MPA Cosmology Lecture Series: Modified Gravity

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1 Goal of this lecture

These lecture notes aim to provide with a broad introduction to the study of modified gravity models in a cosmological context. The main objective is to draw a picture of the research field as a whole, ranging from theoretical to more phenomenological/observational aspects. Owing to the often lengthy calculations, sometimes we will be forced to restrict ourselves to displaying only the final result and interpret the solutions qualitatively. Whenever possible though, we shall go through some calculations with a bit more detail, both for fun and intuition-building purposes. The diagram of Fig. 1 displays the outline (which is also a summary) of this lecture.

For these notes, although it is desirable that one is familiar with variational calculus and tensor algebra, this is not mandatory. It is nearly impossible to avoid discussing modified gravity without resorting to actions and Lagrangians, but care was taken such that whenever and action is written, it is immediately followed by the associated equations of motion (which may perhaps be easier to interpret for some).

It is very unlikely that someone will become a field expert by just reading these notes. Further reading suggestions include the following reviews (on which parts of these notes are based):

- *Clifton, Ferreira, Padilla & Skordis, arXiv:1106.2476.* Being over 300 pages long and referring to over 1300 papers, this is by far the most comprehensive review of modified gravity in cosmology. Most of the focus is on the theoretical aspects of the models, and less so on their observational signatures.
- *Joyce, Jain, Khoury & Trodden, arXiv:1407.0059.* This review organizes the discussion by the different types of screening mechanisms, discussing their theoretical aspects as well as typical observational tests and constraints.
- *Koyama, arXiv:1504.04623.* The structure and scope of this review is similar to the one above, but it pays more attention to the observational aspects of modified gravity (in particular at the nonlinear level of structure formation).

2 Motivation for modified gravity studies

In the current standard ΛCDM cosmological model, the gravitational interaction is described by Einstein's theory of General Relativity (GR). The main reason for this is perhaps related to its remarkable agreement with a wealth of precision tests of gravity done in the Solar System. These include the classical tests of gravitational redshift, the lensing of the light from background stars by the Sun and the anomalous perihelion of Mercury, as well as other tests such as the Shapiro time-delay effect measured by the Cassini spacecraft and Lunar laser ranging experiments which meausure the rate of change of the gravitational strength in the SS. Outside of the SS, GR is also in good agreement with the tests that involve changes in the orbital period of binary pulsars due to the emission of gravitational waves.

Despite of these tremendous successes, however, one can still think of a few reasons to expect/suspect/wish that GR does not provide us with the full picture:

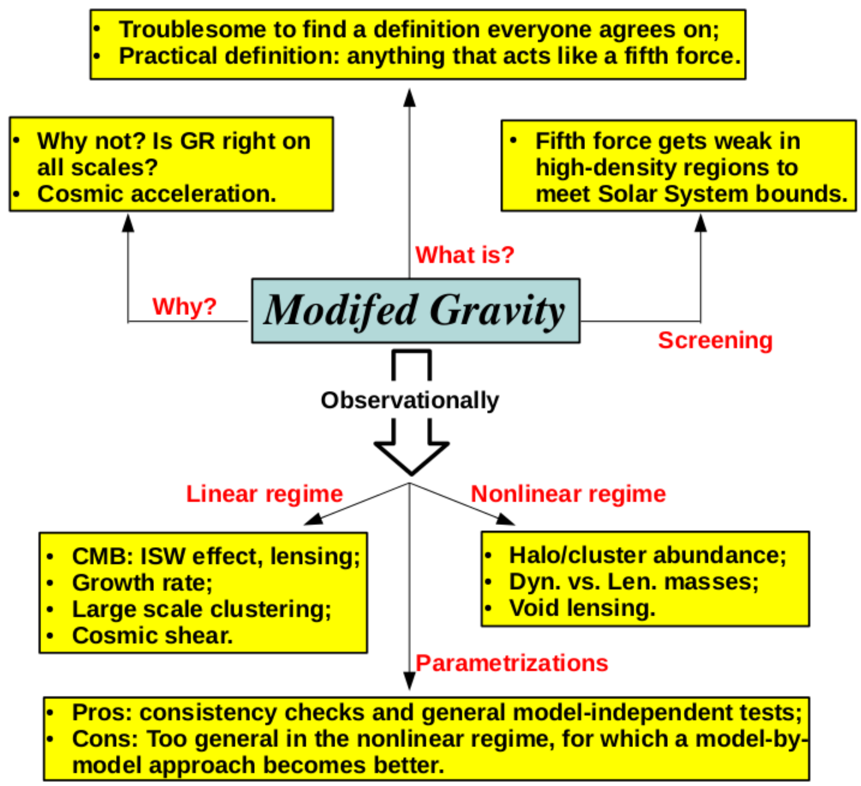

Figure 1: Outline/summary of the content of these notes.

• *Is GR correct on large scales?* The abovementioned tests probe the gravitational law only on scales smaller than the Solar System. This means that the application of GR in any cosmological study constitutes in fact a huge extrapolation of the regime of validity of the theory. In other words, there is room for deviations from GR on cosmological scales, and the size of such deviations should be constrained.

• *Modified gravity can be dark energy.* Dark energy is the general name given to any form of energy with negative enough pressure to have "repulsive" gravity. Its existence is postulated to explain the observed accelerated expansion of the Universe, which is otherwise impossible in a Universe governed by GR and containing only the matter species we know (radiation and matter). The argument for modified gravity is that the need to postulate dark energy may follow from our wrong use of GR as the theory of gravity on large scales. In other words, what we think are effects of dark energy may simply be the effects of the corrections to GR that we are still unaware of.

• *GR has no quantum limit.* A final (slightly more speculative, but enlightening) way to gain courage to go ahead and modify GR is to remind ourselves that this theory does not have a well defined quantum field limit. Taking for granted that all interactions must have a quantum field description, then GR cannot be the final answer and must be corrected. To be fair, these corrections to GR must take place on small scales or in the high-energy limit, whereas in cosmology we are concerned with the opposite end of the energy spectrum: weak fields on large scales. Nevertheless, it is not unreasonable to believe that an eventual quantum field theory of gravity that differs from GR on small scales, should also differ from it on cosmological ones.

3 What is modified gravity?

In this section, we shall try to specify what modified gravity actually means. First of all (and as we could guess from the discussion in the previous section), the phrase "modified gravity" is a slight abuse of language. It is used to describe any theory of gravity that goes beyond GR, and so "modified GR" would be a more appropriate name. As a result, the best way to start defining modified gravity is with a quick recap of GR.

3.1 General Relativity in a nutshell

GR can be described by the action

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{R}{16\pi G} - \mathcal{L}_m\left(\psi, g_{\mu\nu}\right)\right], \tag{1}$$

where g is the determinant of the 2-rank metric tensor field $g_{\mu\nu}$, R is called the Ricci scalar, $\mathcal{L}_m$ is the Lagrangian density that describes the forms of energy we know (dark matter, baryons, radiation, etc., described collectively by the field ψ) and the integration is taken over the whole four-dimensional spacetime x^μ ($\mu = 0, 1, 2, 3$). By varying this action w.r.t. $g_{\mu\nu}$ (in these notes, we will not be bothered with the boring algebra of variational calculus, and will just take the result for granted), we arrive at the famous Einstein field equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T^m_{\mu\nu}, \tag{2}$$

where $R_{\mu\nu}$ is the Ricci tensor and $T_{\mu\nu}$ the energy-momentum tensor associated with $\mathcal{L}_m$. The left hand-side of this equation contains purely geometric terms (i.e., the metric and its derivatives), whereas the right hand-side specifies the energy content that exists in the Universe. For the reader that is least familiar with tensor algebra, the quick way to interpret Eq. (2) is to think of it as a set of 16 equations, each labelled by (μ, ν). In fact, all the above tensors are symmetric, i.e., $T_{\mu\nu} = T_{\nu\mu}$, which means that there can only be 10 different equations (some of these 10 equations can also be redundant, depending on the exact application in mind). Another important aspect of GR is that the Einstein tensor $G_{\mu\nu}$ is divergence-free, which naturally ensures energy-momentum conservation:

$$\nabla_\nu G^{\mu\nu} = 0 \quad \Longrightarrow \quad \nabla_\nu T^{\mu\nu} = 0. \tag{3}$$

In order to arrive at a concrete set of equations to work with, one needs to specify two things: (i) the energy-momentum tensor – to plug in the right-hand side of Eq. (2); and (ii) the metric – to define the curvature tensors on the left-hand side.

In cosmology, it is common to take the form of $T_{\mu\nu}$ to be that of a pefect fluid

$$T_{\mu\nu} = \left(\rho + P\right)u_\mu u_\nu - P g_{\mu\nu}, \tag{4}$$

where ρ, P and u_μ are, respectively, the density, the pressure and the four-velocity of the fluid. For completeness, we note that we are neglecting the fluid's heat flux (a vector) and anisotropic stress (a tensor) in the above equation, but this is not critical for these notes.

Motivated by the cosmological principle, the line element of the metric field is taken to be that of a Friedmann-Robertson-Walker (FRW) spacetime

$$\mathrm{d}s^2 = \left(1 + 2\Psi\right)\mathrm{d}t^2 - a(t)^2\left(1 - 2\Phi\right)\left[\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right], \tag{5}$$

where Ψ, Φ are two gravitational potentials and $a = 1/(1+z)$ is the scale factor (z is the redshift). At the background level, $\Psi = \Phi = 0$, and a becomes the only variable to solve for (the "size" of the spatial sector of the Universe as it expands). Before proceeding, it is important to mention that we are considering only scalar perturbations (Ψ, Φ are scalar fields) to the homogeneous FRW picture. In general, Eq. (5) can contain also vector (which are typically very small and decay rather quickly) and tensor perturbations (gravitational waves). Vector and tensor perturbations are not covered in these notes. We have also assumed that the Universe is spatially flat.

