\frac{\langle\delta_R^2\rangle}{2\pi\delta_c^2}} \exp\left(-\frac{\delta_c^2}{2\langle\delta_R^2\rangle}\right)$$

Where does this factor of 2 come from?

Short answer: The PS formalism was derived for LSS. If $\delta_c \to 0$, then all matter in the universe should be contained in compact objects. However $\int_0^{\infty} d\delta_R P_G(\delta_R) = \frac{1}{2}$. So an extra factor of 2 is normally 'fudged'

→ This is not relevant to PBHs, which are extremely rare, and shouldn't be included.

⇒ Constraints have historically been derived using this method, by assuming all PBHs form with the horizon mass.

In reality, its a lot more complicated. The mass function depends on the shape of P_S, β depends on the mass function, and constraints also depend on the mass function.

5.3: A simple derivation of constraints on the power spectrum

Assumptions:

- DM is made entirely of $1M_\odot$ PBHs, $\beta \lesssim 10^{-9}$
- All PBHs form with exactly $M_{PBH} = M_H$
- The power spectrum P_ζ is scale invariant at the relevant scales, $P_\zeta = A_s$
- ζ is related linearly to δ, $\delta(k,t) = \frac{4}{9} T(k,t) \left(\frac{k}{aH}\right)^2 \zeta(k)$
- Perturbations evolve linearly until horizon crossing

$$P_\delta(k,t) = \frac{16}{81}\left(\frac{k}{aH}\right)^4 T^2(k,t)\, P_\zeta(k)$$

$(aH)^{-1}$ is time dependant

$$T(k,t) = 3\,\frac{\sin\left(\frac{k}{\sqrt{3}aH}\right) - \frac{k}{\sqrt{3}aH}\cos\left(\frac{k}{\sqrt{3}aH}\right)}{\left(\frac{k}{\sqrt{3}aH}\right)^3}$$

- a simple derivation of this is to assume that all the matter initially at a point spreads itself equally over a sphere with radius of the sound horizon.

A real-space top-hat smoothing function in Fourier space becomes

$$\tilde{W}(k,(aH)^{-1}) = 3\,\frac{\left(\sin\left(\frac{k}{aH}\right) - \frac{k}{aH}\cos\left(\frac{k}{aH}\right)\right)}{\left(\frac{k}{aH}\right)^3}$$

(note the similarity to $T(k,t)$)

$$\Rightarrow \langle \delta_R^2 \rangle = \int \frac{dk}{k} \tilde{W}^2(k,(aH)^{-1})\, P_\delta = \int \frac{dk}{k}\frac{16}{81}\left(\frac{k}{aH}\right)^4 T^2(k,t)\, \tilde{W}^2(k,(aH)^{-1})\, A_s$$

$$\simeq 1.0607 A_s$$

(23)

Now return to the Press-Schecter

$$\beta \approx \sqrt{\frac{\langle \delta_R^2 \rangle}{2\pi \delta_c^2}} \exp\left(\frac{-\delta_c^2}{2\langle \delta_R^2 \rangle}\right)$$

$$\Longrightarrow \langle \delta_R^2 \rangle \approx \frac{\delta_c^2}{\ln(1/\beta)}$$

- constraints depend only logarithmically on constraints on β

Using $\delta_c = 0.5$, $\beta < 10^{-9}$, $\langle \delta_R^2 \rangle = 1.0607 A_s$ gives

$$A_s \lesssim 6.5 \times 10^{-3}$$

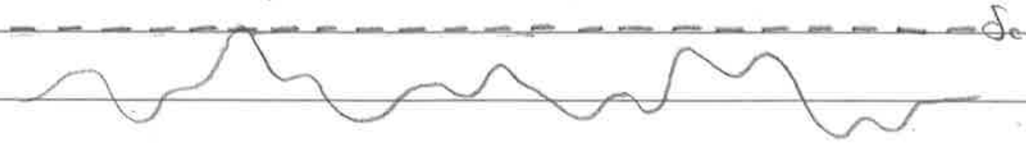
5.4: The theory of peaks (BBKS (1986))

Basic idea: pick a peak at random and look at the density. If it is over δ_c, a PBH forms.

Naively, I expected this to predict less PBHs due to needing to fulfill extra requirements.

PS: $\delta > \delta_c$

Peaks: $\delta > \delta_c$, $\bar{\nabla}\delta = 0$, $\bar{\nabla}^2 \delta < 0$



PS: ~1% is above δ_c

Peaks: 1 peak in 10 is above δ_c

(NB. oversimplification)

NB. Peaks theory can be applied to a smoothed or unsmoothed field. The only difference is changes to P

In the high peak limit, the number density of peaks in the range $\nu \to \nu + d\nu$ is

$$n = \frac{1}{4\pi^2}\left(\frac{\sigma_0}{\sigma_1}\right)^3 \nu^3 \underbrace{\exp\left(-\frac{\nu^2}{2}\right)}_{\text{From Gaussian pdf}}$$

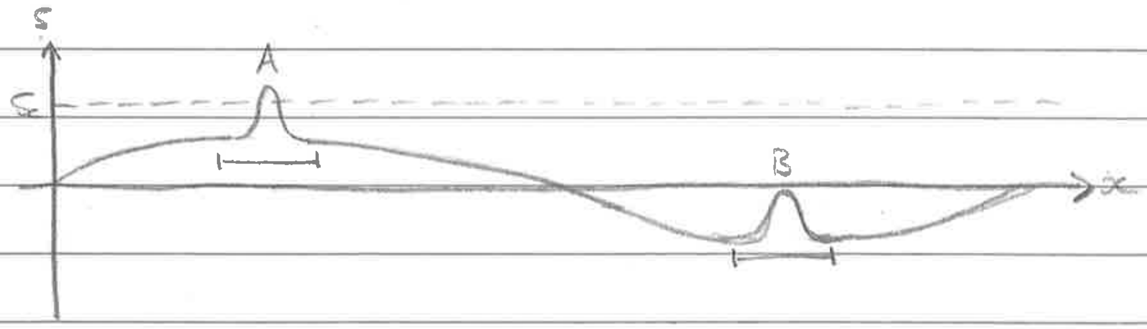
where $\nu = \frac{\delta}{\sigma_0}$

$$\sigma_n^2 = \int \frac{dk}{k} k^n P_\delta(k)$$

To find the total abundance of peaks over ν_c, integrate:

$$\beta = \int_{\nu_c}^{\infty} d\nu \, n$$

Based on Shibata and Sasaki (1999), peaks theory was first applied to PBHs by Green, Liddle, Malik and Sasaki (2004), which became the standard for ~10 years. However, it used peaks in ζ...



