Problem Sets on "Cosmology and Cosmic Microwave Background"

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1 Expansion of the Universe

In this section, we will use Einstein's General Relativity to derive the equations that describe the expanding universe. Einstein's General Relativity describes the evolution of gravitational fields for a given source of energy density, momentum, and stress (e.g., pressure). Schematically,

$$[\text{Curvature of Space-time}] = \frac{8\pi G}{c^4}[\text{Energy density, Momentum, and Stress}]$$

Here, the dimension of "curvature of space-time" is $1/(\text{length})^2$, as the curvature is usually defined as the second derivative of a function with respect to independent variables, and for our application the independent variables are space-time coordinates: $x^\mu = (ct, x^1, x^2, x^3)$ for $\mu = 0, 1, 2, 3$.

1.1 Space-time Curvature: Left Hand Side of Einstein's Equation

The coefficient on the right hand side, $8\pi G/c^4$, is chosen such that Einstein's gravitational field equations reduce to the familiar Poisson equation when gravitational fields are weak and static, and the space is not expanding: $\nabla^2\phi_N = 4\pi G\rho_M$, where ϕ_N is the usual Newtonian potential, and ρ_M is the mass density. Let us rewrite it in the following suggestive form:

$$\nabla^2\left(2\frac{\phi_N}{c^2}\right) = \frac{8\pi G}{c^4}(\rho_M c^2).$$

Here, as ϕ_N/c^2 is dimensionless, and thus the left hand side has the dimension of curvature, i.e., $1/(\text{length})^2$. The right hand side contains $\rho_M c^2$, which is energy density; thus, G/c^4 correctly converts energy density into curvature. Now, this equation tells us something Newton did not know but Einstein finally figured out: the second derivative of the dimensionless Newtonian potential times 2 with respect to space coordinates is the curvature of space, and mass deforms space.

In order to calculate curvature of space-time, we need to know how to calculate a distance between two points. Of course, everyone knows that, in Cartesian coordinates, the distance between two points in flat space separated by $dx^i = (dx^1, dx^2, dx^3)$ is given by $dl = \sqrt{(dx^1)^2 + (dx^2)^2 + (dx^3)^2}$, or

$$dl^2 = \sum_{i=1}^{3}\sum_{j=1}^{3}\delta_{ij}dx^i dx^j, \tag{1}$$

where $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$. Since space is flat, the curvature of this space is zero. This is a consequence of the coefficients of $dx^i dx^j$ on the right hand side of equation (1) being independent of coordinates. In general, when space is not flat but curved, the distance between two points can be written as

$$dl^2 = \sum_{i=1}^{3} \sum_{j=1}^{3} g_{ij}(x) dx^i dx^j, \tag{2}$$

where $g_{ij}(x)$ is known as the **metric tensor**. Schematically, the curvature of space is given by the second derivatives of the metric tensor with respect to space coordinates:

$$\text{Curvature of Space} \sim \frac{\partial^2 g_{ij}}{\partial x^k \partial x^l}.$$

In General Relativity, we extend this to the curvature of *space-time*. The distance between two points in space and time separated by $dx^\mu = (cdt, dx^1, dx^2, dx^3)$ is given by

$$ds^2 = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} g_{\mu\nu}(x) dx^\mu dx^\nu, \tag{3}$$

and

$$\text{Curvature of Space-time} \sim \frac{\partial^2 g_{\mu\nu}}{\partial x^\mu \partial x^\nu}.$$

Now, let us get into the gory details! The precise definition of space-time curvature, known as the **Riemann curvature tensor**, is given by[1]

$$R^\mu_{\nu\rho\sigma} \equiv \frac{\partial \Gamma^\mu_{\nu\sigma}}{\partial x^\rho} - \frac{\partial \Gamma^\mu_{\nu\rho}}{\partial x^\sigma} + \sum_\alpha \Gamma^\alpha_{\nu\sigma} \Gamma^\mu_{\alpha\rho} - \sum_\alpha \Gamma^\alpha_{\nu\rho} \Gamma^\mu_{\alpha\sigma}, \tag{4}$$

where Γ is the so-called Christoffel symbol, also known as the **affine connection**:

$$\Gamma^\mu_{\nu\rho} \equiv \frac{1}{2} \sum_\alpha g^{\mu\alpha} \left(\frac{\partial g_{\alpha\rho}}{\partial x^\nu} + \frac{\partial g_{\nu\alpha}}{\partial x^\rho} - \frac{\partial g_{\nu\rho}}{\partial x^\alpha} \right). \tag{5}$$

The metric tensor with the superscripts, $g^{\mu\alpha}$, is the inverse of the metric tensor, in the sense that

$$\sum_\alpha g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu,$$

where $\delta^\mu_\nu = 1$ for $\mu = \nu$ and zero otherwise.

