Lectures on the Effective Field Theory of Inflation

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References

These lectures are based mainly on the two EFTI papers:

- "The Effective Field Theory of Inflation" – JHEP 0803 (2008) 014, arXiv:0709.0293;
- "On the consistency relation of the 3-point function in single field inflation" – JCAP 0802 (2008) 021, arXiv:0709.0295.

Two useful papers for Section 2.3 are:

- "Non-Gaussian features of primordial fluctuations in single field inflationary models" – JHEP 0305 (2003) 013, arXiv:astro-ph/0210603 (referred as "Maldacena's paper" paper in the text: it is useful also in general for a review of the ADM formalism and the computation of correlation functions in a quasi-de Sitter background);
- "Observational signatures and non-Gaussianities of general single field inflation" – JCAP 0701 (2007) 002, arXiv:hep-th/0605045.

Section 2.4 is based on:

- "Simplifying the EFT of Inflation: generalized disformal transformations and redundant couplings" – JCAP 1709 (2017) no.09, 043, arXiv:1706.03758.

Two very good references for the "$3+1$" formalism and the ADM formalism are:

- the book "General Relativity" by Robert M. Wald;
- "$3+1$ formalism and bases of numerical relativity" – arXiv:gr-qc/0703035. Notice that this reference uses the conventions of the numerical relativity community on the sign of the extrinsic curvature and other things.

Notation and conventions

- We use natural units in which $\hbar = c = 1$. By $M_{\rm P}$ we define the reduced Planck mass: $M_{\rm P} = 1/\sqrt{8\pi G_{\rm N}}$.
- Greek indices $\mu, \nu, \rho, \dots$ run from 0 to 3. Latin indices $i, j, k \dots$ run from 1 to 3.
- We use the mostly-plus signature: $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$.
- Given a scalar s, we define $g^{\mu\nu}\partial_\mu s \partial_\mu s \equiv (\partial_\mu s)^2$ and $\delta^{ij}\partial_i s \partial_j s \equiv (\partial s)^2$.

- ∇_μ denotes the covariant derivative, while ${}^{(3)}\nabla_\mu$ denotes the covariant derivative on some spacelike hypersurface of codimension 1.
- the symmetrizing and antisymmetrizing brackets include the $1/n!$, *e.g.*:

$$\mathcal{T}_{(\mu\nu)} = \frac{\mathcal{T}_{\mu\nu} + \mathcal{T}_{\nu\mu}}{2} \ .$$

- Given some field ϕ, we denote by $\bar{\phi}$ its vacuum expectation value $\langle\phi\rangle$.

Introduction

1.1 - EFFECTIVE FIELD THEORIES

The idea of EFT is, roughly speaking, that every Lagrangian that we write is valid below some energy scale: take for example the Lagrangian $\mathcal{L} = -\frac{(\partial_\mu \phi)^2}{2} + \frac{(\partial_\mu \phi)^4}{\Lambda^4}$:
as long as we are interested in field fluctuations that vary slowly over scales $\sim \Lambda$, we can treat the additional term as a perturbation around $-(\partial_\mu \phi)^2/2$.

Then, we have several consequences:

·) we can include all terms allowed by symmetries, also the non-renormalizable ones;

·) we can treat these terms as perturbations as long as we are interested in predictions at energies $E \ll \Lambda$;

·) at Λ we expect new degrees of freedom to come in (e.g. to unitarize some $\phi\phi$ scattering: just as in Fermi's theory and the W boson ∇) -> "UV completion";

·) matching between UV and IR: at tree-level, at 1-loop level, etc.;

·) naturalness: we expect the coefficients of the operators to be of order 1 unless we recover a (global) symmetry by putting them to zero. E.g.: fermion masses and chiral symmetry, $V'(\phi)/V(\phi)^2 \ll 1$ and shift symmetry;

·) symmetries protect the form of the Lagrangian. E.g. $F(R)$ is not protected by any symmetry, as aren't some inflationary potentials. DBI inflation, instead, is an example of a model protected by symmetries;

·) specific forms of the Lagrangian can also be obtained if one knows the UV theory ~> e.g. models of inflation derived from string theory.

1.2 - EFTs IN COSMOLOGY

Cosmology on large scales : GR + fluid dynamics → COSMOLOGICAL FLUIDS. What implications for QFT? Go back to textbook case : fields ϕ, A_μ, ... and Poincaré invariance :

$$\text{Poincaré invariance} \Rightarrow \begin{cases} \langle \phi \rangle = v \, ; \; \partial_\mu v = 0 \\ \langle A_\mu \rangle = 0 \\ \dots \end{cases}$$

What happens for a fluid? The fluid (spontaneously) breaks Poincaré ⇝ we have non-trivial backgrounds and non-trivial Lagrangians!

Consider indeed an irrotational perfect fluid : it can be described by a scalar field ϕ with a $P(X)$ Lagrangian. In Minkowski, $\langle \phi \rangle = \mu^2 t$ solves the e.o.m. for every μ, with different μ^2 describing different values of the background energy density. More precisely ⇒

- $\mathcal{L} = P(X)$; $X = -\partial_\mu \phi \, \partial^\mu \phi$
- $\rho = 2P'X - P$; $p = P$; $U_\mu = \partial_\mu \phi / \sqrt{X}$
- Shift symmetry $\phi \to \phi + c$ ensures conservation of number density $n = 2\sqrt{X}\, P'$
- $\mathcal{L} = X^{\frac{1+w}{2w}} \Rightarrow p = w\rho$ $\quad \boxed{w \to 0 \text{ limit can be taken (0806.1016), but can't have } w = -1 \,!}$

Then, expanding around $\langle \phi \rangle$ as $\phi = \langle \phi \rangle + \varphi$, study the dynamics of the compressional mode of the fluid :

$$\mathcal{L}_2 = P'(\mu^4)\left(\dot\varphi^2 - (\partial\varphi)^2\right) + 2P''(\mu^4)\mu^4\dot\varphi^2 \, , \quad \text{with } \langle X \rangle = \mu^4$$

can have $\mathcal{L}_2 \supset \dot\varphi^2$ even if $P'(\mu^4) = 0$: ghost condensate

$$\Rightarrow \; c_s^2 = \left. \frac{P'(X)}{P'(X) + 2XP''(X)} \right|_{X=\mu^4}$$: different from 1 because Lorentz is spontaneously broken!

What did we learn? Lagrangians in cosmology can be very different from what we are used to (e.g.: $w = \frac{1}{3} \leadsto \mathcal{L} = X^2$!), but also the backgrounds can be very different !