3.2 Key equations

At the background level (recall $\Phi = \Psi = 0$ and $u_\mu = (-1,0,0,0)$), using the $(0,0)$ component and (i,i) component ($i = 1,2$ or 3) of Eq. (2), as well as the 0-component of $\nabla_\mu T^{\mu\nu} = 0$, we arrive at

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = 8\pi G\bar{\rho}, \tag{6}$$

$$\left(\frac{\ddot{a}}{a}\right) = -\frac{4\pi G}{3}\left(\bar{\rho} + 3\bar{P}\right), \tag{7}$$

$$\dot{\bar{\rho}} = -3H\left(\bar{\rho} + \bar{P}\right), \tag{8}$$

where an overdot denotes a derivative w.r.t. physical time t and an overbar indicates background quantities. Given the matter content, these equations specify the rate at which the Universe expands. For instance, in a matter dominated universe ($\bar{\rho} = \bar{\rho}_m$, $\bar{P} = 0$), Eq. (8) tells us that $\bar{\rho}_m = \bar{\rho}_{m0}a^{-3}$, where we define $\bar{\rho}_{m0}$ as the present-day value ($a = 1$) of the matter density. Plugging $\bar{\rho}_m$ into Eq. (6) yields $(\dot{a}/a)^2 = 8\pi G\bar{\rho}_{m0}a^{-3}$, which is a differential equation that can be solved to find $a(t)$. Equation (7) tells an important story about dark energy. From this equation, the requirement that the expansion of the Universe accelerates, $\ddot{a} > 0$, implies that the Universe must be dominated by a form of energy characterized by $\bar{P}/\bar{\rho} < -1/3$. None of the forms of energy currently known display this behaviour, hence the need to invoke dark energy.

The relevant equations for structure formation are obtained by taking into account the metric perturbations (Φ, Ψ in Eq. (5)) and by perturbing also the four-velocity, $u_\mu = (-1, v_i)$. We can make use of combinations of components of the metric field equations and using also the $\nu = i$ component of Eq. (3) to arrive at three equations that will accompany us throughout these notes. These are the Poisson equation, the Slip equation and the geodesic equation

$$\nabla^2\Phi = 4\pi G\delta\rho_m, \tag{9}$$

$$\Phi = \Psi, \tag{10}$$

$$\dot{v}_i + Hv_i = -\frac{1}{a}\nabla_i\Psi, \tag{11}$$

respectively, where $\delta\rho_m = \rho_m - \bar{\rho}_m$. One should bear in mind that the above equations have been subject to a number of approximations. In particular, we have discarded some terms that are negligible on sub-horizon scales (e.g. $\sim \dot{\Phi}$, $H^2\Psi$). Here and throughout we shall also consider a single non-relativistic fluid that sources the gravitational potentials, e.g., ψ in $\mathcal{L}_m$ in Eq. (1) can represent only the dark matter field.

3.3 The somewhat troubled definition of modified gravity

What is modified gravity then? One could define modified gravity as follows:

A) *"A modified gravity model is any model that adds something new to Eqs. (1) or (2)."*

Admittedly, this is not a good definition, but lets look at it nonetheless to try to build some intuition. For instance, in the ΛCDM model, the action and the field equations are given by

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{R}{16\pi G} - \frac{\Lambda}{8\pi G} - \mathcal{L}_m(\psi, g_{\mu\nu})\right] \tag{12}$$

$$\implies G_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G T^m_{\mu\nu}, \tag{13}$$

which do contain "something new". The extra term, however, leads only to changes in Eqs. (6)-(8) – via the contribution of Λ to the total background density and pressure: $\bar{\rho} = \bar{\rho}_m + \bar{\rho}_\Lambda$ and $\bar{P} = \bar{P}_\Lambda = -\bar{\rho}_\Lambda$; but leaves Eqs. (9)-(11) structurally unchanged. Consider further the action of a *Quintessence* model in which a canonical scalar field φ that rolls down a potential $V(\varphi)$ is meant to drive the accelerated expansion (much like in inflationary models)

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{R}{16\pi G} - \frac{1}{2}\nabla_\mu\varphi\nabla^\mu\varphi - V(\varphi) - \mathcal{L}_m(\psi, g_{\mu\nu})\right] \tag{14}$$

$$\Longrightarrow \quad G_{\mu\nu} = 8\pi G\left[T^m_{\mu\nu} + \nabla_\mu\varphi\nabla_\nu\varphi - \left(\frac{1}{2}\nabla_\alpha\varphi\nabla^\alpha\varphi + V(\varphi)\right)g_{\mu\nu}\right]. \tag{15}$$

The density fluctuations of such a scalar field propagate with unity sound speed $c_s^2 = 1$, which in practice means that the field does not have appreciable density fluctuations on sub-horizon scales, $\delta\rho_\varphi = 0$. The presence of the field is therefore manifest only at the backgroud level with density and pressure given by, $\bar{\rho}_\varphi = \dot{\bar{\varphi}}^2/2 + V(\bar{\varphi})$, $\bar{P}_\varphi = \dot{\bar{\varphi}}^2/2 - V(\bar{\varphi})$, respectively. The time evolution of the scalar field is governed by the equation $\ddot{\bar{\varphi}} + 3H\dot{\bar{\varphi}} - \mathrm{d}V/\mathrm{d}\bar{\varphi} = 0$.

In both the Quintessence and Λ cases, only Eqs. (6)-(8) get extra terms and not Eqs. (9)-(11). These two energy components do not impact directly on structure formation – only indirectly via its effects on the expansion rate of the Universe H. In these models, the theory of gravity is still GR, but with homogenous energy species sourcing the energy-momentum tensor. In the literature, these models are called *pure dark energy* or simply *dark energy* models.

Let us attempt what sounds like a more reasonable definition:

B) *"A modified gravity model is any model that modifies the Poisson equation, Eq. (9)."*

If the relation between the matter density fluctuations and the gravitational potential is not the familiar one described by Eq. (9), then this may be reason to raise some eyebrows. To investigate the merits of this putative definition of modified gravity, consider the following theory:

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{R}{16\pi G} - \mathcal{L}_\varphi - \mathcal{L}_m(\psi, g_{\mu\nu})\right] \tag{16}$$

$$\Longrightarrow \quad G_{\mu\nu} = 8\pi G\left[T^m_{\mu\nu} + T^\varphi_{\mu\nu}\right], \tag{17}$$

where $\mathcal{L}_\varphi$ is some general scalar Lagrangian and $T^\varphi_{\mu\nu}$ its associated energy-momentum tensor. By appropriately choosing the functional form of $\mathcal{L}_\varphi$, it is possible to design models in which the sound speed of φ is smaller than unity, $c_s^2 < 1$. In this case, the scalar field can develop density fluctuations on sub-horizon scales, $\rho_\varphi = \bar{\rho}_\varphi + \delta\rho_\varphi$, and so, in addition to the contribution of $\bar{\rho}_\varphi$ to the background Eqs. (6)-(8), the Poisson equation is augmented with the contribution from $\delta\rho_\varphi$

$$\nabla^2\Phi = 4\pi G\left[\delta\rho_m + \delta\rho_\varphi\right]. \tag{18}$$

Is it then reasonable to dub such a scenario modified gravity? Some will argue that not quite. The extra force felt by the particles is purely due to the gravitational influence of φ, which is still GR-like, i.e., $\nabla^2\Phi_m = 4\pi G\delta\rho_m$ and $\nabla^2\Phi_\varphi = 4\pi G\delta\rho_\varphi$, where Φ_m and Φ_φ represent the part of the potential that is due to the matter and the scalar fields, respectively ($\Phi = \Phi_m + \Phi_\varphi$). In the literature, authors call these models *clustering dark energy* and in them it is said that the scalar field is minimally coupled to the metric. More practically, this means that terms involving φ in the action couple only to the metric field via the $\sqrt{-g}$ term.

The final definition that we list here (although certainly not the last one can think of) is

C) *"A modified gravity model is any model where additional degrees of freedom couple nonminimally to the metric."*

An example of an action that complies with this definition is

$$
\begin{aligned}
S &= \int \mathrm{d}^4x\sqrt{-g}\left[\frac{f(\varphi)R}{16\pi G} - \mathcal{L}_m(\psi, g_{\mu\nu}) - \mathcal{L}_\varphi(\varphi)\right] && (19)\\
&\Longrightarrow \quad [1+f(\varphi)]\, G_{\mu\nu} + ... = 8\pi G\left[T^m_{\mu\nu} + T^\varphi_{\mu\nu}\right], && (20)
\end{aligned}
$$

where the dots represent other terms that we do not have to consider now (we shall look at a concrete example below). In these equations, the product $f(\varphi)R$ effectively couples φ with $g_{\mu\nu}$ and its derivatives (that are "inside" R). In such a model, the Poisson equation would look like

$$
(1+f(\varphi))\,\nabla^2\Phi + ... = 4\pi G\left(\delta\rho_m + \delta\rho_\varphi\right), \tag{21}
$$

which is manifestly non-GR, i.e., the total energy density fluctuation (which may include a clustering dark energy component) determines Φ via an equation that contains extra curvature terms, compared to Eq. (9). In this model, the geodesic equation remains structurally as in Eq. (11), just with a modified potential Φ. Explicitly, we can write (assuming $\Phi = \Psi$)

$$
\dot{v}_i + Hv_i = -\frac{1}{a}\nabla_i\Psi_{\rm GR} - \frac{1}{a}\nabla_i\Delta\Psi, \tag{22}
$$

where $\Psi = \Psi_{\rm GR} + \Delta\Psi$, with $\Delta\Psi$ being the correction to the potential that is induced by the coupling to φ. At this point we could think that we have arrived at our desired definition, but there are more complications. Below we shall see that the above action is mathematically equivalent to the following one

$$
\begin{aligned}
S &= \int \mathrm{d}^4x\sqrt{-g}\left[\frac{R}{16\pi G} - \mathcal{L}_m(\psi, g_{\mu\nu}, \varphi) - \mathcal{L}_\varphi(\varphi)\right] && (23)\\
&\Longrightarrow \quad G_{\mu\nu} = 8\pi G\left[T^m_{\mu\nu} + T^\varphi_{\mu\nu}\right]. && (24)
\end{aligned}
$$

In this case, the gravitational part of the action is as in GR, but note that φ appears coupled to the matter field (φ is an argument of $\mathcal{L}_m$). The Poisson equation is given by

$$
\nabla^2\Phi_{\rm GR} = 4\pi G\left(\delta\rho_m + \delta\rho_\varphi\right), \tag{25}
$$

which is GR like. However, the geodesic equation gets an extra term on the right-hand side, by virtue of the explicit coupling of the scalar field to matter:

$$
\dot{v}_i + Hv_i = -\frac{1}{a}\nabla_i\Psi_{\rm GR} + \text{fifth force}. \tag{26}
$$

The fifth force term matches the $\Delta\Psi$ term in Eq. (22), since the two theories are equivalent. In other words, the coupling to matter in Eq. (23) mimics the effects of the modifications to gravity in Eq. (19).

