# Neutrinos in Cosmology

## References

- Lesgourges & Pastor 2006 (astro-ph/0603494)
- Wong 2011 (1111.1436)

## Motivation

Many properties of the neutrino are still unknown:

- mass ?
- mass hierarchy ?
- is it its own antiparticle ?
  (Dirac or Majorana)
- number of mass eigenstates ?

Some of these questions might be answered by cosmological observations (before laboratory experiments will do so).
Neutrinos play an important role at different epochs in the history of the universe. In this lecture, however, I will focus on how the neutrino mass affects cosmological perturbations.

# 1) Neutrino properties

- The neutrino was postulated by Pauli in 1930 to explain how beta decay could conserve energy, momentum, and spin.
- In the Standard Model of particle physics the neutrino is a neutral lepton, which interacts only through the Weak Interaction ($W^\pm$, $Z$ bosons) and is treated as massless.
- Measurements of the $Z$ boson decay fix the number of active neutrinos to $N_\nu = 2.994 \pm 0.012$
$\Rightarrow$ 3 flavours: $\nu_e$, $\nu_\mu$, $\nu_\tau$
- If neutrinos are massive, flavours are not conserved (Pontecorvo 1957; Maki, Nakagawa, and Sakata 1962) $\longrightarrow$ <u>Neutrino oscillations</u>

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = U_{MNS} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$

$U_{MNS}$ has 3 mixing angles and 1 (3) phase if neutrinos are Dirac (Majorana) particles.

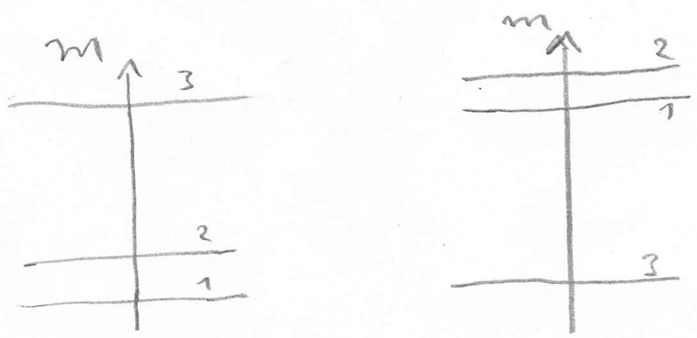
- Neutrino oscillation experiments measure the differences of squared masses

$$\Delta m_{21}^2 = m_2^2 - m_1^2 = (7.65 \pm 0.65) \times 10^{-5}\,\mathrm{eV}$$

$$|\Delta m_{31}^2| = (2.40 \pm 0.35) \times 10^{-3}\,\mathrm{eV}$$

from solar and atmospheric neutrinos

$\Rightarrow$ two possible <u>hierarchies</u>




normal | inverted

$\sum m_\nu \gtrsim 0.05\,\mathrm{eV}$ | $\sum m_\nu \gtrsim 0.1\,\mathrm{eV}$

( • If there are more mass eigenstates than flavour eigenstates, these extra neutrino states are sterile, i.e. insensitive to Weak interactions )

- Mass bounds from laboratory experiments (beta decay, neutrino oscillations)

$$0.05\,\mathrm{eV} \lesssim \sum m_\nu \lesssim 6\,\mathrm{eV}$$

## 2) Cosmic Neutrino Background

At very high temperature the weak interaction rate $\Gamma_\nu$ is larger than the Hubble expansion rate $H$. Hence, the neutrinos are in <u>thermal equilibrium</u> with the cosmic plasma.

They obey the <u>Fermi-Dirac</u> distribution:

$$f_\nu(E,T) = \frac{1}{1 + \exp[(E-\mu)/T]}$$

The chemical potential $\mu$ is expected to be similar to the matter-antimatter asymmetry:

$$\frac{\mu}{T} \sim 10^{-10} \quad ; \quad \text{BBN bounds} \left|\frac{\mu}{T}\right| < 0.05$$

Neutrinos decouple when $\Gamma_\nu \approx G_F^2 T^5$ falls below the expansion rate $H \approx T^2/M_P$

$$\Rightarrow \quad T_{dec} \approx 1 \text{ MeV}$$

$\Rightarrow$ neutrinos are ultra-relativistic at decoupling

$$E = p \longrightarrow f_\nu(p,T_\nu) = \frac{1}{1+\exp(p/T_\nu)}$$

Number density

$$n_\nu = \frac{g_s}{(2\pi)^3} \int d^3p \, f_\nu(p,T)$$

Energy density

$$\rho_\nu = \frac{g_s}{(2\pi)^3} \int d^3p \, E \, f_\nu(p,T)$$

with $g = 2$ counting neutrino and antineutrino.