Question 1.1: For an expanding universe with flat space, the distance between two points in space is given by, perhaps not surprisingly,

$$dl^2 = a^2(t) \sum_{i=1}^{3} \sum_{j=1}^{3} \delta_{ij} dx^i dx^j, \tag{6}$$

[1] Different definitions of curvature are used in the literature. Here, we follow the definition used by Misner, Thorne, and Wheeler, "*Gravitation*" (1973). Steven Weinberg's recent textbook, "*Cosmology*," uses the opposite sign.

where x denotes **comoving coordinates**. The scale factor, $a(t)$, depends only on time t. Then, the distance between two points in *space-time* is given by

$$\begin{aligned} ds^2 &= -c^2dt^2 + dl^2 \\ &= -c^2dt^2 + a^2(t)\sum_{i=1}^{3}\sum_{j=1}^{3}\delta_{ij}dx^i dx^j. \end{aligned} \tag{7}$$

Non-zero components of the metric tensor are

$$g_{00} = -1; \quad g_{ii} = a^2(t) \text{ for } i = 1, 2, 3,$$

and those of the corresponding inverse are

$$g^{00} = -1; \quad g^{ii} = \frac{1}{a^2(t)} \text{ for } i = 1, 2, 3.$$

This metric is known as the **Robertson-Walker metric** (for flat space), and describes the distance between two points in space-time of a homogeneous, isotropic, and expanding universe. For this metric, non-zero components of the affine connection are Γ^i_{j0} and Γ^0_{ij}. $\boxed{\text{Calculate } \Gamma^i_{j0} \text{ and } \Gamma^0_{ij}.}$ The answers will contain a, $\dot{a}/c$, and δ_{ij}. Once again, our space-time coordinates are $x^\mu = (ct, x^1, x^2, x^3)$.

Question 1.2: Einstein's field equations do not use all the components of the Riemann tensor, but only use a part of it. Specifically, they will use the so-called **Ricci tensor**:

$$\begin{aligned} R_{\mu\nu} &\equiv \sum_\alpha R^\alpha_{\mu\alpha\nu} \\ &= \sum_\alpha \left(\frac{\partial\Gamma^\alpha_{\mu\nu}}{\partial x^\alpha} - \frac{\partial\Gamma^\alpha_{\mu\alpha}}{\partial x^\nu}\right) + \sum_{\alpha\beta}\left(\Gamma^\beta_{\mu\nu}\Gamma^\alpha_{\beta\alpha} - \Gamma^\beta_{\mu\alpha}\Gamma^\alpha_{\beta\nu}\right), \end{aligned} \tag{8}$$

and the **Ricci scalar**:

$$R \equiv \sum_{\mu\nu} g^{\mu\nu}R_{\mu\nu}. \tag{9}$$

For the above flat Robertson-Walker metric, non-zero components of the Ricci tensor are R_{00} and R_{ij}. $\boxed{\text{Calculate } R_{00},\ R_{ij}, \text{ and } R.}$ The answers will contain a, $\dot{a}/c$, $\ddot{a}/c^2$, and/or δ_{ij}.

Question 1.3: The left hand side of Einstein's equation is called the **Einstein tensor**, denoted by $G_{\mu\nu}$, and is defined as

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R. \tag{10}$$

$\boxed{\text{Calculate } G_{00} \text{ and } G_{ij}.}$

1.2 Stress-Energy Tensor: Right Hand Side of Einstein's Equation

The precise form of Einstein's field equation is

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{11}$$

where $T_{\mu\nu}$ is called the **stress-energy tensor** (also sometimes called "energy-momentum tensor"). As the name suggests, the components of $T_{\mu\nu}$ represent the following quantities:

- T_{00}: Energy density,
- T_{0i}: Momentum, and
- T_{ij}: Stress (which includes pressure, viscosity, and heat conduction).