Can we then safely say that Eq. (19) is a modified gravity model? At first sight it looked like it, but a closer look will reveal (see below) that it is similar to a theory with GR plus some exotic interaction. In the context of large scale structure formation there are no clear answers to these questions and discussions around them end up being more philosophical than physical. Furthermore, if we interpret observations considering that matter is the only clumpy energy component, then the effects of clustering dark energy models may also be confused with those of coupled scalar fields. For the time being, most of the community has been dubbing any extra term that enters the standard Poisson equation (and therefore the geodesic equation) as a fifth force and progressing with the following plan: *"Lets first determine if there is observational evidence for a fifth force, and worry about its origins only after!"*

4 Example models

In this section, we introduce a few theories of modified gravity that are particularly popular in cosmological studies. This is not meant to be a thorough review of the theory space of modified gravity. Instead, we will introduce only the main representers of two popular types of screening mechanisms (screening mechanisms will be addressed in the next section).

4.1 Scalar-tensor theories

The so-called scalar-tensor theories are perhaps the most well studied models of modified gravity. The action can generically be written as

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{\varphi R}{16\pi G} - \frac{w(\varphi)}{\varphi}\nabla_\mu\varphi\nabla^\mu\varphi - 2U(\varphi) - \mathcal{L}_m(\psi, g_{\mu\nu})\right]. \tag{27}$$

The metric equation, the equation of the scalar field and the conservation equation are given, respectively, as

$$\varphi G_{\mu\nu} + \left[\Box\varphi + \frac{1}{2}\frac{w}{\varphi}\nabla_\mu\varphi\nabla^\mu\varphi + U\right]g_{\mu\nu} - \nabla_\mu\nabla_\nu\varphi - \frac{w}{\varphi}\nabla_\mu\varphi\nabla_\nu\varphi = 8\pi G T^m_{\mu\nu}, \tag{28}$$

$$(2w+3)\Box\varphi + \frac{\mathrm{d}w}{\mathrm{d}\varphi}\nabla_\mu\varphi\nabla^\mu\varphi + 4U - 2\varphi\frac{\mathrm{d}U}{\mathrm{d}\varphi} = 8\pi G T, \tag{29}$$

$$\nabla_\mu T^{\mu\nu} = 0, \tag{30}$$

where $T \equiv T^\alpha_\alpha$ is the trace of the energy-momentum tensor. In the previous section, we have dubbed these equations as being manifestly non-GR because the scalar field φ couples directly to the Einstein tensor, $G_{\mu\nu}$. We had also anticipated that these theories are equivalent to models with GR as the theory of gravity. Next, we make this statement more precise.

The jargon is that *scalar-tensor theories are conformally equivalent to GR*, which means that there is a way to transform the metric so that Eq. (27) becomes GR-like. Such a transformation is called a conformal transformation

$$g_{\mu\nu} = A^2(\varphi)\tilde{g}_{\mu\nu}, \tag{31}$$

with $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ being two metrics that go by the names of Jordan frame and Einstein frame metrics, respectively. The above metric transformation induces transformations in other tensor quantities constructed from $g_{\mu\nu}$. Some transformations that will become useful below are

$$g^{\mu\nu} = A^{-2}\tilde{g}^{\mu\nu}, \tag{32}$$

$$\sqrt{-g} = A^4(\varphi)\sqrt{-\tilde{g}}, \tag{33}$$

$$R = A^{-2}\left[\tilde{R} - 6\tilde{\nabla}_\mu \ln A\tilde{\nabla}^\mu \ln A\right], \tag{34}$$

$$T_{\mu\nu} = A^{-2}\tilde{T}_{\mu\nu} \Leftrightarrow T^{\mu\nu} = A^{-6}\tilde{T}^{\mu\nu}, \tag{35}$$

where the tildes on top of derivative or curvature tensors indicate that they are defined w.r.t. the Einstein frame metric. The first term in Eq. (27) transforms as

$$\begin{aligned}\sqrt{-g}\varphi R &\longrightarrow \varphi A^2\sqrt{-\tilde{g}}\tilde{R} - 6\varphi A^2\sqrt{-\tilde{g}}\tilde{\nabla}_\mu \ln A\tilde{\nabla}^\mu \ln A \\ &= \sqrt{-\tilde{g}}\tilde{R} - 6\sqrt{-\tilde{g}}\tilde{\nabla}_\mu \ln A\tilde{\nabla}^\mu \ln A, \quad \text{for} \quad A^2 = 1/\varphi,\end{aligned} \tag{36}$$

that is, if we set $A = \varphi^{-1/2}$, then the coupling to $\sqrt{-\tilde{g}}\tilde{R}$ disappears. Proceeding similarly for the remaining terms in Eq. (27), one finds that the action can be written as

$$S = \int \mathrm{d}^4x\sqrt{-\tilde{g}}\left[\frac{\tilde{R}}{16\pi G} - \left[6 + 4A^2 w\right]\tilde{\nabla}_\mu \ln A\tilde{\nabla}^\mu \ln A - 2A^4 U - \mathcal{L}_m(\psi, A^2(\varphi)\tilde{g}_{\mu\nu})\right]. \tag{37}$$

We could stop here to make the point we want to make, but lets be a bit pickier to relate a scalar field ϕ to φ implicitly as

$$\frac{\mathrm{d}\phi}{\mathrm{dln}A} = \left(12 + 8A^2 w\right)^{1/2} = \Gamma, \tag{38}$$

and introduce the potential $V(\phi) = 2A^4 U$ to write Eq. (37) as

$$S = \int \mathrm{d}^4x\sqrt{-\tilde{g}}\left[\frac{\tilde{R}}{16\pi G} - \frac{1}{2}\tilde{\nabla}_\mu\phi\tilde{\nabla}^\mu\phi - V(\phi) - \mathcal{L}_m(\psi, A^2(\phi)\tilde{g}_{\mu\nu})\right]. \tag{39}$$

This action is analogous to that of a Quintessence field (cf. Eq. (14)), just with one very important difference: the matter Lagrangian is defined w.r.t. the Jordan frame metric, not the Einstein frame one, as the rest of the action. In other words, the scalar field ϕ is coupled to matter with a coupling function $A(\phi)$. In the literature, these theories are called *coupled scalar field models.* The metric equation, the scalar field equation and the conservation equation are given by

$$G_{\mu\nu} = 8\pi G\left[\tilde{T}_{\mu\nu} + \tilde{\nabla}_\mu\phi\tilde{\nabla}_\nu\phi - \left(\frac{1}{2}\tilde{\nabla}_\alpha\phi\tilde{\nabla}^\alpha\phi + V(\phi)\right)\tilde{g}_{\mu\nu}\right], \tag{40}$$

$$\tilde{\Box}\phi - \frac{\mathrm{d}V}{\mathrm{d}\phi} = -\Gamma\tilde{T}, \tag{41}$$

$$\nabla_\mu T^{\mu\nu} = 0 \Leftrightarrow \nabla_\mu A^{-3}\tilde{T}^{\mu\nu} = 0 \Leftrightarrow ... \Leftrightarrow \tilde{\nabla}_\mu\tilde{T}^{\mu\nu} = \Gamma\tilde{T}\tilde{\nabla}^\nu\phi. \tag{42}$$

This tells us that the metric field equations are like in GR, but the scalar field equation and the conservation equation show that the scalar field is directly coupled to matter. In particular, Eq. (42) implies that the energy-momentum tensor of matter is not conserved and the resulting geodesic equation ($\nu = i$ component) yields a fifth force term that is proportional to the spatial gradient of the scalar field, $\propto \nabla^i\phi$.

To sum up, in the Jordan frame, matter is covariantly conserved but gravity is non-GR; whereas in the Einstein frame, gravity is GR, but by virtue of the coupling to the scalar field, matter is not conserved and it experiences a fifth force mediated by ϕ. At the end of the day, for studies of structure formation it makes little difference whether one works in the Jordan or in the Einstein frames – this choice is most of the times just a matter of analytical convencience[1].