After decoupling particle number is conserved and the physical momentum redshifts as $a^{-1}$

$$\Rightarrow \quad T_\nu = T_{\nu,0}\, a \qquad (a \text{ is the scale factor})$$

$e^\pm$ annihilation after neutrino decoupling heats the photon plasma. Using entropy conservation one can calculate:

$$\frac{T_\gamma}{T_\nu} = \left(\frac{11}{4}\right)^{1/3} \simeq 1.4$$

$$\Rightarrow \quad T_{\nu,0} \simeq 1.95 \text{ K}$$

$$n_\nu = \frac{3}{11} n_\gamma$$

$n_\nu^0 \simeq 113 \text{ cm}^{-3}$ of each flavour

The neutrino energy density has two analytic limits:

$$\rho_\nu (m_\nu \ll T_\nu) = \frac{7}{8} \frac{\pi^2}{15} \left(\frac{4}{11}\right)^{4/3} T_\gamma$$

$$\rho_\nu (m_\nu \gg T_\gamma) = m_\nu n_\nu$$

As long as neutrinos are relativistic they contribute to the total <u>radiation density</u>:

$$\rho_R = \left[1 + \frac{7}{8}\left(\frac{4}{11}\right)^{4/3} N_{eff}\right] \rho_\gamma$$

$$N_{eff} = 3.046 \qquad \left(\begin{array}{l}\text{some } e^\pm \text{, flavour oscillations,} \\ \text{finite temperature QED}\end{array}\right)$$

Depature from this value would be due to non-standard neutrino physics or other relativistic species.

BBN: $N_{eff} = 2.5 \pm 1.0$ (95 % C.L.)

CMB: $N_{eff} = 3.4 \pm 0.7$ (95 % CL)

<u>Neutrino fraction at present</u>

$$\Omega_\nu = \frac{\rho_\nu}{\rho_{crit}} \simeq \frac{\sum m_\nu}{93 \text{ eV}} \frac{1}{h^2}$$

With $\underset{\substack{\uparrow \\ \text{total matter fraction}}}{\Omega_m} < 1 \ (0.3) \Rightarrow m_\nu < 15\text{eV} \ (5\text{eV})$
(with $h = 0.7$)

# 3) Linear cosmological perturbations

Einstein field equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \, T_{\mu\nu}$$

The Ricci tensor $R_{\mu\nu}$ and Ricci scalar $R$ are given by the <u>metric $g_{\mu\nu}$</u> and its derivatives.
The <u>stress-energy tensor $T_{\mu\nu}$</u> describes the density an flux of energy and momentum.

## 3.1) Isotropic and homogenous universe

Friedmann - Lemaître - Robertson - Walker metric:

$$ds^2 = g_{\mu\nu} \, dx^\mu dx^\nu = a^2(\tau) \left( -d\tau^2 + \gamma_{ij} \, dx^i dx^j \right)$$

$\tau$ is the conformal time $d\tau = dt / a(t)$

$\gamma_{ij}$ describes the spatial geometry, which we assume to be flat: $\gamma_{ij} = \delta_{ij}$

Due to homogeneity and isotropy:

$$T^\mu{}_\nu = \begin{pmatrix} -\bar{\rho} & 0 & 0 & 0 \\ 0 & \bar{p} & 0 & 0 \\ 0 & 0 & \bar{p} & 0 \\ 0 & 0 & 0 & \bar{p} \end{pmatrix}$$