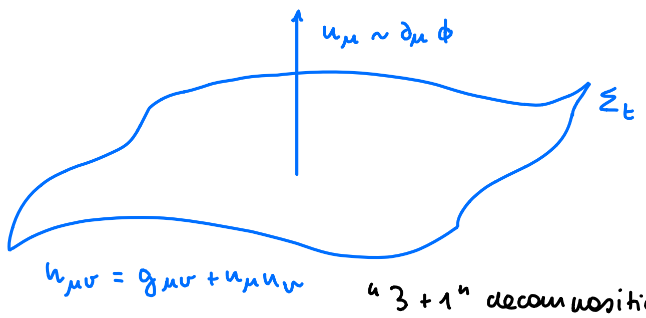
The main point is that we have a background that spontaneously breaks Poincaré : the EFT of inflation (kind of) builds on this ... (We will come back to this at the end of the lectures)

EFT of Inflation

2.1 – INTRODUCTION AND UNITARY GAUGE LAGRANGIAN

Inflation: period of accelerated expansion that smoothly connects to the decelerated HBB expansion ⇝ there is a physical clock (a function ϕ such that $\nabla_\mu \phi$ is timelike) that tells us when inflation ends.

We are interested in writing the action for perturbations to the clock ⇒ employ the gauge symmetry of GR to take $t = f(\phi)$ ... We fix the gauge by going into UNITARY GAUGE



·) spatial diff.s (coordinate changes $x^i \to x^i + \xi^i(t, \vec{x})$) are unbroken;

·) time diff.s $t \to t + \xi^0(t, \vec{x})$ are broken.

"3+1" decomposition:
- •) $ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$;
- •) $u_\mu = -N \partial_\mu t = -N \delta^0_\mu$; •) $g^{00} = -1/N^2$.

Since time diff.s are broken, we can write many more terms in the action:

·) all fully diff-invariant operators;

·) any operator constructed from $\partial_\mu t$, e.g. $g^{00} = g^{\mu\nu}\partial_\mu t\, \partial_\nu t = -1/N^2$;

·) then, all operators constructed from the 3+1 foliation, like:

A) "extrinsic" ones: $K_{\mu\nu} = h_\mu{}^\rho \nabla_\rho u_\nu$; $A_\mu = u^\rho \nabla_\rho u_\mu$ (not to be confused with a gauge field! Called "A" because it is the acceleration of u^μ ...)

B) "intrinsic" ones: ${}^{(3)}R_{\mu\nu}$, ${}^{(3)}R_{\mu\nu\rho\sigma}$, ...

·) coefficients can be functions of time.

Many relations between extrinsic and intrinsic operators: e.g. the various Gauss – Codazzi relations

⇝ e.g. $N^2 K_{\alpha\gamma} R^{\alpha 0 \gamma 0} = -K^{\alpha\gamma} K_\gamma{}^\rho K_{\rho\alpha} + K_\alpha{}^\gamma\, {}^{(3)}\nabla_\gamma A^\alpha + K_\alpha{}^\gamma A_\gamma A^\alpha - K_\alpha{}^\gamma u^\delta \nabla_\delta K_\gamma{}^\alpha$

⇒ see 1706.03758 for details: for now we focus on the operators built from $K_{\mu\nu}$ and g^{00}.

Then, the most general Lagrangian in unitary gauge is given by ↓

$$S = \int d^4x \sqrt{-g} \left[\frac{M_P^2}{2} R - c(t) g^{00} - \Lambda(t) + \sum_{n=2}^{+\infty} \frac{M_n^4(t)}{n!} (g^{00}+1)^n - \frac{\bar{M}_1^3(t)}{2} (g^{00}+1) \delta K_\mu{}^\mu - \frac{\bar{M}_2^2(t)}{2} (\delta K_\mu{}^\mu)^2 - \frac{\bar{M}_3^2(t)}{2} \delta K^\mu{}_\nu \, \delta K^\nu{}_\mu + \dots \right]$$

A very important point: we write things like $(g^{00}+1)$, $\delta K_{\mu\nu} = K_{\mu\nu} - H h_{\mu\nu}$, ... $\longrightarrow$ these vanish on a FLRW background. But the important point is that we can write these perturbations in a way compatible with the residual (unbroken) gauge symmetries! That is, the "unperturbed" value is writable as a function of t times tensors built from $h_{\mu\nu}$, $\partial_\mu t$, etc. This is why the "unperturbed" $\rightsquigarrow$ e.g.: $K_{\mu\nu} - \delta K_{\mu\nu} = H h_{\mu\nu}$ } contains perturbations...

One can show that in FLRW we can do this for any operator... One can also show that this holds ONLY in FLRW, due to the high degree of symmetry. It wouldn't hold if the b.g. is not homogeneous (e.g. Schwarzschild). We will come back to this at the end of the lectures!

This is the reason why the action can be written in that way without loss of generality $\rightarrow$ indeed, assume we had written a term $\mathcal{L} \supset f(t) K_{\mu\nu} K^{\mu\nu}$. This is expanded as:

$$f(t)\,(\delta K_{\mu\nu} + H h_{\mu\nu})(\delta K^{\mu\nu} + H h^{\mu\nu}) = f(t)\, \delta K_{\mu\nu}\, \delta K^{\mu\nu} + 2 f(t) H(t)\, \delta K + 3 H(t) f(t)$$

·) the first term is what we want;

·) the third term renormalizes $\Lambda(t)$;

·) the second term is: $2 f(t) H(t) \delta K = 2 f(t) H(t) (K - 3H)$

$$\propto 3H \text{ goes into } \Lambda(t) \leftarrow \overset{\sim}{=} 2 f(t) H(t) K = 2 f(t) H(t) \nabla_\mu n^\mu$$

$$= \nabla_\mu (2 f(t) H(t) n^\mu) - n^\mu \nabla_\mu (2 f(t) H(t))$$

the first term is a total derivative, and the second is $-\frac{(2f(t)H(t))^\cdot}{N}$: it renormalizes $c(t)$, $\Lambda(t)$, and all the operators $M_n^4(t)$.

This can be done for ANY OPERATOR → very important result that needs the symmetries of FLRW. For a proof see appendix B of 0709.0293.

Therefore, we can now fix $c(t)$ and $\Lambda(t)$ by requiring that a given $H(t)$ is a solution for the b.g., i.e. by requiring that tadpoles cancel around this solution. Then:

- the unperturbed history fixes $c(t)$ and $\Lambda(t)$;
- the difference among different models is encoded in the higher order operators.