4.2 The Horndeski/Galileon machinery

Over recent years, the so-called Horndeski/Galileon models have been gathering considerable attention in the community. The action of the Horndeski model is

$$S = \int \mathrm{d}x^4\sqrt{-g}\left[\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5 - \mathcal{L}_m(\psi, g_{\mu\nu})\right], \tag{43}$$

with

$$\begin{aligned}
\mathcal{L}_2 &= G_2(\varphi, X), \\
\mathcal{L}_3 &= -G_3(\varphi, X)\Box\varphi \\
\mathcal{L}_4 &= -G_4(\varphi, X)R + \frac{\mathrm{d}}{\mathrm{dX}}G_4(\varphi, X)\left[(\Box\varphi)^2 - (\nabla_\mu\nabla_\nu\varphi)(\nabla^\mu\nabla^\nu\varphi)\right] \\
\mathcal{L}_5 &= G_5(\varphi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\varphi \\
&\quad -\frac{1}{6}\frac{\mathrm{d}}{\mathrm{dX}}G_5(\varphi, X)\left[(\Box\varphi)^3 - 3\Box\varphi(\nabla_\mu\nabla_\nu\varphi)(\nabla^\mu\nabla^\nu\varphi) + 2(\nabla_\mu\nabla^\nu\varphi)(\nabla_\nu\nabla^\rho\varphi)(\nabla_\rho\nabla^\mu\varphi)\right],
\end{aligned} \tag{44}$$

[1]Note, however, that certain physical processes can be different in between these two frames and therefore care must be taken when interpreting some observations. For instance, in the Einstein frame, the scalar field can induce a time variation of the mass of the matter particles, which must be taken into account when analysing supernovae light curves at different redshifts (to give an example). This shows that these two frames are actually not completely equivalent.

where G_2, G_3, G_4 and G_5 are functions of φ and $X = -\nabla_\mu\varphi\nabla^\mu\varphi/2$. This four-dimensional action represents all single scalar field models whose equations of motion are kept up to second order in derivatives of the metric and of the scalar fields. As such, this action is general enough to encompass scalar-tensor models, although it is usually invoked to study the phenomenology of derivative couplings of φ. A particularly interesting limit of the Horndeski action is the so-called Covariant Galileon model, which corresponds to the choices

$$G_2 = -c_2 X, \quad G_3 = \frac{2c_3}{M^3}X, \quad G_4 = -\left[\frac{1}{16\pi G} + \frac{c_4}{M^6}X^2\right], \quad G_5 = \frac{3c_5}{M^9}X^2, \tag{45}$$

where c_2, c_3, c_4, c_5 are real constants and M is a mass scale. What is special to this Galileon model is that it enjoys a Galilean symmetry (hence the name) in flat spacetime, i.e., the resulting equations of motion are invariant under shifts of the scalar field derivatives $\partial_\mu\varphi \to \partial_\mu\varphi + b_\mu$ (where b_μ is a constant four-vector). And what is special about this Galileon symmetry? The answer relates more to particle physicists than astrophysicists/cosmologists, but it is related to the fact that the structure of the Galileon model Lagrangians is such that their classical solutions receive no quantum corrections to any loop order in perturbation field theory, i.e., the theory is non-renormalizable. In other words, the classical solutions will not be ruined by some quantum completion of the theory.

For simplicity, lets fix $c_4 = c_5 = 0$, which corresponds to the so-called Cubic Galileon model. In this case, the metric equation is given by

$$\begin{aligned}
G_{\mu\nu} &= 8\pi G\left[T_{\mu\nu} + T^{c_2}_{\mu\nu} + T^{c_3}_{\mu\nu}\right], \\
T^{c_2}_{\mu\nu} &= c_2\left[\nabla_\mu\varphi\nabla_\nu\varphi - \frac{1}{2}g_{\mu\nu}\nabla^\alpha\varphi\nabla_\alpha\varphi\right], \\
T^{c_3}_{\mu\nu} &= \frac{c_3}{M^3}\left[2\nabla_\mu\varphi\nabla_\nu\varphi\Box\varphi + 2g_{\mu\nu}\nabla_\alpha\varphi\nabla_\beta\varphi\nabla^\alpha\nabla^\beta\varphi - 4\nabla^\lambda\varphi\nabla_{(\mu}\varphi\nabla_{\nu)}\nabla_\lambda\varphi\right],
\end{aligned} \tag{46}$$

the equation of the Galileon field is

$$0 = c_2\Box\varphi + 2\frac{c_3}{M^3}\left[(\Box\varphi)^2 - \nabla^\alpha\nabla^\beta\varphi\nabla_\alpha\nabla_\beta\varphi - R_{\alpha\beta}\nabla^\alpha\varphi\nabla^\beta\varphi\right], \tag{47}$$

and the conservation equation is as in GR, because there is no explicit coupling between matter and the scalar field (this is therefore the Jordan frame). The presence of $R_{\mu\nu}$ in Eq. (47) justifies calling this a modified gravity model. The Friedmann equation can be written as

$$H^2 = H_0^2\left[\Omega_{m0}a^{-3} + \sqrt{\Omega_{m0}^2 a^{-6} + 4\left(1 - \Omega_{m0}\right)}\right], \tag{48}$$

(where Ω_{m0} is the fractional matter density today) and the equations that are relevant for structure formation on sub-horizon scales are given by

$$\nabla^2\Phi = 4\pi G\delta\rho_m - 8\pi G\frac{c_3}{M^3}\dot{\varphi}^2\nabla^2\varphi, \tag{49}$$

$$\Psi = \Phi, \tag{50}$$

$$\nabla^2\varphi + \frac{1}{\beta_1(a)}\left[(\nabla^2\varphi)^2 - (\nabla_i\nabla_j\varphi)(\nabla^i\nabla^j\varphi)\right] = \frac{8\pi G}{\beta_2(a)}\delta\rho_m, \tag{51}$$

$$\dot{v}_i + Hv_i = -\frac{1}{a}\nabla_i\Psi, \tag{52}$$

where β_1, β_2 are two functions of time that we do not write here for brevity (note that these are not free functions, they are fixed by the structure of the Lagrangian). These equations tell us that the matter perturbations source the profile of the Galileon field (cf. Eq. (51)), which in turn enters as an extra term in the Poisson equation, Eq. (49). This extra term forcibly leads to potentials that are not the same as in GR (for the same $\delta\rho_m$), which can be represented by an extra (fifth force) term on the right-hand side of the geodesic equation.

4.3 Braneworld scenarios

The idea of exploring higher dimensional spacetimes is another possible way of going beyond GR. Perhaps the simplest and most popular realization of this idea is that of the braneworld Dvali-Gabadadze-Porrati (DGP) model. In this theory, standard GR and the known matter fields are defined on a four-dimensional brane that is embedded in a five-dimensional bulk spacetime containing a five-dimensional generalization of GR. The action of the model can be written as

$$S = \int_{\mathrm{brane}} \mathrm{d}^4x\sqrt{-g}\left(\frac{R}{16\pi G} - \mathcal{L}_m(\psi, g_{\mu\nu})\right) + \int_{\mathrm{bulk}} \mathrm{d}^5x\sqrt{-g^{(5)}}\left(\frac{R^{(5)}}{16\pi G^{(5)}}\right) \tag{53}$$

where the superscript $^{(5)}$ indicates quantities defined w.r.t. the five-dimensional metric, $g^{(5)}_{\mu\nu}$. Note that the matter field ψ belongs to the four-dimensional part of the model. The ratio of the two gravitational strengths, $G^{(5)}$ and G, is a parameter of the model known as the crossover scale, r_c,

$$r_c = \frac{1}{2}\frac{G^{(5)}}{G}. \tag{54}$$

The expansion rate in this model is given by

$$H(a) = H_0\sqrt{\Omega_{m0}a^{-3} + \Omega_{rc}} \pm \sqrt{\Omega_{rc}}, \tag{55}$$

where $\Omega_{rc} = 1/(4H_0^2 r_c^2)$. This model has two branches of solutions, characterized by the sign of the second term on the right-hand side in Eq. (55). The so-called *self-accelerating branch* (sDGP), which corresponds to the $(+)$ sign, is particularly appealing as it allows for solutions in which the acceleration of the Universe arises without adding any pure dark energy component. However, this branch is known to be plagued by ghost problems (degrees of freedom without a well defined minimum energy state). In addition to these theoretical instabilities, the self-accelerating branch is in severe tension with CMB and supernovae data. For these reasons, most of the cosmological studies of DGP gravity have focused on the so-called *normal branch*, nDGP, which is characterized by the $(-)$ sign. This branch requires the addition of an explicit dark energy component, ρ_{de}, on the brane, which appears as $\rho_{de}(a)$ inside the square root in the first term on the right-hand side of Eq. (55).

Being a five dimensional model, the derivation of the relevant equations for structure formation from the action are more involved. In particular, although it is not apparent from the action, there is a scalar degree of freedom φ in this model that is related to the geometry of the spacetime. In these notes, we will not look into these issues with detail and will write down the final result. In the DGP model, the equations that govern structure formation on the brane are given by

$$\nabla^2\Psi = 4\pi G a^2\delta\rho_m + \frac{1}{2}\nabla^2\varphi, \tag{56}$$

$$\Psi = \Phi + \varphi, \tag{57}$$

$$\nabla^2\varphi + \frac{r_c^2}{3\beta(a)a^2}\left[\left(\nabla^2\varphi\right)^2 - \left(\nabla_i\nabla_j\varphi\right)\left(\nabla^i\nabla^j\varphi\right)\right] = \frac{8\pi G}{3\beta(a)}a^2\delta\rho_m, \tag{58}$$

and the conservation equation in this model is as in GR. This set of equations has the same structural form (in terms of scalar field derivatives) as those of the Cubic Galileon model, and therefore they share similar phenomenology[2]. In fact, historically, the interest in Galileon models arose as an attempt to generalize the phenomenology of the DGP model.

[2] Their different relations bebtween Φ and Ψ are very important though for the definition of the lensing potential, $\Phi_{\mathrm{len}} = (\Phi + \Psi)/2$. In particular, for the Galileon model, $\Phi_{\mathrm{len}} = (\Phi + \Psi)/2 = \Psi = \Phi$. That is, the lensing potential is modified in the same way as the dynamical potential. For the DGP model, we can write $\Psi = \Psi_{\mathrm{GR}} + \varphi/2$, and hence, $\Phi_{\mathrm{len}} = (\Psi + \Phi)/2 = \Psi_{\mathrm{GR}}$. That is, in the DGP model, the lensing potential is the same as in GR.

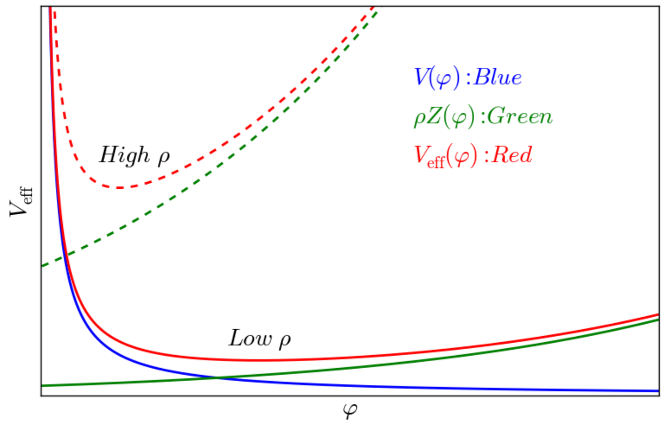

Figure 2: Schematic representation of how the effective potential in Chameleon models reacts to the matter density, $V_{\rm eff}(\varphi) = V(\varphi) + \rho Z(\varphi)$. In the density is low, the effective potential around the minimum has a weak curvature, i.e., the scalar field is light. On the other hand, by increasing the density, the minimum of $V_{\rm eff}$ gets sharper and the scalar field more massive.