$\bar{\rho}$ mean density and $\bar{p}$ mean pressure

From the Einstein equation we get the Friedmann equations:

$$\mathcal{H}^2 \equiv \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3} a^2 \bar\rho$$

$$\dot{\mathcal{H}} = -\frac{4\pi G}{3} a^2 (\bar\rho + 3\bar p) \qquad \dot{} \;\hat{=}\; \frac{d}{d\tau}$$

Conservation of energy-momentum $\nabla_\mu T^{\mu\nu} = 0$

$$\Rightarrow \quad \dot{\bar\rho} + 3\mathcal{H}(\bar\rho + \bar p) = 0$$

non-relativistic matter $\bar p = 0 \Rightarrow \bar\rho_m \propto a^{-3}$

radiation $\bar p = \frac{1}{3}\bar\rho \Rightarrow \bar\rho_r \propto a^{-4}$

## 3.2) Perturbations

$$ds^2 = a^2(\tau)\Big\{-(1+2\psi)\,d\tau^2 - 2B_i\,d\tau\,dx^i + \left[(1-2\phi)\delta_{ij} + 2h_{ij}\right]dx^i dx^j\Big\}$$

$$T^{\mu\nu} = \bar T^{\mu\nu} + \delta T^{\mu\nu} = (\rho + p)u^\mu u^\nu + p g^{\mu\nu} + \Sigma^{\mu\nu}$$

with four-velocity $u^\mu = \frac{dx^\mu}{\sqrt{-ds^2}}$

and shear stress $\Sigma$ : $\Sigma^\mu{}_\mu = 0 = \Sigma^\mu{}_\nu u^\nu$

These perturbations (metric and $\delta T^{\mu\nu}$) can be decomposed into scalars, vectors, and tensors und 3-dim. rotations.

At the linear level the Einstein equations decouple into these three sectors. In this lecture we are only interested in the scalar perturbations:

$$\delta T^0{}_0 = -\delta\rho \qquad \delta T^i{}_0 = -(\bar\rho + \bar p)\, v^{(s)\,i}$$

$$\delta T^0{}_i = (\bar\rho + \bar p)\left(v_i^{(s)} - B_i^{(s)}\right) \qquad \delta T^i{}_j = \delta p\, \delta^i_j + \Sigma^{(s)i}{}_j$$

where $\rho = \bar\rho + \delta\rho$ and $p = \bar p + \delta p$

$(s)$ denotes the scalar component

$v^i$ is the coordinate velocity : $\frac{dx^i}{d\tau}$

Of the 4 scalar d.o.f. in the perturbed metric

$$\psi \,,\ \phi \,,\ B_i^{(s)} \text{ and } h_{ij}^{(s)}$$

only two are physical $\Rightarrow$ we can choose a gauge such that $B_i^{(s)} = 0$ and $h_{ij}^{(s)} = 0$.

This is the <u>conformal Newtonian gauge</u>

$$ds^2 = a^2(\tau)\left\{-(1+2\psi)\,d\tau^2 + (1-2\phi)\,\delta_{ij}\,dx^i dx^j\right\}$$

The perturbed Einstein equations for the scalar modes in Fourier space to linear order:

$$-k^2\phi - 3\mathcal{H}(\dot\phi + \mathcal{H}\psi) = 4\pi G a^2 \delta\rho$$

$$\dot\phi + \mathcal{H}\psi = 4\pi G a^2 (\bar\rho + \bar p)\theta$$

$$\ddot\phi + \mathcal{H}(\dot\psi + 2\dot\phi) + (2\dot{\mathcal{H}} + \mathcal{H}^2)\psi + \frac{k^2}{3}(\phi - \psi) = 4\pi G a^2 \delta p$$

$$k^2(\phi - \psi) = 12\pi G a^2 (\bar\rho + \bar p)\sigma$$

$\theta$ : velocity divergence $\theta \equiv i k^i v_i$

$\sigma$ : anisotropic stress $(\bar\rho + \bar p)\sigma \equiv -(k^{-2} k_i k_j - \frac{1}{3}\delta_{ij})\Sigma_{ij}$

## 3.3) Boltzmann Equation

Phase space density $f(x^i, P_j, \tau)$

number of particles in a phase space volume $d^3x\, d^3P$ : $dN = \frac{g_s}{(2\pi)^3} f(x^i, P_j, \tau)\, d^3x\, d^3P$