One can easily solve for $c(t)$ and $\Lambda(t)$ in a flat FLRW b.g. (one can add curvature as well: see Appendix B of 0709.0293): the Friedmann equations are →

$$H^2 = \frac{c(t)+\Lambda(t)}{3M_P^2} \quad ; \quad \dot{H}+H^2 = \frac{\ddot{a}}{a} = -\frac{1}{3M_P^2}\left(2c(t)-\Lambda(t)\right).$$

Solving for c and Λ gives: $\mathcal{L} = \frac{M_P^2}{2}\left(R + 2\dot{H}g^{00} - 2(\dot{H}+3H^2)\right) + \ldots \equiv \mathcal{L}_0 + \ldots$

What about the time dependence of the coefficients? We are interested in the case where H and $\dot{H}$ vary slowly in time. Then the same can be assumed for the coefficients. What symmetry protects their small time dependence? $\boxed{\text{If}}$ $\ddot{\bar{\phi}}$, with $\langle\phi\rangle = \bar{\phi}$, is small (as in standard attractor models of inflation, but NOT in $P(X)$ / ultra-slow-roll models: see 1802.01580), then we have $\dot{\bar{\phi}} \sim$ const., i.e. $\bar{\phi} \sim t$: therefore, the approximate shift symmetry of the inflaton Lagrangian translates in an approximate $t \to t+c$ symmetry in unitary gauge. Of course, this has nothing to do with the breaking of time diff.s, which are a gauge symmetry.

SOME EXAMPLES

minimal kinetic term: $\int d^4x\sqrt{-g}\left[-\frac{1}{2}(\partial_\mu\phi)^2 - V(\phi)\right] \longrightarrow \int d^4x\sqrt{-g}\left[-\frac{\dot{\bar{\phi}}^2}{2}g^{00} - V(\bar{\phi}(t))\right]$

⇝ since the Friedmann equations give $\dot{\bar{\phi}}^2 = -2M_P^2\dot{H}$ and $\frac{V(\bar{\phi})}{M_P^2} = 3H^2+\dot{H}$ ⇒ $\mathcal{L} = \mathcal{L}_0 \ldots$

$P(X,\phi)$: $\int d^4x \sqrt{-g}\, P(X,\phi) \longrightarrow \int d^4x \sqrt{-g}\, P(\dot{\bar\phi}^2 g^{00}, \bar\phi)$ (see also Appendix A of 0709.0295)

$\rightsquigarrow$ all the operators M_n^4 are turned on, with $M_n^4 = \dot{\bar\phi}^{2n} \left.\frac{\partial^n P}{\partial X^n}\right|_{\phi=\bar\phi}$

Notice that this is a nice example of the above discussed feature of "resumming" the back-ground into $c(t)$ and $\Lambda(t)$: here we see that many terms of the expansion of P affect the b.g., as one could expect. With the EFTI we "don't care" about these terms:

•) advantages: one sees clearly if an operator is important for the dynamics of perturbations, at any given order;

•) disadvantages: one cannot say anything about the background, which must be fixed from the outside (recall that there are no tadpoles). Therefore, one cannot answer the question whether or not a given model can inflate, as one would be able to do with the ϕ Lagrangian at one's disposal. E.g.: above one would need to check if on the b.g. a given $P(X,\phi)$ can yield an inflating solution. For example, $P(X)$ cannot.

Besides (these are technical details):

-) one would need to solve these b.g. equations in an EFT expansion, i.e. treating higher time derivatives of $\bar\phi$ perturbatively unless there is a symmetry that protects the form of the La-grangian (as in DBI inflation : see e.g. Appendix A of 0709.0295)

-) (speculative) $\rightsquigarrow$ still, suppose that we are given a background expansion history $H(t)$ $\rightsquigarrow$ the EFT construction that we carried out above ensures that it is only $\mathcal{L}_0$ that contains terms that start at zeroth order in perturbations $\Rightarrow$ all the higher order terms don't contribute $\rightarrow$ one should still be able to find $\bar\phi(t)$, and then the ϕ Lagrangian, using the equations:

$\underbrace{\dot{\bar\phi}^2 = -2M_P^2 \dot H}_{\text{gives } \bar\phi(t)}$ and $\underbrace{\frac{V(\bar\phi)}{M_P^2} = 3H^2 + \dot H}_{\text{gives } V(\phi)}$ ...

2.2 - STÜCKELBERG TRICK AND DECOUPLING LIMIT

The unitary gauge Lagrangian describes three degrees of freedom: $g_{ij} = a^2 e^{2\zeta}(e^{\gamma})_{ij}$ ⇒ two graviton helicities and a scalar mode. We can make the scalar mode explicit via the Stückelberg trick ⇝ restores gauge invariance (which, unlike a global symmetry, can never really be broken...).

Why would we want to do it? Because of the Goldstone boson equivalence theorem: at high energies / short scales the physics of the additional degree of freedom, the Stückelberg field, decouples from the two graviton helicities ⇒ we expect that its Lagrangian will be also shift-invariant, like that of a Goldstone boson.

Let's see how it works for a U(1) gauge field: $S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right)$: let's break the gauge invariance by adding $S \supset \int d^4x \left(-\frac{1}{2} m^2 A_\mu A^\mu\right)$. Under a gauge transformation, we have:

$$A_\mu \longrightarrow A_\mu + \frac{1}{g} \partial_\mu \pi$$

$$S \to \int d^4x \left[\left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right) - \frac{1}{2} \frac{m^2}{g^2} (\partial_\mu \pi + g A_\mu)^2\right]$$

Then, it is clear that we restore gauge invariance if π also transforms (non-linearly) under the gauge transformation: i.e., $A_\mu \longrightarrow A_\mu + \partial_\mu \varepsilon / g$; $\pi \to \pi - \varepsilon$...

Now: canonically normalise the field π: $\pi_c = m\pi / g$ ⇒

$$S = \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (\partial_\mu \pi_c)^2 - \underbrace{\frac{m^2}{g^2} g A^\mu \partial_\mu \pi}_{m A^\mu \partial_\mu \pi_c} - \frac{1}{2} m^2 A_\mu A^\mu \right]$$

Then, at energies $E \gg m$, the mixing with the vector field is IRRELEVANT WITH RESPECT TO THE KINETIC TERM ⇒ π and A_μ are decoupled, and I can study the physics of the longitudinal mode separately.

A technical detail : we can take $E \gg m$, but at $E \sim 4\pi m/g$ there is strong coupling of the Goldstone (the perturbative unitarity of its scattering is violated). So we are actually restricted to $E \ll 4\pi m/g$ to have a weakly coupled theory. Similar considerations apply to the case of inflation, but we will not discuss them here. E.g.: when the operator M_2^4 is turned on, we have $\Lambda^4 \sim 16\pi^2 M_P^2 |\dot{H}| \frac{c_s^5}{1-c_s^2}$. Using $\Delta_\zeta^2 \sim \frac{H^2}{8\pi^2\varepsilon}\frac{1}{M_P^2 c_s} \sim 2\times 10^{-9}$, we see that for $c_s^2 \ll 1$ we have $\Lambda \sim H$ for $c_s \sim 3\times 10^{-3}$, which is obviously also where non-Gaussianities become of order 1 ("obviously" because the theory is strongly coupled).