5 Screening mechanisms

The idea of modifying the gravitational law inevitably leads to the question of how such modifications can be made compatible with the stringent SS tests. This reconciliation is typically achieved via screening mechanisms which dynamically suppress the modifications to gravity in regions like the SS. Currently, there is a variety of screening mechanisms in the literature which include the Chameleon, Symmetron, Dilaton, Vainshtein and K-mouflage type screenings. In all of these, the implementation of the screening effects relies on the presence of nonlinear terms in the equations of motion, which act to suppress the size of the fifth force in regions where some criterion is met. This criterion typically involves the size of the gravitational potential or of its derivatives, which tend to be higher on smaller length scales than on larger scales[3].

We note in passing that another way to reconcile the SS tests with sizeable fifth forces is to confine them to act only within the dark sector of the Universe, e.g., an interaction that would only be felt by dark matter. In this way, baryons would be unaffected and the SS contraints would be satisfied. In these notes, we focus only on models where the fifth force effects are felt universally by all matter species.

Next, we introduce and briefly describe two of the main types of screening out there: the Chameleon and Vainshtein mechanisms.

5.1 The Chameleon screening mechanism

The Chameleon mechanism is a screening effect that operates in scalar-tensors theories, or in the conformally equivalent coupled scalar field scenarios. To understand the main aspects of this screening mechanism it is best to work in the Einstein frame. From Eq. (42), we get that the scalar field mediates a force that is proportional to the gradient of the scalar field, $F_{\rm 5th} \propto \vec{\nabla}\varphi$. The behavior of the scalar field is set by Eq. (41), which we write here as

$$\nabla^2\varphi = V(\varphi)_{,\varphi} + \rho Z(\varphi)_{,\varphi} \equiv V_{\rm eff}{}_{,\varphi}\,, \tag{59}$$

[3]In the literature, one often encounters this criterion being described as the amplitude of the matter density fluctuations. This is done perhaps to make the explanations simpler, but is inaccurate. For instance, the matter density perturbation between the Sun and the Earth is small (compared to that of each object), but one still requires the fifth force to be screened there. A more correct description would be made in terms of the gravitational potential (or its gradients), which can be sizeable even if there is a vacuum between two massive bodies.

with $_{,\varphi} \equiv \mathrm{d}/\mathrm{d}\varphi$, and where we have assumed a pressureless matter fluid and $\dot{\varphi} = 0$. The last equality serves to illustrate that the behavior of the scalar field depends on an effective potential $V_{\rm eff} = V + \rho Z$, which is density dependent. The key point behind the idea of the Chameleon mechanism lies in adjusting the functions V and Z such that the scalar field acquires a large mass m_φ in high-density regions. The more massive the scalar field, then the smaller the Compton wavelenght $1/m_\varphi$ within which it can mediate a fifth force, and hence, the more chances it gets to evade Solar System bounds.

For the sake of illustration, consider a compact spherical object of size R and constant density $\rho_{\rm in}$, embedded in an ambient density $\rho_{\rm out}$. One can think of this as a golf ball embedded in the Earth's atmosphere, or the Earth embedded in the Milky Way halo, or the Milky Way halo embedded in the Local Group, etc. The idea now is to choose the functions $V(\varphi)$ and $Z(\varphi)$ so that $V_{\rm eff}$ has a minimum at $\varphi = \varphi_{\rm min}$. The typical choices are $V \propto \varphi^{-n}$ and $Z(\varphi) = e^{\beta\varphi/M_{\rm Pl}}$, where $M_{\rm Pl}$ is the reduced Planck mass, n is a positive integer and $\beta > 0$ measures the strength of the coupling between the scalar field and matter. Then, lets assume that the scalar field "sits" around the minimum of the effective potential. In this case, $V_{\rm eff}$ can be approximated as quadratic with $m_\varphi^2 = V_{\rm eff}{}_{,\varphi\varphi}$. Figure 2 illustrates how m_φ depends on ρ. In particular, the larger the density ρ, then the smaller the value of $\varphi_{\rm min}$ and the larger the mass m_φ. Here, we shall skip the details of the derivation and analyse only the final solution to this problem. If

$$\frac{\Delta R}{R} \approx \frac{\varphi_{\rm min,out} - \varphi_{\rm min,in}}{6\beta M_{\rm Pl}\Phi} \equiv \frac{\Delta\varphi}{6\beta M_{\rm Pl}|\Phi|} \ll 1, \tag{60}$$

then it can be shown that the scalar field profile outside of the spherical overdensity can be approximated as

$$\varphi(r) \approx \varphi_{\rm min,out} - \left(\frac{3\beta}{4\pi M_{\rm Pl}}\right)\left(\frac{\Delta R}{R}\right) M \frac{e^{-m_{\rm out}(r-R)}}{r}. \tag{61}$$

In the above two equations, $\varphi_{\rm min,in}$ and $\varphi_{\rm min,out}$ are, respectively, the values of the field at the minimum of the potential for $\rho = \rho_{\rm in}$ and $\rho = \rho_{\rm out}$, $m_{\rm out}$ is the mass of the scalar field for $\rho = \rho_{\rm out}$, $M = 4\pi\rho_{\rm in}R^3/3$ is the mass of the spherical object and $\Phi = -GM/R$ is its gravitational potential. Equation (61) tells us that the force generated by φ is of the Yukawa type, but that only the matter inside a *thin shell* of width ΔR contributes to it. This is because deeper inside the overdensity, the mass of the scalar field, $m_{\rm in}$, is large and the spatial dependence of the scalar field becomes exponentially suppressed, $\propto e^{-m_{\rm in}r}$. Equation (61) is valid only if Eq. (60) holds, which depends on the ratio between $\Delta\varphi$ and $|\Phi|$. This indicates that the efficiency of the screening mechanism depends sensitively on an interplay between the properties of the source (which determine $\varphi_{\rm min,in}$ and Φ) and of its environment (which sets $\varphi_{\rm min,out}$). For fixed density contrast (fixed $\Delta\varphi$), the chameleon mechanism works if the gravitational potential of the object is sufficiently deep. If the condition of Eq. (60) is not met, then the scalar field profile is given by

$$\varphi(r) \approx \varphi_{\rm min,out} - \left(\frac{\beta}{4\pi M_{\rm Pl}}\right) M \frac{e^{-m_{\rm out}(r-R)}}{r}, \tag{62}$$

in which there are no suppression effects and, therefore, the fifth force effects are noticeable.

5.2 The Vainshtein screening mechanism

The Vainshtein screening mechanism relies on the presence of nonlinear derivative terms in the equations of motion of the scalar field, and hence it is at play in models like DGP and Galileon gravity. We shall use the equations of the DGP model. From Eq. (56), we have that the total potential can be writen as

$$\Psi = \Psi_{\rm GR} + \varphi/2, \tag{63}$$

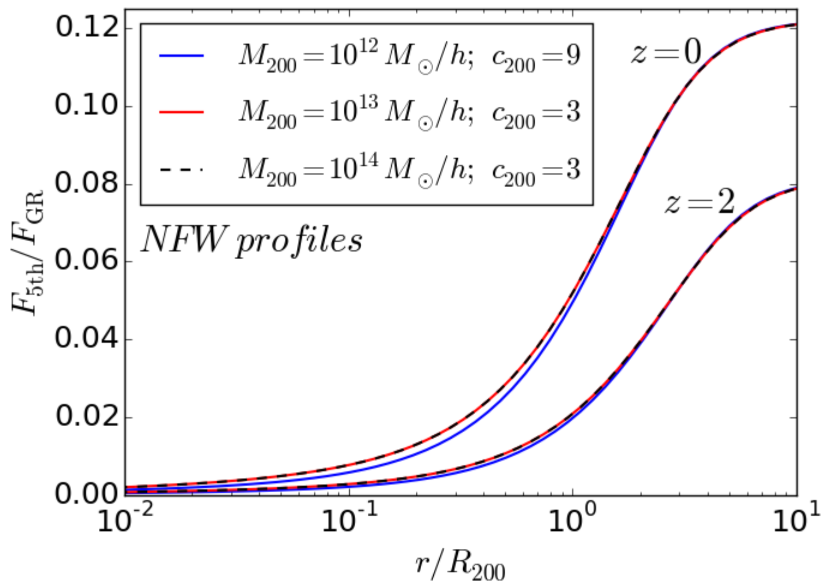

Figure 3: Fifth force to normal gravity ratio for NFW halo profiles in the nDGP model at $z = 0$ and $z = 2$ and for different values of M_{200} and c_{200}, as labelled.