$P_j$ is the canonical conjugate of $x^i$

$$P_i = a(1-\phi)\, p_i \quad (p_i \text{ proper momentum measured by a comoving observer})$$

$$P_0 = -a(1+\psi)E \quad (E \text{ proper energy})$$

4-momentum $P_\mu = (P_0, P_i)$ $\quad P_\mu P^\mu = -m^2$

Covariant expression of the stress-energy tensor:

$$T_{\mu\nu} = \frac{g_s}{(2\pi)^3} \int d^3P \sqrt{-g} \; \frac{P_\mu P_\nu}{P^0} \; f(x^i, P_j, \tau)$$

with $\sqrt{-g} = a^{-4} \left( 1 - 4 + 3\phi \right)$

$$T^0{}_0 = -a^{-4} \int \frac{d^3q}{(2\pi)^3} \; \epsilon(q,\tau) \; f(\vec{x}, \vec{q}, \tau)$$

$$T^0{}_i = a^{-4} \int \frac{d^3q}{(2\pi)^3} \; q \; f(\vec{x}, \vec{q}, \tau) = -T^i{}_0$$

$$T^i{}_j = a^{-4} \int \frac{d^3q}{(2\pi)^3} \; \frac{q^i q_j}{\epsilon} \; f(\vec{x}, \vec{q}, \tau)$$

with $\vec{q} = a\vec{p}$ and $\epsilon(q,\tau) \equiv aE = \sqrt{a^2m^2 + q^2}$

## Boltzmann equation

$$\frac{\partial f}{\partial \tau} + \frac{d\vec{x}}{d\tau} \cdot \frac{\partial f}{\partial \vec{x}} + \frac{d\vec{q}}{d\tau} \cdot \frac{\partial f}{\partial \vec{q}} = C[f]$$

$C[f] = 0$ for collisionless evolution.

Expanding the phase space density around the homogenous and isotropic solution:

$$f(\vec{x}, \vec{q}, \tau) = f_0(q) + f_1(\vec{x}, \vec{q}, \tau)$$

For neutrinos $f_0(q) = \left[ 1 + \exp(q/T_{\nu,0}) \right]^{-1}$

To first order in Fourier space:

$$\boxed{\frac{\partial f_1}{\partial \tau} + i\,\frac{q}{\epsilon}\left(\vec{k}\cdot\hat{n}\right) f_1 + \left[\dot{\phi} - i\,\frac{\epsilon}{q}\left(\vec{k}\cdot\hat{n}\right)\psi\right]\frac{d f_0}{d \ln q} = 0}$$

where $\hat{n} = \dfrac{\vec{q}}{q}$ using $\dfrac{d\vec{x}}{d\tau} = \dfrac{P^i}{P_0} = \dfrac{\vec{q}}{\epsilon}$

and $\dfrac{d\vec{q}}{d\tau} = q\dot{\phi} - i\epsilon\,\vec{k}\cdot\vec{q}\,\psi$ from the geodesic equation $p^0\dot{p}^0 + \Gamma^0_{\mu\nu}p^\mu p^\nu = 0$

## For Cold Dark Matter (CDM)

$\epsilon \simeq am \qquad \dfrac{q}{\epsilon} \ll 1$

$$\delta_c = \frac{1}{\bar{n}_c}\int \frac{d^3q}{(2\pi a)^3}\, f_1 \qquad \left(\rho_c = (1+\delta_c)\,\bar{\rho}_c\right)$$

$$\theta_c = \frac{1}{\bar{n}_c}\int \frac{d^3q}{(2\pi a)^3}\, f_1\,\frac{q}{\epsilon}\left(\vec{k}\cdot\hat{n}\right)$$

$$\xrightarrow{\text{Boltzmann}} \quad \delta_c + \theta_c - 3\dot{\phi} = 0 \qquad \text{(continuity)}$$

$$\dot{\theta}_c + \mathcal{H}\,\theta_c - k^2\psi = 0 \qquad \text{(Euler)}$$

# Boltzmann Hierarchy



Note that the linearized Boltzmann equation does not depend explicitly on the direction of $\vec{k}$ or $\vec{q}$. It only involves $\vec{k}\cdot\vec{q} \equiv kq\mu$