Let's do the same for time diffeomorphisms. Let's focus on $\mathcal{L} = A(t) + B(t) g^{00} \Rightarrow$

$$S = \int d^4x \sqrt{-g}\,\left(A(t) + B(t)\, g^{\mu\nu}\partial_\mu t\, \partial_\nu t\right)$$

$$t = \tilde{t} + \tilde{\pi}(\tilde{x}) \Rightarrow S = \int d^4\tilde{x}\, \sqrt{-\tilde{g}}\,\left(A(\tilde{t}+\tilde{\pi}) + B(\tilde{t}+\tilde{\pi})\, \tilde{g}^{\mu\nu}\partial_\mu(\tilde{t}+\tilde{\pi})\partial_\nu(\tilde{t}+\tilde{\pi})\right),$$

where we use the fact that tensors which do not explicitly contain the time coordinate (i.e. a particular function ("clock") from the manifold $\mathcal{M}$ to $\mathbb{R}$, which I am now changing) do not change under $t = \tilde{t} + \tilde{\pi}(\tilde{x})$, since they are geometric objects defined independently of the clock. For a more "coordinate & component"-based approach, see 0709.0293... See the Appendix A of these notes for a sketch of what happens for the operators involving $K_{\mu\nu}$.

TECHNICAL ASIDE: in the previous calculation we have used Wald's "abstract index" approach. t is just a function from $\mathcal{M}$ to $\mathbb{R}$ that we are redefining inside an integral $\Rightarrow$ the integral can be carried out in any coordinate system (even if, of course, later we are interested in using the coordinate system where $x^0 = \tilde{t}$, and $\tilde{g}_{ij} = a^2(e^\gamma)_{ij}$), and it remains unchanged apart from the effect of the t redefinition. Take g^{00}:

$$\int d^4x \underbrace{\sqrt{-g}}_{\text{volume form : a tensor independent of } t} g^{ab}\nabla_a t \nabla_b t \Big\} \; g^{ab},\ \nabla_a \text{ : also independent of } t$$

THEN: $t = \tilde{t} + \tilde{\pi}$ simply gives $\int d^4x \sqrt{-g}\, g^{ab}\nabla_a(\tilde{t}+\tilde{\pi})\nabla_b(\tilde{t}+\tilde{\pi})$.

Now: the new time coordinate x^0 is $\tilde{t}$: we then drop the tilde everywhere. In this coordinate system, the "longitudinal mode" is explicit, i.e. the scalar mode is not eaten by the metric. $g_{ij} = a^2 (e^{\gamma})_{ij} = a^2 \delta_{ij}$ if we neglect tensors: flat gauge.

Some points:

•) one can find the non-linear relation between ζ and π -> by computing correlation functions of π one knows those of ζ, which are linked to observations (since ζ is conserved). Very useful !

•) the Lagrangian is now made explicitly diff-invariant since π transforms non-linearly under diff.s. The combination $\varphi \equiv t + \pi$ transforms as a scalar, that is -> for $t = t' - \xi^0(x')$, we have $\varphi'(x') = \varphi(x(x')) = \varphi(x)$. This holds if π transforms as $\pi'(x') = \pi(x(x')) + \xi^0(x')$ $\Rightarrow$ it transforms non-linearly ! As a scalar plus an additional shift.

The Lagrangian has now a complicated form : what did we gain ? We can go in the decoupling limit ! How do we see what is this limit ? We need to look at the quadratic mixing between π and the metric. Because of the SVT decomposition, only the coupling with g_{00} and the (scalar part of) g_{0i} matters. Let's consider two cases :

•) single-field slow-roll $\Rightarrow$ the mixing is of the form $\sim M_P^2 \dot{H} \dot{\pi} \delta g^{00}$: after canonical normalization ($\pi_c \sim M_P \sqrt{|\dot{H}|}\, \pi$, $\delta g^{00}_c \sim M_P \delta g^{00}$), we see that $E_{mix} \sim \varepsilon^{1/2} H$, where $\varepsilon = -\frac{\dot{H}}{H^2}$ is the usual slow-roll parameter;

•) operator M_2^4 large $\Rightarrow$ the mixing is $\sim M_2^4 \dot{\pi} \delta g^{00}$: after canonical normalization ($\delta g^{00}_c \sim M_P \delta g^{00}$, $\pi_c \sim M_2^2 \pi$), $E_{mix} \sim M_2^2 / M_P$.

This can be seen also in another way: theories of the type $P(X,\phi)$ have an explicit solution of the constraints ($N_i \sim g_{0i}$, $\delta N \sim \delta g_{00}$), i.e. the non-dynamical Einstein equations, in terms of π, ε, and the speed of sound c_s^2 (which can be related to the coefficient M_2^4 by $c_s^{-2} = 1 + \frac{2M_2^4}{H^2 M_P^2 \varepsilon}$ $\Rightarrow$ if M_2^4 is large enough, $c_s^2 \sim \frac{H^2 M_P^2 \varepsilon}{M_2^4}$. The solution of the constraints is (hep-th/0605045, 0709.0295):

$\delta N = \varepsilon H \pi$; $\partial^i N_i = -\frac{\varepsilon H \dot{\pi}}{c_s^2} \sim -\frac{M_2^4 \dot{\pi}}{H M_P^2}$: then, we see that large M_2^4 increases the mixing with gravity / the mixing is negligible for $\varepsilon / c_s^2 \ll 1$.

For $E \gg E_{mix}$, then, the Lagrangian takes a very simple form, since we neglect metric fluctuations. Putting $g_{00} = -1$, $g_{0i} = 0$, $g_{ij} = a^2 \delta_{ij}$ everywhere, we get:

$$S_\pi = \int d^4x \sqrt{-g} \left\{ -M_P^2 \dot{H} \left(\dot{\pi}^2 - \frac{(\partial\pi)^2}{a^2} \right) + 2M_2^4 \left(\dot{\pi}^2 + \dot{\pi}^3 - \dot{\pi} \frac{(\partial\pi)^2}{a^2} \right) - \frac{4}{3} M_3^4 \dot{\pi}^3 + \dots \right\}$$

•) shift symmetric action, as we expected in the decoupling limit from the Goldstone boson equivalence theorem. Notice that this assumes that the time dependence of the coefficients of the operators is small so that we can neglect terms from $\sim f(t+\pi)$!

•) in the decoupling limit the (non-perturbative) relation between ζ and π is super simple: $\boxed{\zeta = -H\pi}$;

•) notice that for S.F.S.R. the cubic π Lagrangian is zero, so there's no 3-point function for ζ. This is expected, since in this case the interactions come from the mixing w/ gravity (i.e. by solving the constraints), as shown in Maldacena's calculation.

2.3 - SOME (QUICK) PHENOMENOLOGY

Small speed of sound and non-Gaussianities: we see that the quadratic Lagrangian of π is $\mathcal{L}_2 = (-M_P^2 \dot{H} + 2M_2^4)\dot{\pi}^2 + M_P^2 \dot{H} \frac{(\partial\pi)^2}{a^2}$:

·) $\dot{H} < 0$ if we don't want instabilities (NEC is satisfied);

·) $(-M_P^2\dot{H} + 2M_2^4) > 0$ if we don't want ghosts;

·) $c_s^{-2} = 1 + \frac{2M_2^4}{H^2 M_P^2 \varepsilon}$: for $\dot{H} < 0$, we need $M_2^4 > 0$ to have $c_s^2 < 1$ (subluminality).