ζ_c was calculated assuming a flat background ($\zeta_{Bg} = 0$), but this isn't generally true.

Separate Hubble regions evolve as separate universes, so A and B should follow the same behaviour until the horizon is large enough to see the long-wavelength mode.

(and thats my most cited paper)

5.5: Critical mass scaling relationship

Basic idea: density = mass x number density.

If there is one PBH-forming peak per billion Hubble volumes ($n=10^{-9}$), and a PBH forms with 10% of the horizon mass ($\frac{M_{PBH}}{M_H}=0.1$), then $\beta=10^{-10}$

$$M_{PBH} = K M_H (\delta-\delta_c)^{\gamma}$$

$$\Downarrow$$

$$\beta = \int_{\delta_c}^{\infty} d\delta \, K(\delta-\delta_c) \times \frac{1}{4\pi^2}\left(\frac{\sigma_0}{\sigma_1}\right)^3 \left(\frac{\delta}{\sigma_0}\right)^3 \exp\left(-\frac{\delta^2}{2\sigma_0^2}\right)$$

→ This is almost always ignored

5.6: non-linear relationship between ζ and δ

Basic idea: even if ζ is Gaussian, δ won't be

$[ds^2 = dt^2 + a^2(t)e^{2\zeta}dX^2$

$$\delta = -\underbrace{\frac{4(1+\omega)}{5+3\omega}}_{8/9}\left(\frac{1}{aH}\right)^2 \exp(-5\zeta)\,\bar{\nabla}^2\left[\exp\left(\frac{\zeta}{2}\right)\right]$$

- defined on a comoving uniform-cosmic time slicing in the super-horizon limit

Assuming spherical symmetry (valid for high peaks)

$$\bar{\nabla}^2 = \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}$$

$$\delta = -\frac{4}{9}\left(\frac{1}{aH}\right)^2 \exp(-2\zeta(r))\left[\zeta''(r) + \frac{2}{r}\zeta'(r) + \frac{1}{2}(\zeta'(r))^2\right]$$

- this is very messy. It depends on ζ, ζ' and ζ''
- This is messier than it looks. Areal radius $R = a(t)\exp(\zeta(r))r$
- It looks like it depends on ζ - did we break the separate universe approach?

Lets make the time dependance more explicit:

$$\varepsilon(t) = \frac{\hat{r}_H}{\hat{r}_M} = \frac{1}{aHe^{\zeta(r)} r_m}$$

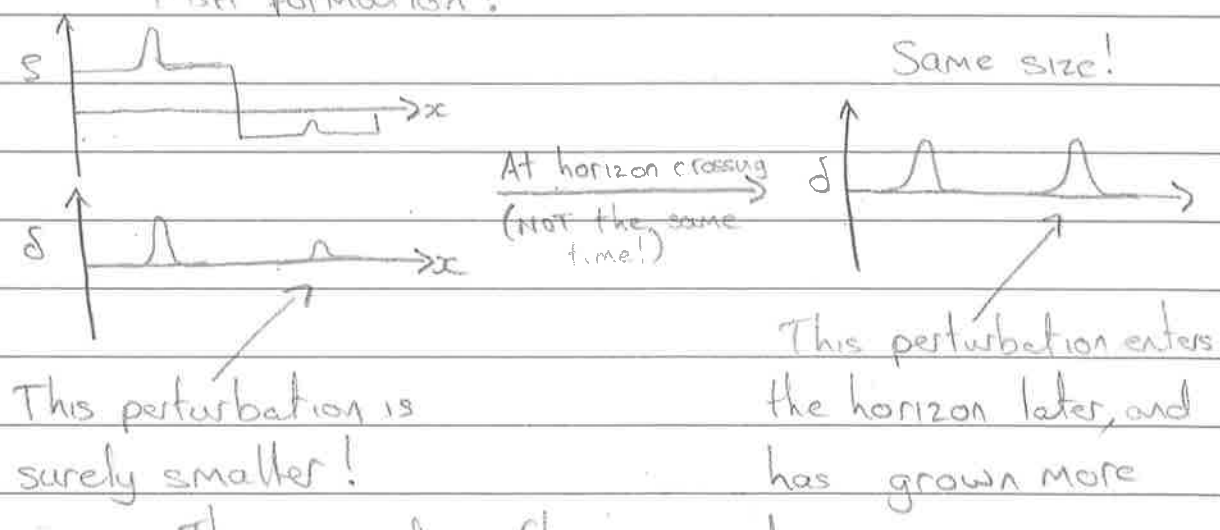
Hubble radius to perturbation scale, $\hat{r} = e^{\zeta(r)} r$

$$\Longrightarrow \delta(r,t) = -\frac{4}{9}\,\varepsilon^2(t)\, e^{2(\zeta(r_m)-\zeta(r))}\, r_m^2 \left[\zeta''(r) + \frac{2}{r}\zeta'(r) + \frac{1}{2}(\zeta'(r))^2\right]$$