For a perfect fluid, the stress-energy tensor takes on the following specific form:

$$T_{\mu\nu} = P g_{\mu\nu} + (\rho + P) \frac{(\sum_\alpha g_{\mu\alpha} u^\alpha)(\sum_\beta g_{\nu\beta} u^\beta)}{c^2}, \tag{12}$$

where ρ and P are the energy density and pressure, respectively, and u^μ is a four-dimensional velocity of a fluid element. The spatial components of a four velocity, u^i, represent the usual 3-dimensional velocity of a fluid element, while the temporal component, u^0, is determined by the normalization condition of u^μ:

$$g_{\mu\nu} u^\mu u^\nu = -c^2. \tag{13}$$

Note that the 3-dimensional velocity, u^i, does not contain the apparent motion due to the expansion of the universe, but only contains the true motion of fluid elements.

Question 1.4: In a homogeneous, isotropic, and expanding universe, fluid elements simply move along the expansion of the universe, and the 3-dimensional velocity vanishes. (In other words, fluids are comoving with expansion.) Therefore, such a fluid element has $u^i = 0$, and the normalization condition gives $u^0 = c$. Non-zero components of the stress-energy tensor are T_{00} and T_{ij}. $\boxed{\text{Calculate } T_{00} \text{ and } T_{ij} \text{ for the flat Robertson-Walker metric and comoving fluid.}}$

Question 1.5: Now, we are ready to obtain Einstein's equations. First, write down $G_{00} = (8\pi G/c^4)T_{00}$ and $G_{ij} = (8\pi G/c^4)T_{ij}$ for the flat Robertson-Walker metric and comoving fluid in terms of a, $\dot{a}/c$, $\ddot{a}/c^2$, and/or δ_{ij}. Then, by combining these equations, $\boxed{\text{obtain the right hand side of}}$

$$\frac{\dot{a}^2}{a^2} =$$

$$\frac{\ddot{a}}{a} =$$

The first equation is the Friedmann equation, and the second one is the acceleration equation that we have learned in class (with $c = 1$).

1.3 Energy Conservation

Combining the above equations for $\dot{a}/a$ and $\ddot{a}/a$ will yield the energy conservation equation, $\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0$. In other words, the energy conservation is already built into Einstein's equations.

Question 1.6: Alternatively, one can derive the energy conservation equation directly from the conservation of the stress-energy tensor. In General Relativity, the "conservation" means that the **covariant derivative** (rather than the partial derivative) of the stress-energy tensor vanishes.

$$0 = \sum_{\alpha\beta} g^{\alpha\beta} T_{\mu\alpha;\beta} \equiv \sum_{\alpha\beta} g^{\alpha\beta} \left(\frac{\partial T_{\mu\alpha}}{\partial x^\beta} - \sum_\lambda \Gamma^\lambda_{\alpha\beta} T_{\mu\lambda} - \sum_\lambda \Gamma^\lambda_{\mu\beta} T_{\lambda\alpha} \right). \tag{14}$$

The energy conservation equation is $\sum_{\alpha\beta} g^{\alpha\beta} T_{0\alpha;\beta} = 0$, while the momentum conservation equation is $\sum_{\alpha\beta} g^{\alpha\beta} T_{i\alpha;\beta} = 0$. $\boxed{\text{Reproduce } \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0 \text{ from } \sum_{\alpha\beta} g^{\alpha\beta} T_{0\alpha;\beta} = 0.}$

1.4 Cosmological Redshift

Consider a non-relativistic particle, which is moving in a gravitational field with a 3-dimensional velocity of $u^i \ll c$. The other external forces (such as the electromagnetic force) are absent. According to General Relativity, the equation of motion of such a particle is

$$\frac{du^i}{d\tau} + \sum_{\alpha\beta} \Gamma^i_{\alpha\beta} u^\alpha u^\beta = 0, \tag{15}$$

where $d\tau \equiv \sqrt{-ds^2}/c$ is called the **proper time**. The four-dimensional velocity is given by $u^\mu = dx^\mu/d\tau$; thus, $u^0 = cdt/d\tau$ and $u^i = dx^i/d\tau$.

Question 1.7: Using the affine connection for the flat Robertson-Walker metric, rewrite the equation of motion in terms of $\dot{u}^i = du^i/dt$, $\dot{a}/a$ and u^i. $\boxed{\text{Show how } u^i \text{ changes with the scale factor, } a(t).}$

2 Cosmic Microwave Background - I

While the speed of light is kept for completeness below, you may set $c = 1$ if you wish.