where $\nabla^2\Psi_{\mathrm{GR}} = 4\pi G\delta\rho_m$. Hence, the total force can be split as well into a contribution coming from the standard term, plus a fifth force $F_{5\mathrm{th}} = \vec{\nabla}\varphi/2$. To allow for an analytical illustration of the Vainshtein screening effect, let us work assuming spherical symmetry. In this case, Eq. (58) becomes

$$\frac{1}{r^2}\left(r^2\varphi_{,r}\right)_{,r} + \frac{2r_c^2}{3\beta}\frac{1}{r^2}\left(r\varphi_{,r}^2\right)_{,r} - \frac{8\pi G}{3\beta}\delta\rho_m = 0, \tag{64}$$

where $_{,r} \equiv \mathrm{d}/\mathrm{d}r$. Integrating this equation as $\int r^2\mathrm{d}r$, and then dividing by r^3 we get

$$\left(\frac{\varphi_{,r}}{r}\right) + \frac{2r_c^2}{3\beta}\left(\frac{\varphi_{,r}}{r}\right)^2 - \frac{2G}{3\beta}\frac{M(<r)}{r^3} = 0 \tag{65}$$

where $M(<r) = 4\pi\int_0^r \xi^2\delta\rho_m(\xi)\mathrm{d}\xi$ is the mass enclosed by radius r. This is an algebraic equation for $\varphi_{,r}/r$, which can be solved to yield

$$\varphi_{,r} = \frac{4}{3\beta}\left(\frac{r}{r_V}\right)^3\left[-1 + \sqrt{1 + \left(\frac{r_V}{r}\right)^3}\right]\frac{GM(<r)}{r^2}, \tag{66}$$

$$\Longrightarrow F_{5\mathrm{th}} = \frac{2}{3\beta}\left(\frac{r}{r_V}\right)^3\left[-1 + \sqrt{1 + \left(\frac{r_V}{r}\right)^3}\right]F_{\mathrm{GR}}, \tag{67}$$

where F_{GR} is the standard Newtonian force and we have defined a distance scale r_V known as the *Vainshtein radius*, which is given by

$$r_V^3 = \frac{16r_c^2}{9\beta^2}GM(<r). \tag{68}$$

This Vainshtein radius sets the scale where the screening effects become important. If $r \ll r_V$, then

$$F_{5\mathrm{th}} \approx \frac{2}{3\beta}\left(\frac{r}{r_V}\right)^{3/2}F_{\mathrm{GR}} \ll F_{\mathrm{GR}}, \tag{69}$$

i.e., the fifth force is negligible compared to the standard term. On the other hand, for $r \gg r_V$

$$F_{5\mathrm{th}} \approx \frac{2}{3\beta}\left(\frac{r}{r_V}\right)^3\left[-1+1+\frac{1}{2}\left(\frac{r_V}{r}\right)^3\right] = \frac{1}{3\beta}F_{\mathrm{GR}}, \tag{70}$$

i.e., the fifth force gives a contribution that is comparable to F_{GR} ($\beta \sim \mathcal{O}(1)$ at late times).

An important difference to the Chameleon mechanism is that, in the Vainshtein case, there is no environmental dependence: the efficiency of the screening depends only on the properties of the source, in particular, its enclosed mass. For $r_c = H_0^{-1}$, a quick order of magnitude estimate yields that the Vainshtein radius of the Sun is of order a few kpc. This is indeed much larger that the size of the Sun, $\sim 10^{-14}$kpc, or the distance of the Earth to the Sun, $\sim 10^{-9}$kpc, which is why these models can meet the stringent Solar System bounds.

In a more cosmological setup, we can look at the profile of the ratio $F_{5\mathrm{th}}/F_{\mathrm{GR}}$ for NFW haloes

$$\rho_{\mathrm{NFW}} = \frac{\rho_s}{(r/r_s)\,(1+r/r_s)^2}, \tag{71}$$

$$M_{\mathrm{NFW}}(<r) = 4\pi\rho_s r_s^3\left[\ln\left(1+r/r_s\right) - \frac{r/r_s}{1+r/r_s}\right], \tag{72}$$

where the parameters ρ_s and r_s relate to the halo mass M_{200} and halo concentration c_{200} as

$$\begin{aligned} \rho_s &= \frac{200}{3}\rho_c c_{200}^3\left[\ln(1+c_{200}) - \frac{c_{200}}{1+c_{200}}\right]^{-1}, \\ r_s &= R_{200}/c_{200}. \end{aligned} \tag{73}$$

The result is shown in Fig. 3. The radial coordinate is scaled by R_{200},

$$R_{200} = \left(\frac{3M_{200}}{4\pi 200\rho_c(z)}\right)^{-3}, \tag{74}$$

which is the radius above which the mean density of the halo drops below 200 times the critical matter density at a given redshift. The result is shown for three combinations of M_{200} and c_{200} at $z=0$ and $z=2$, as labelled. The figure shows that the suppresional effects of the screening mechanism start to become important on scales $\lesssim 10R_{200}$. Moreover, the figure also illustrates that the fifth force to normal gravity ratio is independent of M_{200} when plotted as a function of r/R_{200} (this can be checked by noting that Eq. (66) depends on the combination r_V^3/r^3, which depends only on r/R_{200} and c_{200} for NFW halos). This is why the dashed black and solid red curves are overlapping in Fig. 3. The concentration has a slight impact on the ratio $F_{5\mathrm{th}}/F_{\mathrm{GR}}$. Namely, the higher c_{200}, the higher $M(r)$, and hence r_V, which strengthens the screening mechanism (this is why the blue curves have a lower amplitude, although barely noticeable).

6 Looking out for modified gravity

In this section, we comment briefly on a number of typical cosmological signatures that one can expect in modified gravity theories. To obtain some of these predictions one often has to recourse to numerics (Boltzmann-Einstein or N-body simulation codes), and as a result, in these notes there is little more we can aim for other than simply displaying some results and interpret them qualitatively.

6.1 Consistency tests with parametrized approaches

One approach to constraining modified gravity models consists of parametrizing the modifications to gravity at the level of the equations. The idea is then to conduct observational constraints on the free functions that enter into the parametrization, and the observational viability of several specific models

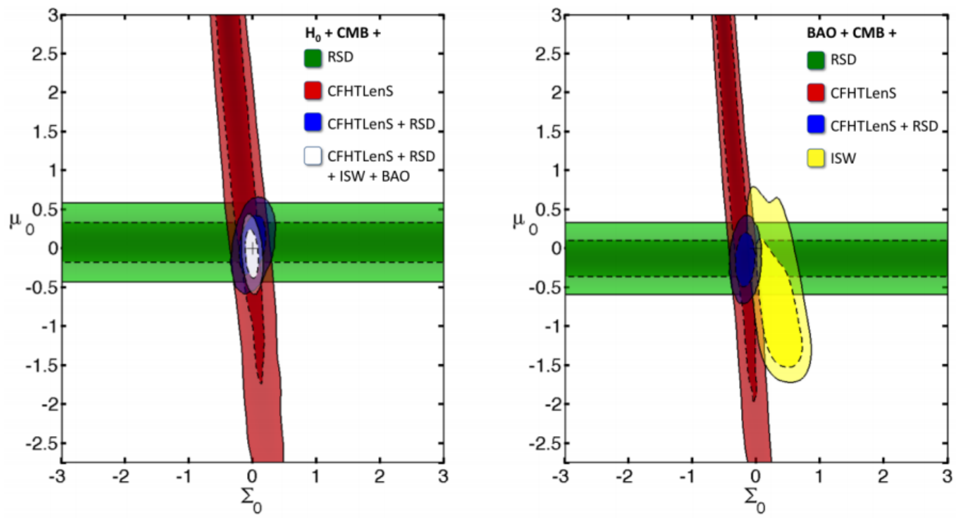

Figure 4: Constraints on the $\mu - \Sigma$ parametrization (assuming constant parameters) using different dataset combinations. Taken from Simpson et al, arXiv:1212.3339.

can be assessed by mapping their equations onto the parametrized framework. To give a simple example, one can parametrize the Poisson and the Slip equations as

$$\begin{aligned} \nabla^2\Phi &= 4\pi G\left[1+\mu(a,\vec{r})\right]\delta\rho_m, \\ \Psi+\Phi &= \left[1+\Sigma(a,\vec{r})\right](\Psi+\Phi)_{\rm GR}, \end{aligned} \tag{75}$$

where $\mu(a,\vec{r})$ and $\Sigma(a,\vec{r})$ are two free functions of time and space. The model-independent nature of the parametrized approach, however, comes at the price that the unspecified time- and space-dependence of the free functions renders them too general to be tightly constrained. Nevertheless, by making simplistic assumptions about the form of the free functions (e.g. that they are scale or time independent), the parametrized framework can prove very useful in identifying observational tensions. For sake of argument, imagine that some dataset combination constrains $\mu(a) \neq 0$ or $\Sigma(a) \neq 0$. Then this might alert us to a tension with ΛCDM. Moreover, the parametrized framework can be useful in determining which dataset combination is the best at breaking eventual degeneracies between the free functions. Figure 4 displays constraints on μ and Σ (assumed constant) from different datasets, which illustrates the types of analysis that can be performed within parametrized frameworks. Further steps could involve describing the free functions in a piecewise manner, in both time and scale, and treat the amplitude of each piece as a free parameter.

Another example of a parametrized framework that we mention only in passing is that of the *effective field theory* (EFT) formalism, in which the parametrization is made at the level of the action instead of the equations of motion. The idea consists in writting down the most general linearized action that satisfies some desired set of properties (e.g. second order field equations of motion). This is reminiscent of the Horndeski action, but made simpler by restricting the analysis to the linear regime. A particularly useful aspect of EFT is that it can provide guides for the time and scale dependence of the μ and Σ functions in Eqs. (75).

The parametrized frameworks are generically employed to place fairly model-independent constraints on modified gravity using data that is sensitive to the evolution of linear density fluctuations only. However, in the linear regime, the predictions of modified gravity models cannot be explored to their full extent because the scale-dependent effects of the screening mechanisms are not at play. In the nonlinear regime, however, the majority of the parametrized frameworks becomes less useful because the equations of the models become significantly more complicated. To study nonlinear structure formation in modified gravity, it becomes almost imperative to proceed on a model-by-model basis.

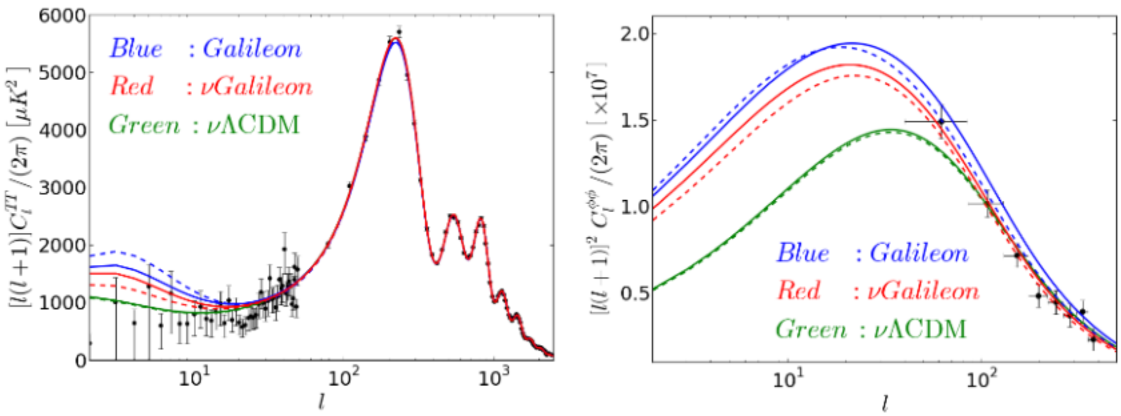

Figure 5: CMB temperature (left) and CMB lensing potential (right) power spectra for ΛCDM (green) and Cubic Galileon models (blue without massive neutrinos and red with massive neutrinos).