Then, we see that the non-linear realization of time diff.s forces a relation between the speed of sound and the size of the cubic Lagrangian. Indeed, we have (putting to zero $M_3^4, \dots$):

$\mathcal{L}_3 \supset \frac{M_P^2}{c_s^2} \dot{H} (c_s^2 - 1)\left[\dot{\pi}^3 - \dot{\pi} \frac{(\partial\pi)^2}{a^2}\right]$. $\mathcal{L}_2$, instead, is $-\frac{M_P^2}{c_s^2} \dot{H} (\dot{\pi}^2 - c_s^2 (\partial\pi)^2/a^2)$.

As usual, we need to compute correlation functions around the time when modes exit the horizon, since there ζ freezes. While $\dot{\pi} \sim H\pi$, at crossing $\frac{k}{a} \sim H/c_s$, since now the spatial term has the form $\frac{c_s^2 (\partial\pi)^2}{a^2}$. Then it is clear that $\frac{\dot{\pi}(\partial\pi)^2}{a^2}$ dominates over $\dot{\pi}^3$. We have that the level of NG is controlled by $\frac{\mathcal{L}_3}{\mathcal{L}_2}$, i.e.

$$\frac{\dot{\pi}(\partial\pi)^2/a^2}{\dot{\pi}^2 - c_s^2(\partial\pi)^2/a^2} \sim \frac{H\pi\left(\frac{H}{c_s}\right)^2 \pi^2}{(H\pi)^2} \sim \frac{H\pi}{c_s^2} \sim \frac{\zeta}{c_s^2} .$$

This corresponds to $f_{NL}^{\text{equil.}} \sim 1/c_s^2$. It is equilateral because the presence of derivatives tells us that the interaction dies quickly outside the horizon, and then it will be the largest when the modes have all roughly the same wavelength.

Operators at higher order in derivatives can be used for other things. E.g.: recall that the sign of $(\partial\pi)^2/a^2$ was fixed by $\dot{H}$, so that $\dot{H} > 0$ would result in instabilities. However, it is possible to consider the operators δK^2 and $\delta K_\mu{}^\nu \delta K_\nu{}^\mu$, which can give the right sign to the spatial kinetic term and avoid gradient instabilities by means of a contribution $\omega \propto k^2$ to the dispersion relation: Ghost Inflation!

2.4 - REDUNDANT COUPLINGS IN THE EFTI

We have many operators: can we find an organizing principle? We can classify them by the order in perturbations at which they start, and how many derivatives on the metric they carry. E.g.:

·) $(g^{00}+1)^3$ starts cubic in perturbations, and has zero derivatives acting on the metric;

·) $\delta K_{\mu\nu}\delta K^{\mu\nu}$ starts quadratic in perturbations, and has two derivatives acting on $g_{\mu\nu}$.

The rationale is the following. The classification in perturbations is self-explanatory. The one in derivatives is based on the fact that an operator that, according to our definitions, is of order n in perturbations and m in derivatives, will affect the Lagrangian of π at order $(\partial^{m/n+1}\pi)^n$, if we neglect metric perturbations.

Then, we have already seen that the Gauss-Codazzi and similar relations tell us that some operators can be related to each other. Besides, we have integration by parts (IBP). E.g. → consider $V \equiv n^\mu \nabla_\mu N$: if we consider the operator $V\delta N$, where $\delta N = N-1$, we can rewrite it in terms of δN and δK ⇝ indeed:

$$
\begin{aligned}
V\delta N = (n^\mu\nabla_\mu \delta N)\delta N &= \nabla_\mu(\delta N n^\mu)\delta N - \delta N^2 \nabla_\mu n^\mu = \\
&= \nabla_\mu(\delta N^2 n^\mu) - \delta N n^\mu \nabla_\mu \delta N - \delta N^2 \delta K - 3H\delta N^2 \\
&= -\delta N V + \nabla_\mu(\delta N^2 n^\mu) - \delta N^2 \delta K - 3H\delta N^2
\end{aligned}
$$

$$
\Rightarrow \quad V\delta N = -\frac{\delta K \delta N^2}{2} - \frac{3H}{2}\delta N^2 + \text{four-divergence}
$$

We can list all the independent operators after IBP and "use of geometrical identities", but there is still something we can do: field redefinitions.

Let's recall what happens in the textbook case: there we can also do field redefinitions. The correlation functions of the field change, but the observables, i.e. the S-matrix elements, are invariant. Then: imagine $\mathcal{L} = a_1 \mathcal{O}_1 + a_2 \mathcal{O}_2 + a_3 \mathcal{O}_3$, and do a field redefinition $\phi \longrightarrow f(\phi)$. Assume, e.g., that:

$$\left.\begin{aligned} \mathcal{O}_1 &\longrightarrow \mathcal{O}_1 + \lambda \mathcal{O}_2 \\ \mathcal{O}_2 &\longrightarrow \mathcal{O}_2 - \lambda \mathcal{O}_3 \\ \mathcal{O}_3 &\longrightarrow \mathcal{O}_3 - 2\lambda \mathcal{O}_1 \end{aligned}\right\} \quad \mathcal{L} \longrightarrow \mathcal{O}_1 (a_1 - 2\lambda) + \mathcal{O}_2 (a_2 + \lambda) + \mathcal{O}_3 (a_3 - \lambda)$$

$\leadsto$ then, taking e.g. $\lambda = a_3$ gets rid of $\mathcal{O}_3$ in $\mathcal{L}$, i.e. a_3 is a redundant coupling. Of course which coupling to set to zero is a matter of convenience, but the message is that PHYSICALLY there are only 2 operators / 2 combinations of a_1, a_2, a_3 that contribute to the observables.

In cosmology things are different: we don't compute S-matrix elements, but we are really interested in correlation functions. Can we still do these field redefinitions during inflation? Let's see: a generic field redefinition takes the form $\rightarrow$

$$g_{\mu\nu} \longrightarrow C(t, N, K, \dots)\, g_{\mu\nu} + D(t, N, K, \dots)\, u_\mu u_\nu + E(t, N, K, \dots)\, K_{\mu\nu} + \dots$$

In inflation we are interested in correlation functions at <u>late times</u>: these are the asymptotic values relevant for observations. On super-Hubble scales:

- all derivatives of metric perturbations decay;
- the lapse N goes to 1 (see, e.g., Maldacena's paper).

Then, we remain with $g_{\mu\nu} \longrightarrow C(t)\, g_{\mu\nu} + D(t)\, u_\mu u_\nu$: these redefine the scale factor and cosmic time of the background FLRW, but scalar and tensor perturbations remain unchanged at late times. I.e., the transformation above will change the form of the action, but will leave observables unchanged.