→ We can see that, instead of depending on $\zeta(r)$, δ depends on $\zeta(r) - \zeta(r_m)$

→ It is therefore insensitive to constant changes in ζ

→ Also shows a problem with using δ to determine PBH formation:



. . . This is also often ignored

Lets calculate the peak height of the volume-averaged density by integrating over the areal radius,

$$R = a(t)\exp(\zeta(r))\,r$$

$$\delta_R = \frac{3}{4\pi R_m^3}\int_0^{R_m} dR\, 4\pi R^2\, \delta(R)$$

$\Downarrow$ Some magic happens

$$\delta_R = \varepsilon^2(t)\left[\left(-\frac{2}{3} r_m \zeta'(r_m)\right) - \frac{3}{8}\left(-\frac{2}{3} r_m \zeta'(r_m)\right)^2\right]$$

- where $r=0$ is the location of the peak and spherical symmetry is assumed

Much simpler! Depends only on ζ'

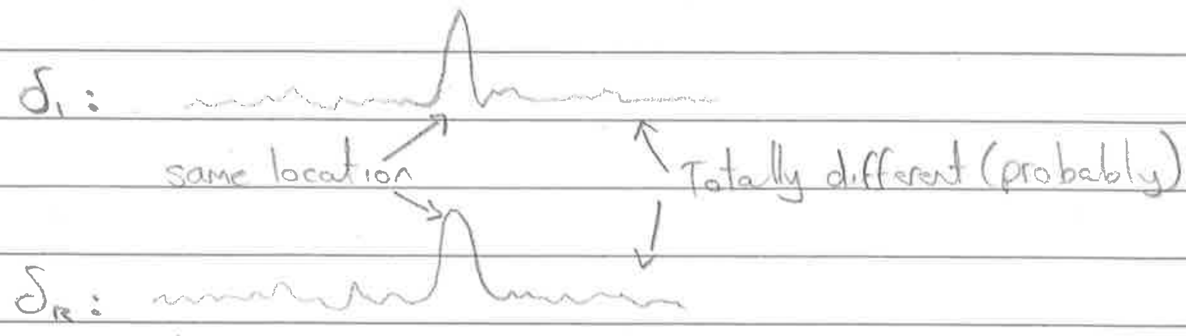
Even better: $\delta_1 = -\frac{2}{3} r_m \zeta'(r_m)$ is the prediction from the linear calculation

we'll factor this out

$$\delta_R = \varepsilon^2(t)\left(\delta_1 - \delta_1^{\,2}\right)$$

This gives a simple relationship between the height of (spherically symmetric) peaks in the Gaussian δ_1, and the non-Gaussian δ_R.

The correspondance of peaks is guaranteed (ish) by the rarity of relevant peaks



$\rightarrow$ verified numerically as well

28

Peaks theory gives us the number density of peaks in the range $\delta_R \to \delta_R + d\delta_R$

$$n = \frac{1}{4\pi^2}\left(\frac{\sigma_0}{\sigma_1}\right)^3 \left(\frac{\delta_R}{\sigma_0}\right)^3 \exp\left(-\frac{\delta_R^2}{2\sigma_0^2}\right)$$

$\to$ Changes to previous calculation

- need to worry about change of variables $\delta_R \to \delta_1$
- PBH mass

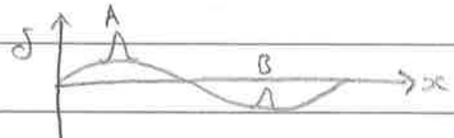
$$M_{PBH} = kM_H(\delta_R - \delta_c)^\gamma = kM_H\left(\delta_1 - \frac{3}{8}\delta_1^2 - \delta_c\right)^\gamma$$

- δ_R has a maximum at $\frac{2}{3}$, corresponding to $\delta_1 = 4/3$
  $\to$ not very important for final numbers

$$\beta = \int_{\delta_{c,1}}^{\infty} d\delta_1 \frac{k}{4\pi^2}\left(\delta_1 - \frac{3}{8}\delta_1^2\right)^\gamma \left(\frac{\sigma_1}{\sigma_0}\right)^3 \left(\frac{\delta_1}{\sigma_0}\right)^3 \exp\left(-\frac{\delta_1^2}{2\sigma_0^2}\right)$$

5.7: Initial clustering

Basic idea: in order to form in clusters on large scales, forming PBHs need to know something about those large scales. They don't (in a Gaussian universe).

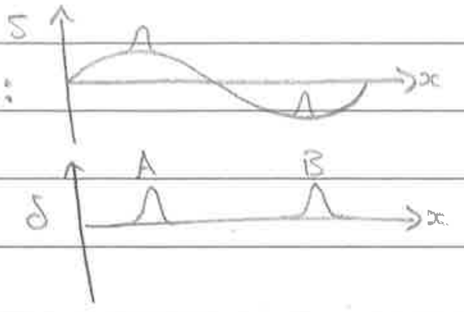
LSS: 

Perturbation A "knows", it is in a large-scale overdense region, so a galaxy forms. Perturbation B does not

$\to$ scale-independent bias

$\to$ Not true for PBHs, where large perturbations are suppressed.

PBHs:



- Perturbations A and B still don't know if they will be in an overdense region or not.

⟹ Same probability to collapse

There should be no clustering beyond random Poisson fluctuations

→ community has now come to a consensus but this has been hotly debated for several years.