6.2 Signatures in the linear regime

- CMB data

The most powerful and reliable datasets in cosmology are CMB related ones, and as a result, it seems natural to first identify how can modified gravity affect these data. In terms of the temperature power spectrum, apart from eventual changes in the background cosmology (which are constrained by the angular position of the acoustic peaks), modified gravity models typically impact on the large scale amplitude of the power spectrum, which is dominated by the ISW effect. The latter is determined by the integrated effect of time variations of interveening lensing gravitational potentials, $\Phi_{\rm len} = (\Phi + \Psi)/2$, which can evolve differently in modified gravity, compared to GR. CMB measurements allow also to infer the power spectrum of the projected lensing potential, which can be affected by modified gravity. Figure 5 shows how the Cubic Galileon model impacts these two CMB observables. The right panel shows also that the modified gravity effects are degenerate with massive neutrino physics: the boosting effects of modified gravity can be compensated by the suppresional effects of massive neutrinos on structure formation.

- Growth rate of structure

On sub-horizon scales, the time evolution of a linear density constrast mode $\delta = \rho_m/\bar{\rho}_m - 1$ is governed by

$$\ddot{\delta_k} + 2H\dot{\delta_k} - 4\pi G\mu(t,k)\bar{\rho}_m\delta_k = 0. \tag{76}$$

This equation has the same well known structure as in GR, but we have inspired ourselves in Eqs. (75) to modify the source term of this equation to account for modified gravity. This equation tells us that if $\mu > 1$, then the amplitude of the linear fluctuations grows more rapidly, compared to GR. The growth rate, defined as $f = {\rm dln}\delta/{\rm dln}a$, can be determined observationally, most notably via galaxy redshift space distortion modelling. These types of measurements have matured quite substantially in recent years, which has triggered a lot of interest in using these data to test gravity on large scales.

- DM/Galaxy clustering

Figure 6 shows the relative difference of the nonlinear matter power spectrum in $f(R)$ models to ΛCDM. We did not showed this explicitly, but $f(R)$ theories are scalar-tensor theories (and therefore conformally equivalent to coupled scalar field models). A few takeaway points from the figure are that (i) even on scales $k \approx 0.1h$Mpc, the growth of the fluctuations in the $f(R)$ model is not scale-independent (noted by the fact that the relative diference is not flat there); (ii) mode coupling on smaller scales amplifies the boosting effects of modified gravity.

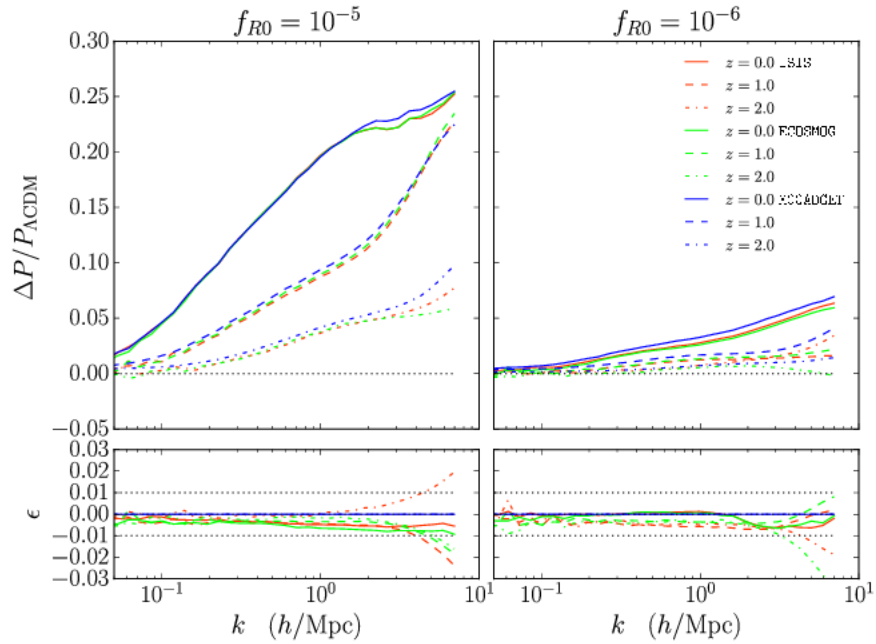

Figure 6: Relative difference of the nonlinear matter power spectrum in scalar-tensor $f(R)$ theories w.r.t. ΛCDM. The result is shown at different epochs and for three N-body codes (green, blue and red for `ECOSMOG`, `MGGadget` and `ISIS`, respectively). The lower panels show the relative difference of the different code results. Taken from Winther et al, arXiv:1506.06384.

The galaxy distribution traces that of the underlying dark matter field, and as a result, some of the effects depicted in Fig. 6 for dark matter are expected to be present also in the clustering of galaxies. However, some complications arise. In particular, galaxies trace the dark matter field with some bias and there are also redshift space distortion effects that contribute to differences between the observed clustering pattern of galaxies and the predicted dark matter power spectrum. In investigations of galaxy clustering in modified gravity, one should therefore expect the fifth forces to impact also on the way galaxies trace dark matter. This is not a problem *per se*, it is just that the analysis has to be more involved than just looking at the dark matter power spectrum in Fig. 6 (but this is true for standard GR studies anyway).

- *Cosmic shear*

A way to probe the total matter clustering (and try to circumvent the difficulties associated with galaxy bias) is to look at the power spectrum of the lensing convergence, κ, or lensing shear, γ, fields. These probe the total projected *effective density*[4] distribution between us and some source galaxies. If the typical redshift of the sources is around $z_s \approx 1$, then the lensing signal peaks at about $z_s \approx 0.5$. Compared to lensing of CMB photons, cosmic shear surveys therefore have better sensitivity to late-time structure formation, which is where modified gravity effects are typically stronger (in models that attempt to explain cosmic acceleration).

6.3 Signatures in the nonlinear regime

- *Halo counts*

It is intuitive to expect that the abundance of gravitationally bound structures changes if one changes the growth rate of structure formation. A quick way to obtain an estimate for the result is to rely on

[4] In GR, $\kappa \sim \int \nabla^2 \Phi_{\rm len} {\rm d}z \sim \int \delta\rho_m {\rm d}z$. We use the phrase "effective density" to remind ourselves that in modified gravity, in addition to $\delta\rho_m$, there might be other terms sourcing the lensing potential.

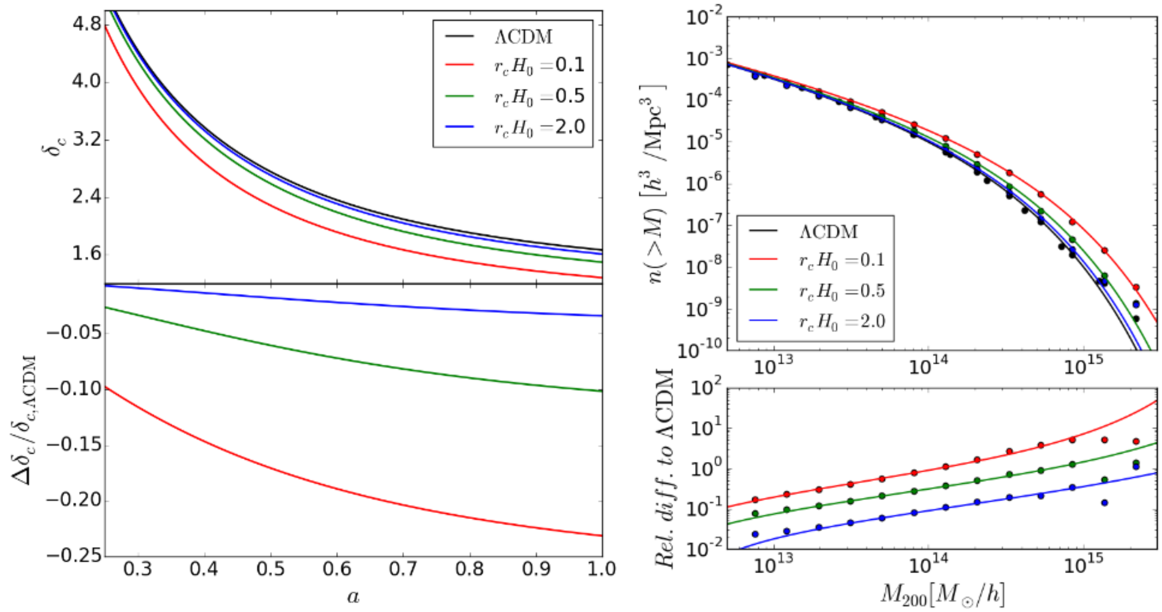

Figure 7: The left panel shows the critical overdensity for spherical collapse at redshift z, $\delta_c(z)$, for ΛCDM and three nDGP models. The right panel shows the corresponding cumulative halo mass functions obtained in the Sheth-Tormen formalism (solid lines) and from simulations (dots).

simple spherical collapse models. In the Sheth-Tormen formalism, the mass function is given by

$$\begin{aligned} \frac{\mathrm{d}n(M)}{\mathrm{dln}M} &= \frac{\bar{\rho}_m}{M} f(S) \frac{\mathrm{d}S}{\mathrm{dln}M}, \\ f(S) &= A\sqrt{\frac{q}{2\pi}} \frac{\delta_c}{S^{3/2}} \left[1 + \left(\frac{q\delta_c^2}{S}\right)^{-p}\right] \exp\left[-\frac{q\delta_c^2}{2S}\right], \end{aligned} \tag{77}$$

where A is determined by $\int f(S)\mathrm{d}S = 1$, $(q,p) \approx (0.7, 0.3)$, S is the variance of the linear matter power spectrum smoothed on the scales of the size of haloes with mass M and $\delta_c \equiv \delta_c(z)$ is the initial critical overdensity for collapse at redshift z. The value of δ_c is determined by solving the spherical collapse equation

$$\frac{\ddot{R}}{R} - \left(\dot{H} + H^2\right) = -\mu(a,\delta)\frac{H_0^2 \Omega_m \delta a^{-3}}{2}, \tag{78}$$

to find the initial conditions (i.e. value of δ_c) that lead to collapse at the desired redshift. The effects of modified gravity enter via eventual changes to the expansion rate, H, and via the $\mu(a,\delta)$ factor in the source term. The latter can now have a nontrivial dependence on the density to take screening effects into account. Figure 7 summarizes this exercise for the normal branch of the DGP model. The figure shows that the values of δ_c in the modified gravity model are lower than in ΛCDM (i.e. GR). This is expected in models where gravity gets stronger: if structure formation occurs faster, then the initial overdensities have to be smaller for the collapse to occur at the same time. As one would expect, the enhanced structure formation translates into a higher abundance of massive haloes, as shown in the right panel of Fig. 7. This occurs at the expense of a suppression in the number density of low mass haloes (that merge to form more massive haloes), although the scale of the figure does not allow to see this low-mass effect.