Thus one can decompose $f_1$ in terms of a Legendre series:

$$f_1(k,\mu,q,\tau) = \sum_{n=0}^{\infty} (-i)^n (2n+1)\,\Psi_n(k,q,\tau)\,P_n(\mu)$$

($P_n(\mu)$ are the Legendre polynomials)

With this decomposition in multipoles $\Psi_n$ one can write the Boltzmann equation as an infinite series of differential equations:

$$\dot{\Psi}_0 = -\frac{qk}{\epsilon}\Psi_1 - \dot{\phi}\frac{df_0}{d\ln q}$$

$$\dot{\Psi}_1 = \frac{qk}{3\epsilon}\left(\Psi_0 - 2\Psi_2\right) - \frac{\epsilon k}{3q}\psi\frac{df_0}{d\ln q}$$

$$\dot{\Psi}_{n\geq 2} = \frac{qk}{(2n+1)\epsilon}\left[n\Psi_{n-1} - (n+1)\Psi_{n+1}\right]$$

The multipoles $\Psi_n$ are related to the bulk quantities of the fluid by

$$\delta\rho = a^{-4} \int \frac{d^3q}{(2\pi)^3} \, \epsilon \, \Psi_0$$

$$\delta p = \frac{1}{3} a^{-4} \int \frac{d^3q}{(2\pi)^3} \, \epsilon \left(\frac{q}{\epsilon}\right)^2 \Psi_0$$

$$\left(\bar\rho + \bar p\right) \theta = k a^{-4} \int \frac{d^3q}{(2\pi)^3} \, \epsilon \left(\frac{q}{\epsilon}\right) \Psi_1$$

$$\left(\rho + p\right) \sigma = \frac{2}{3} a^{-4} \int \frac{d^3q}{(2\pi)^3} \, \epsilon \left(\frac{q}{\epsilon}\right)^2 \Psi_2$$

Using the adibiatic initial conditions set by inflation:

$$\delta_\gamma = \delta_\nu = \frac{4}{3}\delta_c = \frac{4}{3}\delta_b = -2\psi$$

$$\theta_\gamma = \theta_\nu = \theta_c = \theta_b = \frac{1}{2} k^2 \tau \psi$$

$$\sigma_\nu = \frac{1}{15} (k\tau)^2 \psi \, , \quad \psi = \frac{20C}{15 + R_\nu}$$

$$\phi = \left(1 + \frac{2}{5} R_\nu\right) \psi \quad \text{with} \quad R_\nu = \frac{\bar\rho_\nu}{\bar\rho_\nu + \bar\rho_\gamma}$$

one can numerically integate the linearized Einstein - Boltzmann equations

$\Rightarrow$ CAMB, CLASS, CMBFAST

# 4) A simple physical picture

## Free streaming length

analog to the Jeans scale:

$$k_{FS} = \left( \frac{4\pi G \bar{\rho}(a) a^2}{v_{th}(a)} \right)^{\frac{1}{2}}$$

$$\lambda_{FS} = 2\pi \frac{a}{k_{FS}} = 2\pi \sqrt{\frac{2}{3}} \frac{v_{th}(a)}{H(a)}$$

average thermal velocity when non-relativistic

$$v_{th} = \frac{\langle p \rangle}{m} \simeq \frac{3 T_\nu}{m} \simeq 150 (1+z) \left( \frac{1eV}{m} \right) \frac{km}{s}$$

$$\Rightarrow k_{FS} \approx 0.8 \frac{\sqrt{\Omega_\Lambda + \Omega_m (1+z)^3}}{(1+z)^2} \left( \frac{m}{1eV} \right) h\, Mpc^{-1}$$

when relativistic: $v_{th} \approx c$ and the free-streaming length is equal to the Hubble radius.