5.8: The total PBH abundance and the mass function

Basic idea: β tells you about PBHs which formed at a single time → you need to add up all the times at which PBHs form

$$\Omega_{PBH} = \int_{M_{min}}^{M_{max}} d(\ln M_H) \left(\frac{M_{eq}}{M_H}\right)^{1/2} \beta(M_H)$$

PBHs forming at a time the horizon mass is M_H

In principle, the limits can be $0 \rightarrow \infty$

Redshift of matter density

$$f(M_{PBH}) = \frac{1}{\Omega_{CDM}} \frac{d\Omega_{PBH}}{d(\ln M_{PBH})}$$

→ NB. There are lots of different definitions for the mass function.

$$\psi(M_{PBH}) = \frac{1}{\Omega_{PBH}} \frac{d\Omega_{PBH}}{dM_{PBH}}$$

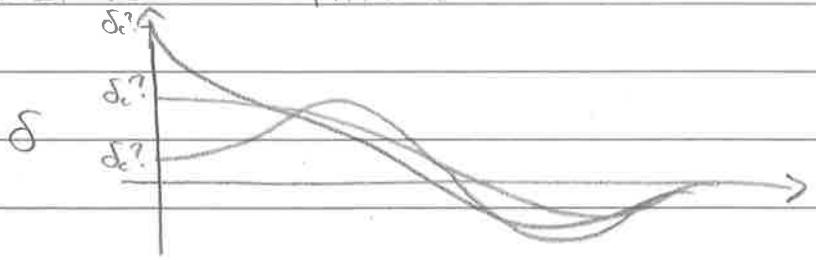
5.9: The formation criterion

Curvature or density?

We've already discussed the use of ζ, and argued against it. However, it works reasonably well if you only look at narrowly peaked power spectra, so you avoid any pesky large-scale modes. However, the PBH mass now depends a lot more on the profile shape, and you should worry about the co-ord transform $r \to e^{\zeta(r)} r$

Peak value, or volume-average?

① Consider the following profiles, which lie at the critical value to form a PBH:



We can argue that $\delta_{pk,c}$ lies in the range $-\infty < \delta_{pk,c} < \infty$

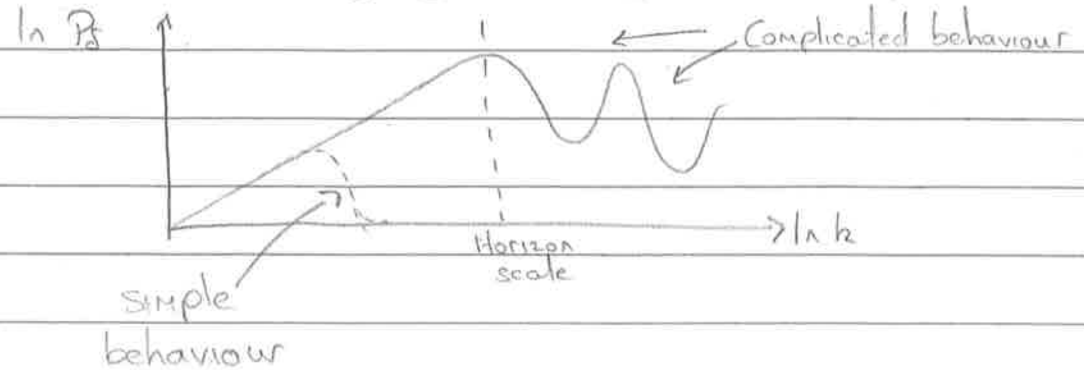
For realistic shapes, however, $\frac{2}{3} \lesssim \delta_{pk,c} \lesssim 5$

For the volume-averaged density, $0.41 \lesssim \delta_{R,c} \lesssim \frac{2}{3}$

$\rightarrow$ much less room for error.

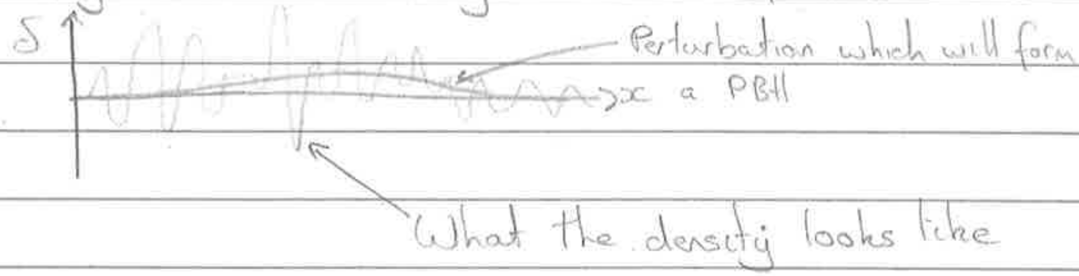
(31)

(2) You want to do your statistics on super-horizon scales, before things get complicated.



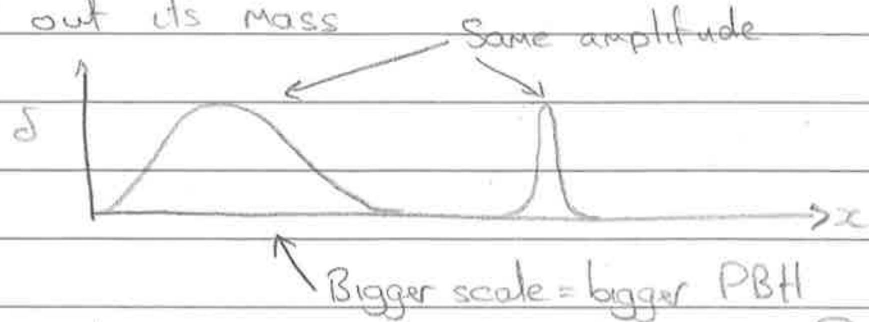
Smoothing means you can isolate perturbations which are super-horizon. (NB. A top hat function doesn't converge fast enough)
→ You don't need to smooth if P_δ has a narrow peak

(3) The density is dominated by small-scale perturbations



→ Not a problem for narrow peaks in P

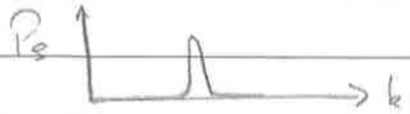
(4) You want to know the scale of a perturbation to work out its mass