A technical aside: we can do these field redefinitions because in the predictions of inflation that we are considering the coupling with matter does not enter (i.e. late time physics is completely determined by the adiabatic mode ζ on cosmological scales). Doing them in the EFT of Dark Energy (which studies DE models in a similar way to the EFTI) is less powerful, since there the coupling with matter is changed by the redefinition. If in our case the coupling with matter (e.g. during reheating) is important for the evolution of ζ and its correlators, then also for us the redefinitions lose their power since we should follow the coupling with matter and how it is changed.

What is the strategy? Let's stop at order $n \leq 3$ in perturbations (of course, recall that here we mean that operators START at order n in perturbations. An operator can start at order 2 and contain all orders > 2 in perturbations, e.g. consider g^{00} & δN...). Regarding derivatives, we stop at order $m \leq 2$. After geometric identities and IBPs we have 17 operators.

Let's consider the generic metric transformation discussed above. We see that, at this order, it can only be of the form:

$$g_{\mu\nu} \longrightarrow (f_1(t) + f_3(t)\,\delta N + f_5(t)\,\delta N^2)\,g_{\mu\nu} + (f_2(t) + f_4(t)\,\delta N + f_6(t)\,\delta N^2)\,u_\mu u_\nu$$

The reason is the following: in the action there is always the Einstein-Hilbert term, which contains already two derivatives. Then, the change of the action will be of the form $\delta S \sim \int d^4x \sqrt{-g}\, G_{\mu\nu}\,\delta g^{\mu\nu}$, i.e. it always generates operators with at least $m = 2$. Then, $\delta g_{\mu\nu}$ cannot contain derivatives of the metric. Moreover, we know that tadpoles cancel $\Rightarrow$ terms $\sim \delta N^3$ in the transformation are dropped.

Then, we conclude that at this order there are $17 - 6 = 11$ independent couplings that can affect the 2- and 3-point functions of ζ and γ_{ij}.

There is something more that we can do, however, if we assume that the dynamics is dominated by the lower derivative operators, like δN^2 and δN^3 (besides of course R)

⇝ then, we can consider transformations of the form:

$$g_{\mu\nu} \longrightarrow g_{\mu\nu} + \nabla_{(\mu}\xi_{\nu)} \quad \text{, with} \quad \xi_\mu = F(t, \delta N, V, \delta K)\, n_\mu$$

Indeed, this transformation does not change the Einstein-Hilbert term ! It starts at 1st order in derivatives, so the variation of the operators with two derivatives can be neglected since it is of order three in derivatives. Then, we can write:

$$F = \frac{g_1(t)\,\delta N}{H} + \frac{g_2(t)\,\delta N^2}{H} + \frac{g_3(t)\,V}{H^2} + \frac{g_4(t)\,\delta N V}{H^2} + \frac{g_5(t)\,\delta K}{H^2} + \frac{g_6(t)\,\delta N \delta K}{H^2}$$

We cannot go higher in derivatives in F, since a derivative is already outside ($\nabla_\mu \xi_\nu$): if we did, the operators δN^2 and δN^3 would generate operators with $m = 3$.

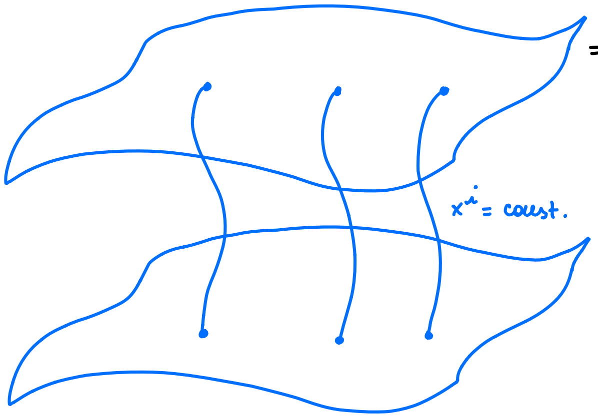
This tells us that we have 6 additional transformations at our disposal that can be used to set to zero other 6 couplings (of those with $m = 1$ or $m = 2$), leaving us with 5 non-redundant couplings. An operator that cannot be set to zero and hasn't yet been studied in the literature is $A_\mu A^\mu$.

Notice that this approach cannot be employed when the dynamics is dominated by the higher-derivative operators (like those involving the extrinsic curvature, like e.g. Ghost Inflation and Galileons).

Open questions and work in progress

3.1 – EFT FOR BROKEN SPATIAL DIFFEOMORPHISMS

Recall that the main point from which we started was that there was a preferred clock, that gave a preferred SLICING of spacetime. Here we are now interested in three rulers that give a preferred THREADING of spacetime.



⇒ note: the surface is just for visualization purposes: we don't care about a slicing

These rulers are three spacetime scalars, that we can identify with three spatial coordinates x^i in unitary gauge...

Which operators can we write? First of all, notice that the coefficients of the operators can be generic functions of x^i. Then, we can write fully diff.-invariant operators, as usual, and any operator contracted with $\partial_\mu x^i$ as, e.g., $g^{\mu\nu}\partial_\mu x^i \partial_\nu x^j = g^{ij}$. Geometrically, as before we had Σ_t and the projectors $-u_\mu u^\nu$ and $h_\mu{}^\nu = \delta_\mu{}^\nu + u_\mu u^\nu$, now we have the projectors: $X^{(i)}_{\mu\nu} = \partial_\mu x^i \partial_\nu x^i / g^{\rho\sigma}\partial_\rho x^i \partial_\sigma x^i$

These are the equivalent of $-u_\mu u_\nu$: it is straightforward to show that they satisfy $g^{\rho\sigma} X^{(i)}_{\mu\rho} X^{(i)}_{\nu\sigma} = X^{(i)}_{\mu\nu}\ \forall i$, and they are symmetric tensors. That is, they are projectors. They project on the space orthogonal to $\{\vec{x} = \text{const.}\}$.

⇒ What is the equivalent of the projector $h_{\mu\nu}$, i.e. the projector parallel to $\{\vec{x} = \text{const.}\}$? In Appendix B we show that such projector, $P_{\mu\nu}$, can be obtained from a four-vector:

$$P_{\mu\nu} = \frac{U_\mu U_\nu}{U_\rho U^\rho}\ , \quad \text{where} \quad U^\mu = \underbrace{\sqrt{-g}\,\varepsilon^{\mu\nu\rho\sigma}}_{\text{volume form}}\ \underbrace{\varepsilon_{ijk}}_{\text{3D Levi-Civita symbol}} \nabla_\nu x^i \nabla_\rho x^j \nabla_\sigma x^k$$

This makes even clearer how x^i define a threading.

We can now do a "1+3", instead of a "3+1" decomposition ⇝ and use same results for the extrinsic and intrinsic geometry of CODIMENSION 3 surfaces, i.e. CURVES ⇝ many results known from the fathers of differential geometry.