The maximum comoving free-streaming length is reached at the transition when neutrinos become non-relativistic: $m \approx \langle p \rangle = 3 T_\nu$

$$\Rightarrow z_{nr} \approx 2000 \left( \frac{m}{1eV} \right)$$

The corresponding minimum wavenumber is

$$k_{nr} \simeq 0.02 \sqrt{\Omega_m} \sqrt{\frac{m}{1\,eV}} \; h\, Mpc^{-1}$$

Modes with: $k \ll k_{nr}$ behave like CDM

$k \gg k_{nr}$ : $\delta_\nu \longrightarrow 0$

The power spectrum of all matter species:

$$P(k) = \left\langle \left| \frac{\delta\rho_c + \delta\rho_b + \delta\rho_\nu}{\bar\rho_c + \bar\rho_b + \bar\rho_\nu} \right|^2 \right\rangle$$

$$= \begin{cases} \langle |\delta_c|^2 \rangle & \text{for } k \ll k_{nr} \\ (1 - f_\nu)^2 \langle |\delta_c|^2 \rangle & \text{for } k \gg k_{nr} \end{cases}$$

with $f_\nu = \dfrac{\Omega_\nu}{\Omega_m}$

## Neutrino backreaction

1) neutrinos change the background evolution

2) - perturbation on scales $k \ll k_{nr}$

- neutrino shear $\sigma_\nu$ during radiation domination

During matter domination for modes well inside the Hubble radius:

$$-\frac{k^2}{a^2}\phi = 4\pi G\,\delta\rho \qquad \text{(Poisson)}$$

$$\ddot{\delta}_c + \frac{\dot{a}}{a}\dot{\delta}_c = -k^2\phi \qquad \text{(continuity + Euler)}$$

$$\Rightarrow \quad \ddot{\delta}_c + \frac{\dot{a}}{a}\dot{\delta}_c = 4\pi G a^2 \delta\rho$$

$$\delta\rho_\nu = 0 \quad \text{for } k \gg k_{nr}$$

$$\Rightarrow \quad \delta\rho = (\bar{\rho}_c + \bar{\rho}_b)\,\delta_c$$

but $$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} a^2 \left(\bar{\rho}_c + \bar{\rho}_b + \bar{\rho}_\nu\right)$$

$$\Rightarrow \quad a \propto \tau^2$$

$$\ddot{\delta}_c + \frac{2}{\tau}\dot{\delta}_c - \frac{6}{\tau^2}(1 - f_\nu)\,\delta_c = 0$$

The growing solution is

$$\delta_c \propto a^{1-\frac{3}{5}f_\nu} \qquad (f_\nu \ll 1)$$

The potential decays as

$$-k^2\phi \propto a^{-\frac{3}{5}f_\nu}$$

With Dark Energy: $\delta_c \propto (a\,g(a))^{1-\frac{3}{5}f_\nu}$

Matter power spectrum for massive vs. massless neutrinos: (holding $\Omega_m$ fixed)

1) for $k < k_{nr}$ identical

2) $k \gg k_{nr}$
time of matter - radiation equality is delayed

$$\frac{a_{eq}^{f_\nu}}{a_{eq}} = (1 - f_\nu)^{-1}$$

For $a < a_{nr}$ : $\delta_c^{f_\nu}(a) = \delta_c\left(a = (1-f_\nu)\, a\right)$

For $a = a_0$ : $\dfrac{\delta_c^{f_\nu}}{\delta_c} = (1-f_\nu)^{\frac{1}{2}} \left( \dfrac{a_0\, g(a_0)}{a_{nr}} \right)^{-\frac{3}{5} f_\nu}$