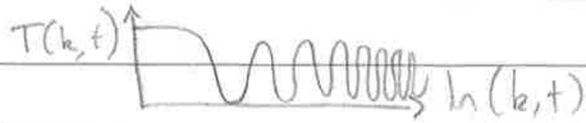


- Again, not a problem for narrow peaks in P

5 The (linear) transfer function oscillates

Lets assume a narrow P_s:





This means $\langle \delta^2 \rangle$ also oscillates over time, as P_s matches up alternately with peaks and troughs

Linear calculation:

Reality:

Conclusion: its much easier to smooth

SECTION 6: NON-GAUSSIANITY

The dominant term when calculating PBH abundance is the exponential one. Any small changes have a linear effect on most of the terms, but an exponential effect on that term — so lets ignore all the other terms for simplicity and focus on how things change. We'll also assume a narrow peak in P_ζ and use ζ...

6.1: Skewness and kurtosis

Lets consider quadratic local-type non-Gaussianity

$$\zeta = \zeta_G + \frac{3}{5} f_{NL} \left(\zeta_G^2 - \langle \zeta_G^2 \rangle \right)$$

This term ensures $\langle \zeta \rangle = 0$

This is inverted to give:

$$\zeta_{G\pm}(\zeta) = \frac{5}{6 f_{NL}} \left[-1 \pm \sqrt{1 + \frac{12 f_{NL}}{5} \left(\frac{3 f_{NL} \langle \zeta_G^2 \rangle + 5\zeta}{5} \right)} \right]$$

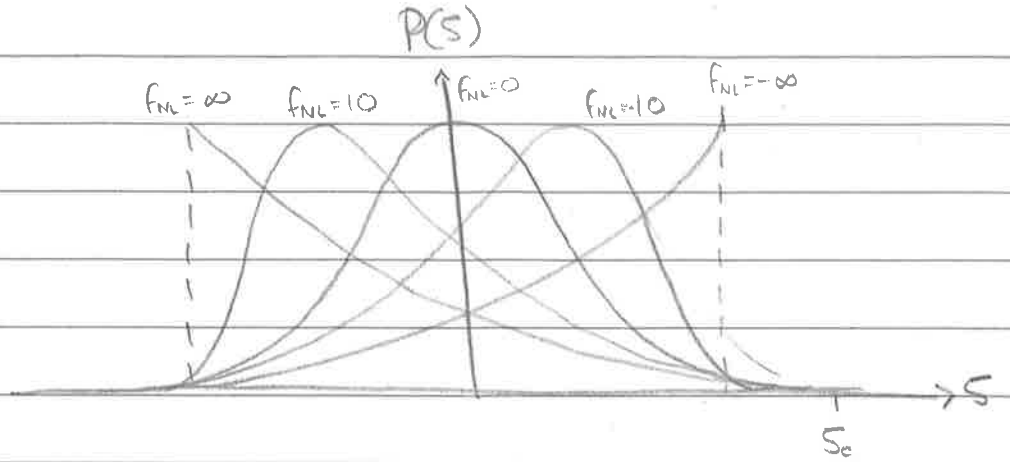
The PDF of ζ can be found by making a formal change of variables:

$$P_{NG}(\zeta) d\zeta = \sum_{i=1}^{n} \left| \frac{d\zeta_{G,i}(\zeta)}{d\zeta} \right| \overset{\text{Gaussian PDF}}{P_G}(\zeta_{G,i}(\zeta)) d\zeta$$

$$= \left(2\pi \langle \zeta_G^2 \rangle \left(1 + \frac{12 f_{NL}}{5} \left(\frac{3 f_{NL} \langle \zeta_G^2 \rangle}{5} + \zeta \right) \right) \right)^{-1/2} \left(\mathcal{E}_+ + \mathcal{E}_- \right) d\zeta$$

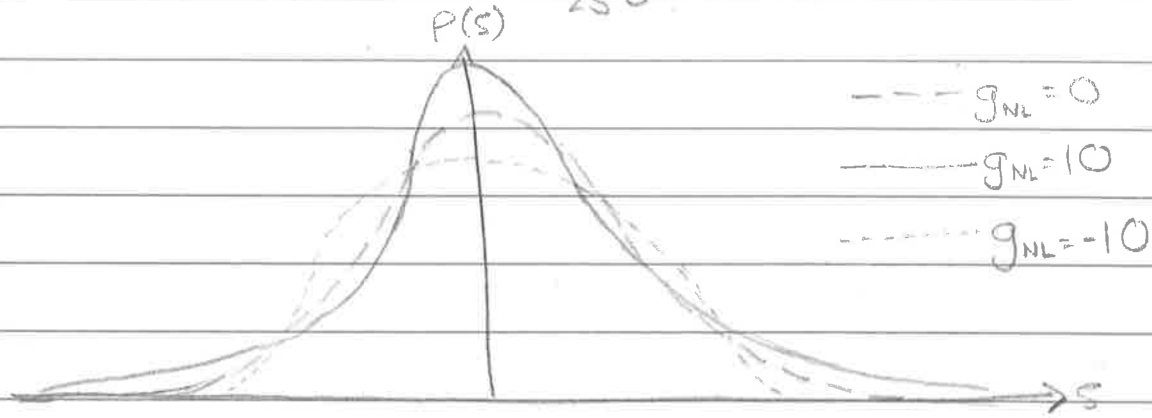
where $\mathcal{E}_\pm = \exp\left(- \frac{\zeta_{G\pm}^2(\zeta)}{2 \langle \zeta_G^2 \rangle} \right)$

(The derivation of this is left as an exercise for the class)



Positive (negative) f_{NL} induces positive (negative) skew. This has an order 1 effect on the central region, but an exponential effect in the tail where PBHs form.