Now, this is nice, but recall that the geometric part was not all that we needed for the EFT I: we need to consider the background. More precisely:

•) we needed to be able to rewrite all operators on the background in a way compatible w/ the residual gauge invariance. This is why we could do it around FLRW and not Schwarzschild, even if both have a preferred slicing;

•) we need to be able to rewrite all operators as perturbations starting at 2nd order plus a FINITE number of tadpoles (and, correspondingly, a finite number of coefficients, which are then fixed by requiring that a given background is a solution).

Let us focus on the first point. The intuition is the following ⇒ in the case of the EFT I, we saw that basically we needed that the preferred slicing Σ_t, on the background, was maximally symmetric. Then, here we require the following -> we need a background where the preferred threading $\{x^i = \text{const.}\} \equiv \gamma_x$ is maximally symmetric. I.e., we need a timelike Killing vector: we need a stationary spacetime (as Kerr) or a static spacetime (as Schwarzschild). See Appendix C for a sketch of proof.

Killing is hypersurface orthogonal / non-vortical

This is why considering an FLRW background, even if it has a preferred threading, is not going to work. Indeed, $Z^\mu = \delta^\mu_{(t)}$ in an FLRW universe is NOT a Killing vector (there is no time translation symmetry)... But de Sitter is maximally symmetric, so there are indeed stationary (actually static) observers: works in inflation up to $\mathcal{O}(\varepsilon)$?
(THANKS @ FABIAN ❣)

3.2 - HIGGS MECHANISM, GOLDSTONE THEOREM, AND ALL THAT

Let's go back to the example of the U(1) gauge field : we had ⇝

$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu$: after reintroducing the Stückelberg field, we have ⇝

$\mathcal{L} = \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right) - \frac{1}{2}\frac{m^2}{g^2}(\partial_\mu \pi + g A_\mu)^2$: re-expressing the Stückelberg field we recover gauge invariance. Gauge symmetries, being merely redundancies, cannot be broken. A more accurate terminology would be gauge theories in a non-linearly realized phase (or Higgs phase).

In the decoupling limit $m \to 0$ and $g \to 0$, but m/g fixed (so that $\pi_c = m\pi/g$ is well defined), the U(1) symmetry becomes global, the gauge field decouples, and :

$\mathcal{L}_\pi = -\frac{1}{2}(\partial_\mu \pi_c)^2$.

Here, π_c transforms non-linearly under the U(1) global symmetry (it shifts) and the Lagrangian is invariant under this non-linearly realized global symmetry.

NON-ABELIAN VERSION ⟶ the same thing can be done for a non-Abelian gauge theory: consider $\mathcal{L} = -\frac{1}{4} \mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu} - \frac{m^2}{2} \mathrm{Tr}\, A_\mu A^\mu$, where $A_\mu = A_\mu^a T^a$. Then, in the decoupling limit we would have the Lagrangian for the Stueckelberg fields given by :

$\mathcal{L} = -\frac{f_\pi^2}{2} \partial_\mu U \partial^\mu U^\dagger$, where $U = e^{i\pi_c/f_\pi} = e^{i\pi}$, with $\pi = \pi^a T^a$ and $f_\pi = \frac{m}{g}$.

In general, higher derivative terms will be also present, since they are not forbidden by symmetry : e.g. in the U(1) case ⇝ $(\partial_\mu \pi)^4$, $(\Box\pi)^2$, etc. For the non-Abelian case, we see that $\mathcal{L} = -\frac{f_\pi^2}{2}\partial_\mu U \partial^\mu U^\dagger$ already contains higher derivative interactions, suppressed by powers of f_π once we canonically normalize $\mathcal{L} \supset -\frac{1}{2}(\partial_\mu \pi_c)^2 \ldots$

The Lagrangian for the pions, in the decoupling limit, would have been captured by the coset construction, a procedure that requires only to know the group G and the pattern of its breaking into a subgroup H. It yields the building blocks (in the above cases $\partial_\mu \pi$ and $U^+ \partial_\mu U$) for the Lagrangian of these GOLDSTONE BOSONS. The Goldstones will transform non-linearly under the broken global transformations in the coset G/H (composed of the full generators of G minus those of H), and the form of their Lagrangian is fixed by $G \longrightarrow G/H$ (and is fully invariant under G).

The examples above have $H = \emptyset$, which of course is not necessary. The simplest example is $G = G_1 \times G_2$, and $H = G_1$ (for something similar see the derivation of the chiral Lagrangian). Or, of course, the breaking of $SU(2)_L \times U(1)_Y$ into $U(1)_{em}$ of the electroweak sector of the Standard Model.

The question is: what is G and H (and then G/H) for the EFTI? Difficult to answer, since we start from time diffeomorphisms that are NOT a finite dimensional group. They are an infinite dimensional Lie group generated by all timelike vector fields. The commutator is the Lie braket between vector fields. Can we write the action for gravity via the coset construction, and then break time diffeomorphism and recover the EFTI? This would be a start...

Appendices

4.1 - STÜCKELBERG TRICK FOR $K_{\mu\nu}$

In this Appendix we briefly sketch how to reintroduce the Stückelberg field in $K_{\mu\nu}$.

Recall that g^{00} transforms as $g^{00} \longrightarrow g^{\mu\nu}\partial_\mu(t+\pi)\partial_\nu(t+\pi) \equiv \tilde{g}^{00}$. We also know from the 3+1 decomposition that:

·) $N = (-g^{00})^{-1/2}$; ·) $u_\mu = -N\partial_\mu t$.

Then : $\tilde{N} = (-g^{00} - 2g^{0\mu}\partial_\mu\pi - (\partial_\mu\pi)^2)^{-1/2} = (-g^{00})^{-1/2}\left[1 + \frac{2g^{0\mu}\partial_\mu\pi}{g^{00}} + (\partial_\mu\pi)^2\right]^{-1/2}$

$$= N \times \xi(\pi, g^{\mu\nu}, \dots)$$

$$\tilde{u}_\mu = -N\xi\partial_\mu t - N\xi\partial_\mu\pi = \xi(u_\mu - N\partial_\mu\pi) \quad [\text{with } \tilde{u}^\mu = \tilde{g}^{\mu\nu}\tilde{u}_\nu = g^{\mu\nu}\tilde{u}_\nu]$$