$$\Rightarrow \quad \frac{P^{f_\nu}}{P} \simeq -8 f_\nu$$

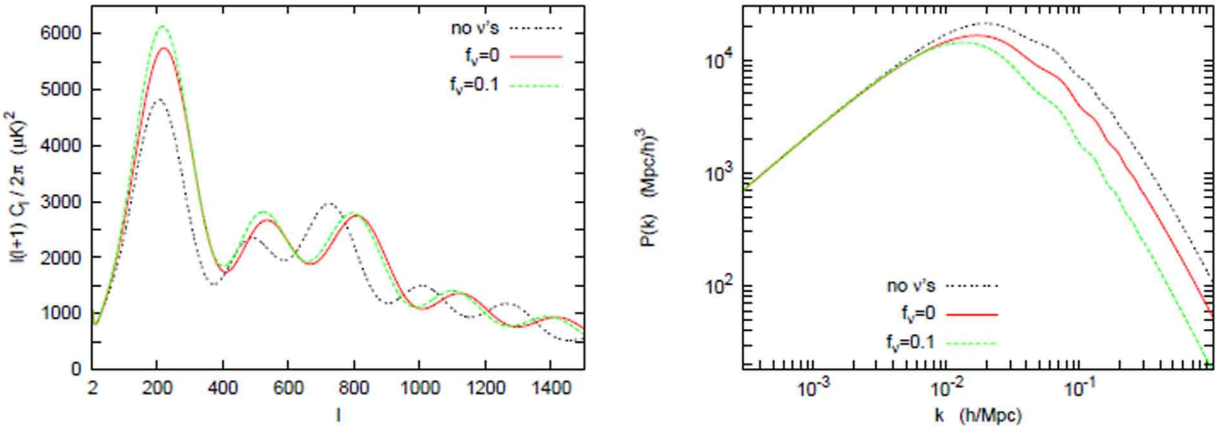



Fig. 14. CMB temperature anisotropy spectrum $C_l^T$ and matter power spectrum $P(k)$ for three models: the neutrinoless ΛCDM model of section 4.4.6, a more realistic ΛCDM model with three massless neutrinos ($f_\nu \simeq 0$), and finally a ΛMDM model with three massive degenerate neutrinos and a total density fraction $f_\nu = 0.1$. In all models, the values of ($\omega_\mathrm{b}$, $\omega_\mathrm{m}$, $\Omega_\Lambda$, $A_s$, $n$, $\tau$) have been kept fixed.

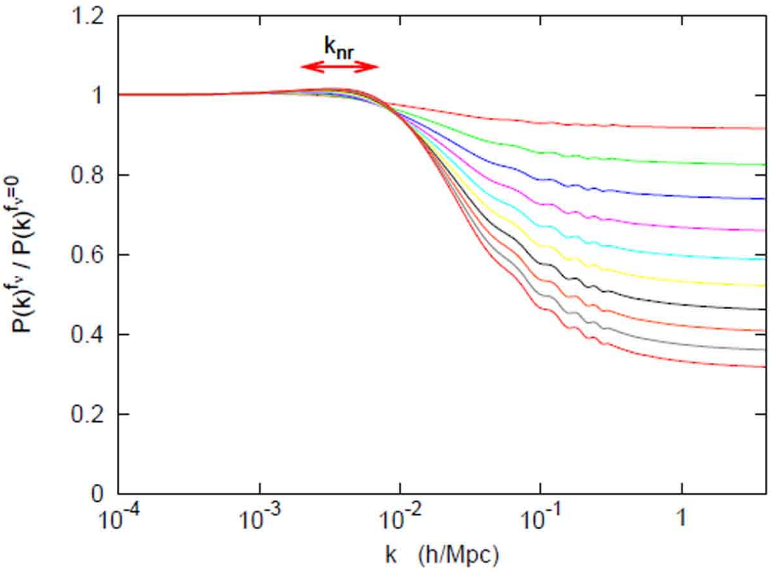



Fig. 13. Ratio of the matter power spectrum including three degenerate massive neutrinos with density fraction $f_\nu$ to that with three massless neutrinos. The parameters $(\omega_\mathrm{m}, \Omega_\Lambda) = (0.147, 0.70)$ are kept fixed, and from top to bottom the curves correspond to $f_\nu = 0.01, 0.02, 0.03, \ldots, 0.10$. The individual masses $m_\nu$ range from 0.046 eV to 0.46 eV, and the scale $k_\mathrm{nr}$ from $2.1 \times 10^{-3} h\,\mathrm{Mpc}^{-1}$ to $6.7 \times 10^{-3} h\,\mathrm{Mpc}^{-1}$ as shown on the top of the figure. $k_\mathrm{eq}$ is approximately equal to $1.5 \times 10^{-2} h\,\mathrm{Mpc}^{-1}$.