Third-order : $\zeta = \zeta_G + \frac{9}{25} g_{NL} \zeta_G^3 \rightarrow$ kurtosis



$$P_\zeta \approx \langle \zeta^2 \rangle \approx \langle \zeta_G^2 \rangle + 4\left(\frac{3f_{NL}}{5}\right)^2 \langle \zeta_G^2 \rangle^2 \ln(kL)$$

$$+ 6\left(\frac{9g_{NL}}{25}\right)\langle \zeta_G^2 \rangle^2 \ln(kL) + 27\left(\frac{9g_{NL}}{25}\right)^2 \langle \zeta_G^2 \rangle^3 \ln^2(kL)$$

$$+ \ldots$$

Using a Press-Schechter approach, the PDFs can then be integrated to give β

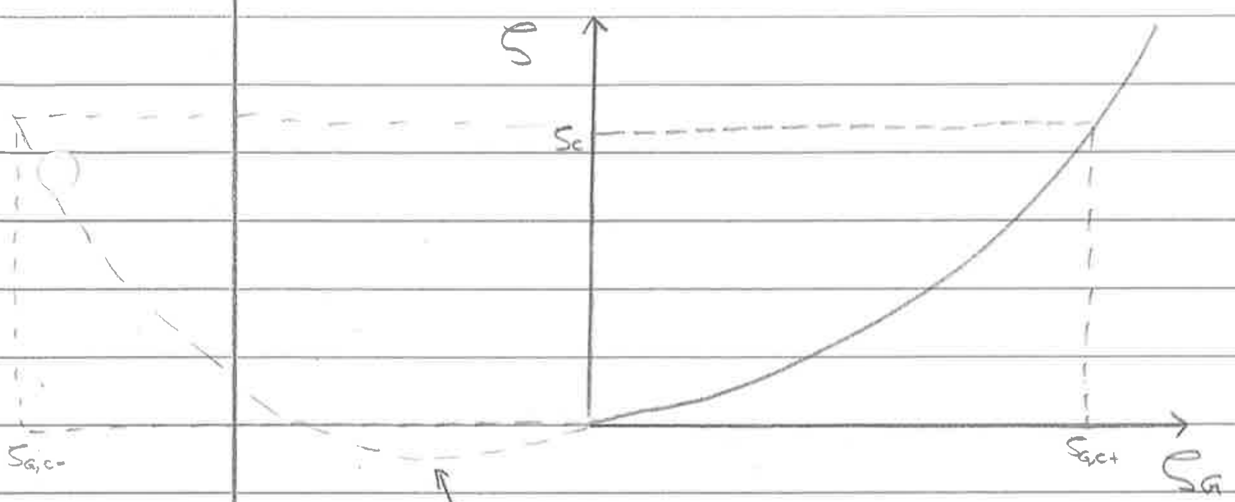
$$\beta = \int_{\zeta_c}^{\infty} d\zeta \, P_{NG}(\zeta)$$

6.2: Effect of non-Gaussianity on PBH abundance

$$\beta = \int_{\zeta_c}^{\infty} d\zeta \, P_{NG}(\zeta)$$

To (over)simplify this, lets abuse the fact that

$$\int P_{NG}(\zeta)\, d\zeta = \int P_G(\zeta)\, d\zeta_G$$



This region is unimportant unless f_{NL} is large. The calculation is dominated by the smallest solution for $\zeta_{G,c}$. In this case, $\zeta_{G,c+}$

$$\zeta_{G,c+} = \frac{-5 + \sqrt{25 + 60 f_{NL}\zeta_c + 36 f_{NL}^2 \langle \zeta_G^2 \rangle}}{6 f_{NL}}$$

$\downarrow$ Expand to 1st order in f_{NL}

$$\zeta_{G,c+} = \zeta_c - \frac{3}{5} f_{NL}\left(\zeta_c^2 - \langle \zeta_G^2 \rangle\right)$$

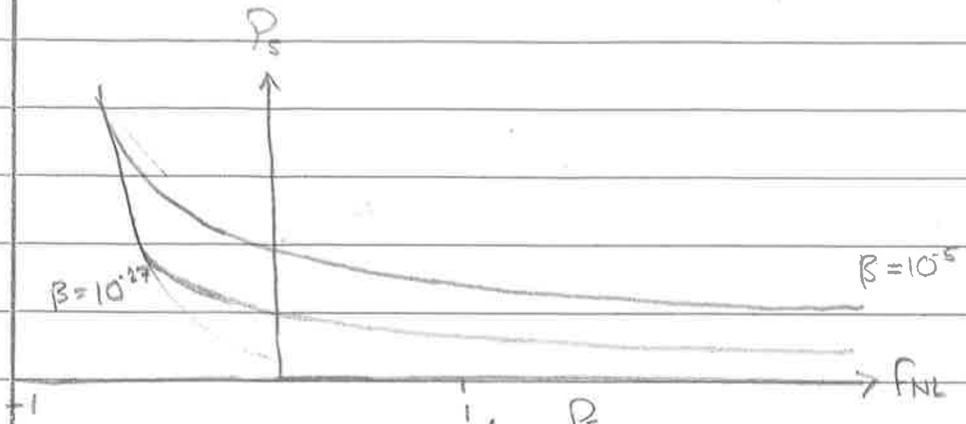
$\rightarrow$ larger f_{NL} = smaller $\zeta_{G,c}$