We can check that $u_\mu u^\mu$ doesn't transform, as it shouldn't since it is equal to -1:

$$\tilde{u}_\mu\tilde{u}^\mu = \xi^2(u^\mu - N\partial^\mu\pi)(u_\mu - N\partial_\mu\pi) = \frac{-1 - 2N^2 u_\mu\partial^\mu\pi + (\partial_\mu\pi)^2}{1 + \frac{2g^{0\mu}\partial_\mu\pi}{g^{00}} + (\partial_\mu\pi)^2}$$

$$g^{0\mu} = g^{\mu\nu}\partial_\nu t = \frac{-N}{-N}g^{\mu\nu}\partial_\nu t = -\frac{1}{N}g^{\mu\nu}u_\nu \Rightarrow \frac{g^{0\mu}\partial_\mu\pi}{\underbrace{g^{00}}_{-1/N^2}} = \frac{-N^2}{-N}g^{\mu\nu}\partial_\mu\pi\, u_\nu$$

$$\rightsquigarrow \frac{-1 - 2N^2 u_\mu\partial^\mu\pi + (\partial_\mu\pi)^2}{1 + 2N g^{\mu\nu}\partial_\mu\pi\, u_\nu + (\partial_\mu\pi)^2} = -1 \;!$$

Then, we can move to the transformation of $K_{\mu\nu}$: we actually need $\delta K_{\mu\nu} = K_{\mu\nu} - H h_{\mu\nu}$, where $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$. Knowing $\tilde{u}_\mu$, $\tilde{h}_{\mu\nu}$ is easy to compute. $\tilde{H}$ is just $H(t+\pi) = H(t) + \dot{H}(t)\pi + \dots$ Then, we move to $K_{\mu\nu} = h_\mu{}^\rho\nabla_\rho u_\nu = \nabla_\mu u_\nu + u_\mu \underbrace{u^\rho\nabla_\rho u_\nu}_{A_\nu}$: we stop at first order in perturbations for simplicity →

·) $\tilde{\nabla}_\mu\tilde{u}_\nu = \nabla_\mu\tilde{u}_\nu = u_\nu\nabla_\mu\xi + \xi\nabla_\mu u_\nu - \nabla_\mu\nabla_\nu\pi$ [using $N = 1 + \delta N$, $\xi = 1 + \mathcal{O}(1)$]

·) $\tilde{u}_\mu\tilde{A}_\nu = u_\mu u_\nu u^\rho\nabla_\rho\xi + \xi u_\mu A_\nu - H u_\mu{}^{(3)}\nabla_\nu\pi - u_\mu u^\rho\nabla_\rho\nabla_\nu\pi$ [using $A_\mu = \mathcal{O}(1)$]

⇒ $\tilde{K}_{\mu\nu} = \xi K_{\mu\nu} + u_\nu{}^{(3)}\nabla_\mu\xi - H u_\mu{}^{(3)}\nabla_\nu\pi - h_\mu{}^\rho\nabla_\rho\nabla_\nu\pi$, where ${}^{(3)}\nabla_\mu = h_\mu{}^\rho\nabla_\rho$.

Notice that there is no reason for $\tilde{K}_{\mu\nu}$ to be orthogonal to u^μ. It is orthogonal to $\tilde{u}^\mu$, though.

4.2 - EFT OF BROKEN SPATIAL DIFF. S : "1+3" PROJECTOR

In this Appendix we derive an identity for the projector on the hypersurface of codimension 3 defined by $\gamma_x = \{ x^i = \text{const.} \}$.

First, we show that $U^\mu = \sqrt{-g}\, \varepsilon^{\mu\nu\rho\sigma}\, \varepsilon_{ijk} \nabla_\nu x^i \nabla_\rho x^j \nabla_\sigma x^k$ is the vector tangent to γ_x. To show this, it's enough to show that x^i don't change along the integral lines of U^μ: indeed: $U^\mu \nabla_\mu x^i = \sqrt{-g}\, \varepsilon^{\mu\nu\rho\sigma}\, \varepsilon_{lmn} \nabla_\nu x^l \nabla_\rho x^m \nabla_\sigma x^n \nabla_\mu x^i = 0 \quad \forall\, i$.

The projector along U^μ can then be defined as $P_{\mu\nu} = \dfrac{U_\mu U_\nu}{U_\rho U^\rho}$. Indeed we have that $P_\mu{}^\rho P_\rho{}^\nu = P_\mu{}^\nu$ (idempotent), and $P_{\mu\nu} = P_{(\mu\nu)}$ (hermitian). Moreover, we have $\rightarrow$ $P_{\mu\nu} U^\nu = U_\mu$. Then, $P_{\mu\nu}$ is the projector on the submanifold γ_x.

4.3 - EFT OF BROKEN SPATIAL DIFF.S : STATIONARY SPACETIMES

In this Appendix we briefly sketch the proof of why we expect that the EFT for broken spatial diff.s will work in stationary backgrounds.

Recall that in the "time diff.s" case, we could do the "resumming" of perturbations only around FLRW, and not any b.g. w/ a preferred slicing. So: what is so special about FLRW? The slicing is maximally symmetric! I.e., given the constant-cosmic time hypersurfaces, there are 6 Killing vectors of the induced metric on Σ_t (i.e. these vectors live on Σ_t).

Now we have a preferred threading γ_x. In the previous section we have seen that the "induced metric" on γ_x is $P_{\mu\nu} = \frac{U_\mu U_\nu}{U_\rho U^\rho}$, where $U^\mu = \sqrt{-g}\, \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{ijk} \nabla_\nu x^i \nabla_\rho x^j \nabla_\sigma x^k$.

Then, we require that there is a $K^\mu \parallel U^\mu$ (i.e. "$\in \gamma_x$") such that $\Rightarrow$ $\mathcal{L}_{\vec{K}} P^{\mu\nu} = 0$, where $\mathcal{L}$ is the Lie derivative (notice that $\mathcal{L}_{\vec{K}} P^{\mu\nu} \in \gamma_x$ as well).

Notice that we consider $P^{\mu\nu}$ and not $P_{\mu\nu}$ since it makes calculations easier. Indeed, let us take $K^\mu = U^\mu$: then, $\mathcal{L}_{\vec{U}} P^{\mu\nu} = \frac{\mathcal{L}_{\vec{U}}(U^\mu U^\nu)}{U_\rho U^\rho} - \frac{U^\mu U^\nu}{(U_\rho U^\rho)^2} \mathcal{L}_{\vec{U}}(g_{\alpha\beta} U^\alpha U^\beta) =$

$$= \frac{(\mathcal{L}_{\vec{U}} U^\mu) U^\nu + U^\mu (\mathcal{L}_{\vec{U}} U^\nu)}{U_\rho U^\rho} - \frac{U^\mu U^\nu U^\alpha U^\beta \mathcal{L}_{\vec{U}} g_{\alpha\beta}}{(U_\rho U^\rho)^2} - \frac{U^\mu U^\nu g_{\alpha\beta} \mathcal{L}_{\vec{U}}(U^\alpha U^\beta)}{(U_\rho U^\rho)^2}$$

Now: $\mathcal{L}_{\vec{V}} V^\mu = 0 \ \forall V^\mu$, so we remain with $-P^{\mu\nu} P^{\alpha\beta} \mathcal{L}_{\vec{U}} g_{\alpha\beta}$, which vanishes if U^μ is a Killing vector... Then, we need a background with a timelike Killing vector to carry out the construction successfully.