- *Dynamical vs. lensing mass estimates*

Recently, several authors have investigated the merits to test gravity on Mpc scales by using information from the galaxy velocity field in the infall regions around massive clusters. These techniques were designed for models of gravity that modify the dynamical potential, $\mu \neq 0$ in Eqs. (75), but do

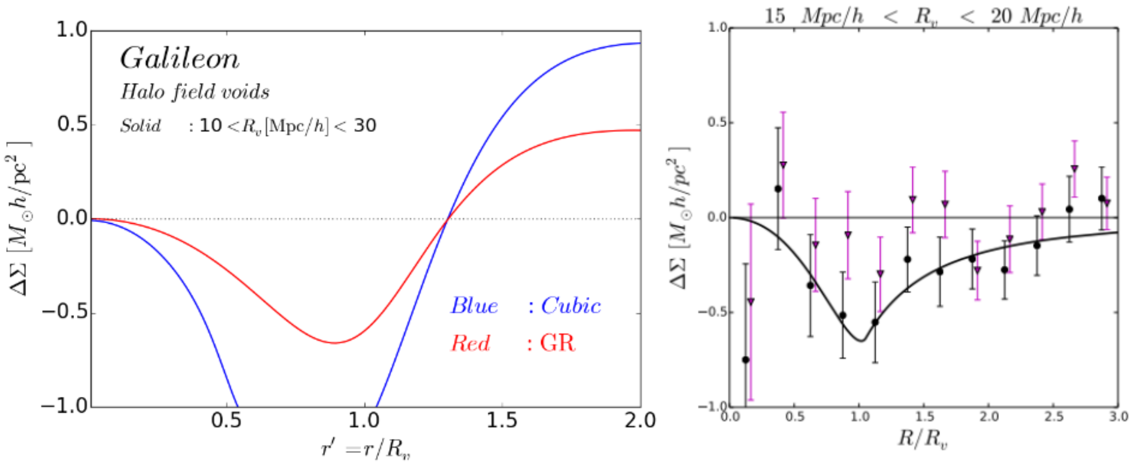

Figure 8: The left panel shows the lensing shear profiles for voids found in simulations of the Cubic Galileon model (taken from arXiv:1505.05809). The right panel shows measurements of the lensing signal associated with voids found in SDSS LRG galaxies (taken from Clampitt&Jain, arXiv:1404.1834). Although a proper comparison between the left and right panels requires extra modelling, the figure does help to show that the effects of modified gravity can be quite pronounced and are likely to be in tension with the data.

not modify the lensing potential $\Sigma = 0$ in Eqs. (75). Scalar-tensor theories and the DGP model fall in the above category, and as such, the lensing mass estimates, $M_{\rm len}$, for these models would automatically be the same as in GR. On the other hand, the velocity dispersion of surrounding galaxies as they fall towards the clusters would be affected by the modifications to the dynamical potential. Therefore, if one would interpret these observations assuming GR, then one would infer dynamical masses, $M_{\rm dyn}$, which are different from those estimated using lensing. In particular, if the dynamical force gets stronger, then the $M_{\rm dyn}$ estimates would be biased-high, relative to $M_{\rm len}$ – that is, one would infer a larger dynamical mass to compensate for the boosting effects of the fifth force, which are not being taken into account if we interpret the data assuming GR. A mismatch in the estimates of the lensing and dynamical masses would therefore be a smoking gun for modified gravity.

This argument is very powerful in theory, but as usual, in practice things tend to get more complicated. For instance, the above reasoning assumes that the galaxy velocity field that surrounds the clusters is solely determined by their mass, which is not necessarily true. Moreover, the merit of this test of gravity becomes less clear when applied to models that also modify the lensing potential, in which case both $M_{\rm dyn}$ and $M_{\rm len}$ can be different than in GR, but consistent with each other. Nevertheless, despite some complications, the "smoking-gun" nature of this test warrants keeping it in mind.

- *Void properties: explore weak screening effects*

A number of recent observational efforts have been able to detect the lensing signal associated with voids [5]. These types of measurements are naturally interesting in a broad context of large scale structure studies, but acquire particular relevance when it comes to testing modified gravity models. The reason for this can be traced back to what we learned already about screening mechanisms: in high density regions, the effects of modified gravity get suppressed; but in low-density regions they do not. This therefore warrants trying to design ways to test gravity using void properties, in particular their lensing profiles.

Modified gravity effects can change the lensing signal in two main ways. First, the modifications to the dynamical potential lead to different void density profiles, making them deeper in the center and denser at the surrounding matter ridges. Second, in case the lensing potential is also modified ($\Sigma \neq 0$ in

[5] The reason why this took so long is mostly due to the fact that the void lensing signal is very weak, compared to that of clusters for instance. All measurements done to data involved stacking methods to increase the signal-to-noise ratio.

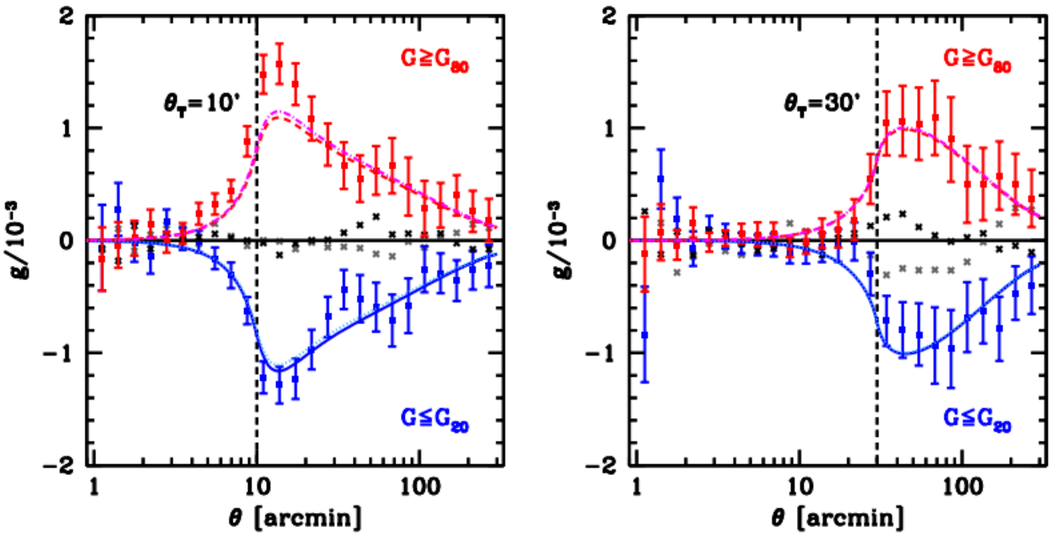

Figure 9: Measurements of the lensing signal around overdense (red) and underdense (blue) projected cylinders (along the line of sight) in the galaxy distribution from DES (taken from Gruen et al, arXiv:1507.05090). Owing to the screening mechanisms, it is a general prediction of modified gravity that the lensing signal of overdense/underdense regions is not symmetric. Measurements such as these can help to put interesting constraints on modified gravity.

Eqs. (75)), the lensing signal gets directly a contribution from the fifth force (and not just indirectly via modified matter distributions). The Cubic Galileon model is an example of a model that modifies the lensing potential, and as a result, it can leave stong inprints in the void lensing signal. This is shown in the left panel of Fig. 8, which can be compared to the observed signal that is depicted in the right panel. Taken at face value, we can conclude that the Galileon model probably produces too much lensing to be compatible with the data (we say probably because there are extra modelling steps that need to be taken into account, even though they are unlikely to modify this conclusion).

Finally, even more recently, the DES survey measured the lensing signal associated with overdense and underdense cylinders along lines of sight in the galaxy distribution. This is shown in Fig. 9. In modified gravity, it seems reasonable to expect that the signal due to underdense lines of sight may be more pronounced than that of overdense ones, by virtue of the screening mechanisms. Currently, the size of the errorbars still does not allow for concrete assessments of the degree of symmetry in these measurements, but this a field that is currently in development. One should therefore keep an eye out to check on the progress of similar observational studies, given its potentially strong power to constrain modified gravity models.

7 Concluding remark

Modified gravity scenarios are one of the most popular theoretical alternatives to ΛCDM and constitute one of the main scientific goals of current and planned observational missions. These surveys will reach unprecedented statistical precision in measurements of the expansion rate and large-scale structure of the Universe. As a result, it is essential to ensure that we understand the physics of ΛCDM and competing models to the same extent. This should serve to improve current techniques in the theoretical modeling and data analysis to design new observational tests and avoid catastrophic systematic errors in the interpretation of these upcoming surveys. This is one of the reasons behind the recent interest (which is expected to remain for a few more years) in theories of gravity beyond General Relativity.