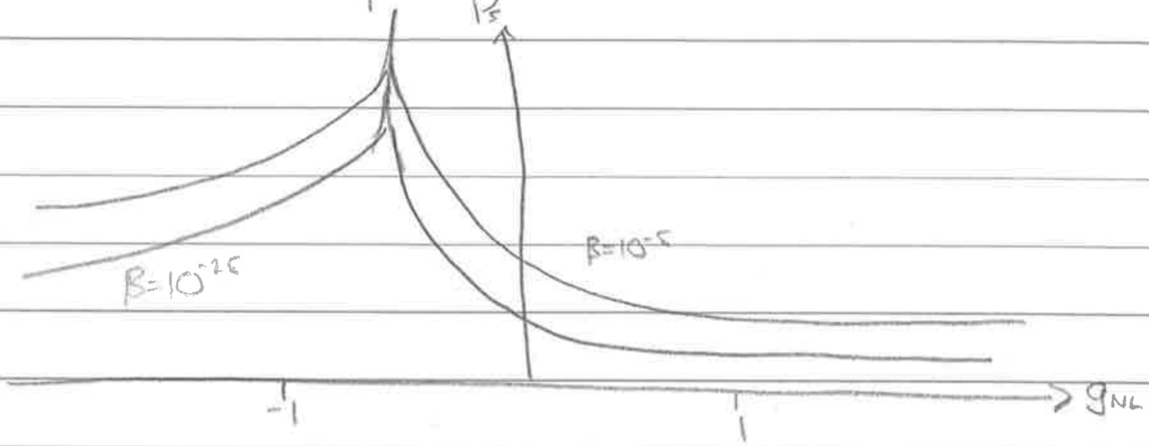
(Assuming small non-Gaussianity parameters) β is well approximated by

$$\beta = \text{erfc}\left(\frac{\zeta_{G,c}}{\sqrt{2\langle \zeta_G^2 \rangle}}\right)$$

$\rightarrow$ larger f_{NL} = smaller $\zeta_{G,c}$ $\Rightarrow$ exponentially more PBHs

$\Rightarrow$ linearly tighter constraints



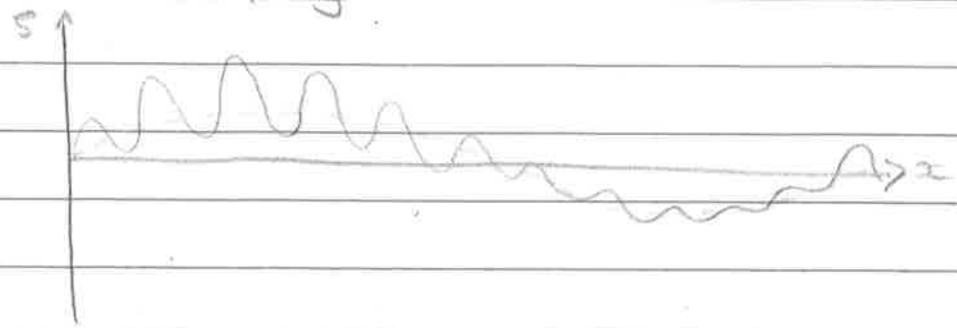


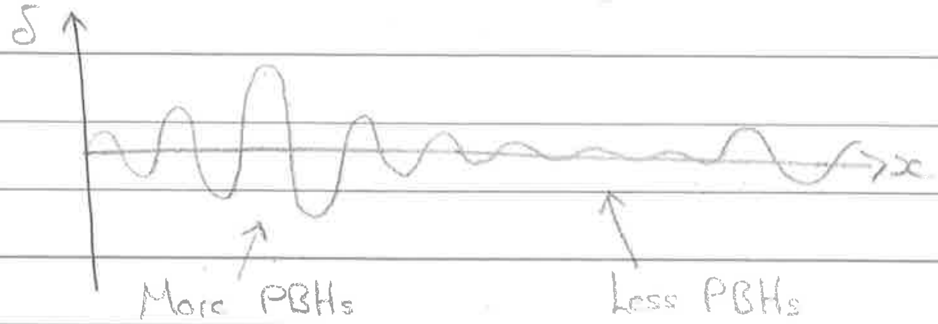
Conclusion: Order 1 non-Gaussianity parameters have a larger effect on constraints on P_s than tightening constraints on β by more than 20 orders of magnitude.

WARNING: may not converge when higher-order terms are considered

$$\zeta = \zeta_G + \frac{3}{5} f_{NL}\left(\zeta_G^2 - \langle \zeta_G^2 \rangle\right) + \frac{9}{25} g_{NL} \zeta_G^3 + \frac{27}{125} h_{NL}\left(\zeta_G^4 - \langle \zeta_G^4 \rangle\right)$$
$$+ \dots$$

6.3: Modal coupling





Lets split ζ into a short-wavelengh "peak" component ζ_s and a long-wavelength "background" component ζ_L, which are uncorrelated

$$\zeta_G = \zeta_s + \zeta_L$$

We can neglect any terms dependant only on ζ_L

$$\Rightarrow \zeta = \zeta_G = \zeta_s + \cancel{\zeta_L}$$

In a Gaussian universe, we can neglect ζ_L completely

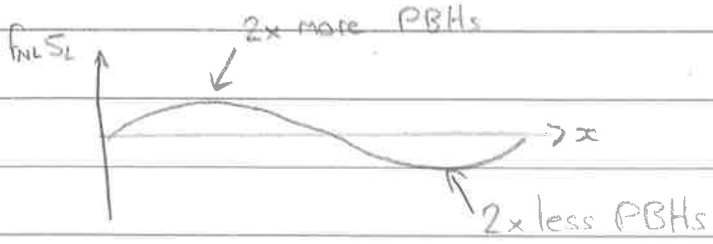
$$\zeta = (\zeta_s + \zeta_L) + \frac{3}{5} f_{NL} \left((\zeta_s + \zeta_L)^2 - \langle \zeta_G^2 \rangle \right) + \frac{9}{25} g_{NL} (\zeta_s + \zeta_L)^3$$

$$= \left(1 + \frac{6}{5} f_{NL} \zeta_L + \frac{27}{25} g_{NL} \zeta_L^2 \right) \zeta_s + \left(\frac{3}{5} f_{NL} + \frac{27}{25} g_{NL} \zeta_L \right) \zeta_s^2 + \frac{9}{25} g_{NL} \zeta_s^3 + \dots$$

Lets just focus on f_{NL} for now:

$$S_s \longrightarrow \Big(1 + \underbrace{\tfrac{6}{5} f_{NL} S_L}_{\text{linear}}\Big) S_s$$

This has a linear effect on the amplitude of the small-scale perturbations, but an exponential effect on β:



$$\frac{2 + 0.5}{2} = 1.25 \longrightarrow 25\% \text{ more PBHs}$$

Accounting for modal coupling always predicts more PBHs, even for negative non-Gaussianity parameters.

Similarly, g_{NL} has a linear effect on the quadratic term

$$\tfrac{3}{5} f_{NL} S_s^2 \longrightarrow \left(\tfrac{3}{5} f_{NL} + \tfrac{27}{25} g_{NL} S_L\right) S_s^2$$

$\longrightarrow$ linear effect on skewness $\longrightarrow$ exponential effect on β.

6.4: Initial clustering and DM isocurvature

This modal coupling causes a bias in the formation. To first order in ζ_L:

$$\rho_{PBH} = (1 + \underbrace{b\zeta}_{\text{bias term}} + \overbrace{3\zeta}^{\text{adiabatic } (\rho \propto (ae^{-\zeta})^{-3} \propto 3\zeta)})\bar{\rho}_{PBH}$$

We can write down the full expression and then expand to 1[st] order in f_{NL}, g_{NL} and ζ_L

(we'll see later why I assume f/g_{NL} is small)

$$b = \frac{6}{5}\overbrace{\left(1+\frac{\zeta_c^2}{\langle\zeta_s^2\rangle}\right)}^{\sim\frac{\zeta_c^2}{\langle\zeta_s^2\rangle}} f_{NL} + \overbrace{\frac{27(\zeta_c^2 - \langle\zeta_s^2\rangle)(\zeta_c^2 + \langle\zeta_s^2\rangle)}{25\zeta_c\langle\zeta_s^2\rangle}}^{\sim\frac{27}{25}\frac{\zeta_c^3}{\langle\zeta_s^2\rangle}} g_{NL} + \dots$$

The power spectrum of the PBH density is then:

$$P_{PBH} = \underbrace{9P_\zeta}_{\text{adiabatic}} + \underbrace{b^2 P_\zeta}_{\text{bias/isocurvature}} + \underbrace{P}_{\text{Poisson fluctuations}}$$

If DM (or part of it) is made of PBHs, then these perturbations to ρ_{PBH} are seen as dark matter isocurvature modes

→ strongly constrained by Planck

$$|f_{NL}| \lesssim 10^{-3} \leftarrow \text{surprisingly strong!}$$

$$|g_{NL}| \lesssim 10^{-3} \leftarrow \text{even more surprisingly strong!}$$

→ The constraints are weakly dependant on PBH mass

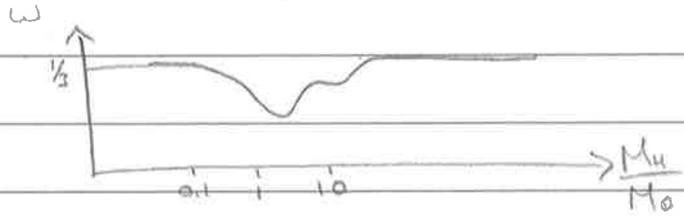
→ Constraints are weaker if PBH abundance is lower

SECTION 7: PHASE TRANSITIONS

During a phase transition, the equation of state decreases temporarily.
As an extreme example, lets consider a curvaton-type model.

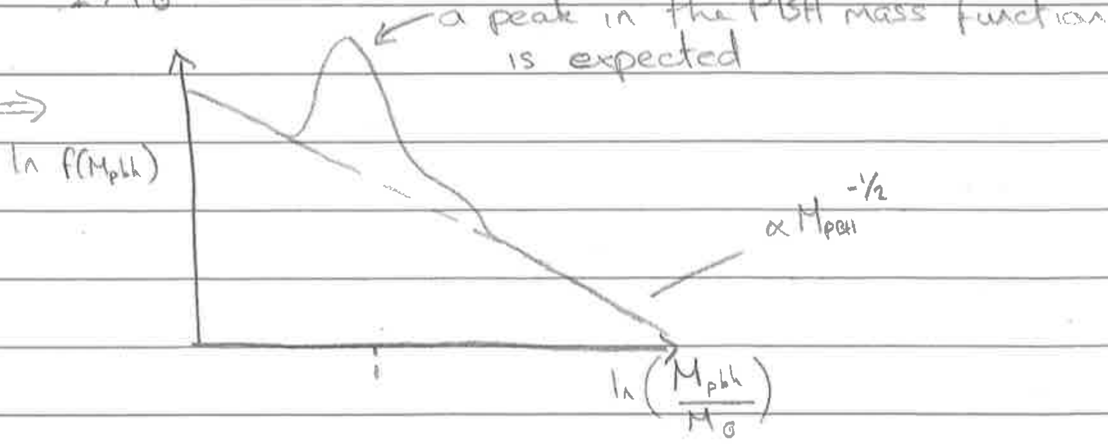
1. The universe is filled with relativistic particles (including the curvaton), $\omega = 1/3$
2. The temperature drops, and the curvatons become non-relativistic
3. $\rho_{rad} \propto a^{-4}$, $\rho_{curv} \propto a^{-3}$. Eventually the curvaton dominates the energy density of the universe. $\omega = 0$
4. The curvaton decays into relativistic particles. $\omega = 1/3$

The lower equation of state means less pressure, which reduces the threshold for PBH formation



The QCD phase transition happens when the horizon mass is about $1 M_\odot$

$\Longrightarrow$



(BYRNES, HINDMARSH, YOUNG, HAWKINS (2018))