2.1 Propagation of photons in a clumpy universe

How does the momentum of photons change as photons propagate through space? First, every photon suffers from the mean cosmological redshift, and thus its magnitude, p, will decrease as $p \propto 1/a$. In addition, as photons pass through potential wells and troughs, they gain or lose momentum. Finally, not only the magnitude, p, but also the direction of momentum, γ^i, will change when photons are deflected gravitationally.

We can calculate the evolution of four-dimensional momentum, $p^\mu \equiv dx^\mu/d\lambda$, using the following **geodesic equation**:

$$\frac{dp^\mu}{d\lambda} + \sum_{\alpha\beta} \Gamma^\mu_{\alpha\beta} p^\alpha p^\beta = 0. \tag{16}$$

Here, λ is a parameter which gives the location along the path of photons. Using $p^0 = d(ct)/d\lambda$, one may rewrite the geodesic equation in terms of the total time derivative of p^μ:

$$\frac{dp^\mu}{dt} + c \sum_{\alpha\beta} \Gamma^\mu_{\alpha\beta} \frac{p^\alpha p^\beta}{p^0} = 0. \tag{17}$$

In order to calculate $\Gamma^\mu_{\alpha\beta}$, we need to specify the metric. To describe a clumpy universe, we perturb the Robertson-Walker metric in the following way:

$$ds^2 = -[1 + 2\Psi(t, x^i)]c^2 dt^2 + a^2(t)[1 + 2\Phi(t, x^i)] \sum_{ij} \delta_{ij} dx^i dx^j. \tag{18}$$

Here, Ψ is the usual Newtonian potential (divided by c^2 to make it dimensionless), and Φ is called the **curvature perturbation**. For this metric, all of the components of $\Gamma^\mu_{\alpha\beta}$ are non-zero.

From now on, we will assume that the magnitudes of these variables are small: $|\Psi| \ll 1$ and $|\Phi| \ll 1$, and <u>calculate everything only up to the first order in these variables.</u>

Question 2.1: $\boxed{\text{Calculate } \Gamma^0_{00},\ \Gamma^0_{0i},\ \Gamma^0_{ij},\ \Gamma^i_{00},\ \Gamma^i_{0j}, \text{ and } \Gamma^i_{jk}, \text{ up to the first order in } \Phi \text{ and } \Psi.}$ You may use the short-hand notation such as

$$\dot{\Psi} \equiv \frac{\partial \Psi}{\partial t}, \quad \Psi_{,i} \equiv \frac{\partial \Psi}{\partial x^i}.$$

The components of the metric and its inverse are given by

$$g_{00} = -(1 + 2\Psi); \quad g^{00} = -(1 - 2\Psi); \quad g_{ij} = a^2(1 + 2\Phi)\delta_{ij}; \quad g^{ij} = \frac{1}{a^2}(1 - 2\Phi)\delta^{ij}. \tag{19}$$

Question 2.2: $\boxed{\text{Write down the geodesic equations in the following form:}}$

$$\begin{aligned} \frac{dp^0}{dt} &= \dots, \\ \frac{dp^i}{dt} &= \dots, \end{aligned}$$

up to the first order in Φ and Ψ. The final answers should not contain $\sum_{ij} \delta_{ij} p^i p^j$. You can eliminate this by using the normalization condition for momentum of massless particles, $\sum_{\alpha\beta} g_{\alpha\beta} p^\alpha p^\beta = 0$, which gives, for the above perturbed metric,

$$a^2 \sum_{ij} \delta_{ij} p^i p^j = (1 - 2\Phi + 2\Psi)(p^0)^2. \tag{20}$$

Question 2.3: Now, we want to derive the evolution equations for the magnitude of momentum, p, and its direction, γ^i. First, we define the magnitude as

$$p^2 \equiv \sum_{ij} g_{ij} p^i p^j. \tag{21}$$

Also, we normalize the direction such that

$$\sum_{ij} \delta_{ij} \gamma^i \gamma^j = 1. \tag{22}$$

Using this information, $\boxed{\text{write } p \text{ in terms of } p_0 \text{ and } \Psi}$, and $\boxed{\text{write } \gamma^i \text{ in terms of } p,\ p^i,\ a, \text{ and } \Phi}$, up to the first order in Φ and Ψ.

Question 2.4: $\boxed{\text{Write down the geodesic equations in the following form:}}$

$$\begin{aligned} \frac{dp}{dt} &= \dots, \\ \frac{d\gamma^i}{dt} &= \dots, \end{aligned}$$

up to the first order in Φ and Ψ. The answers should not contain p^0 or p^i. Whenever you find them, replace them with p and γ^i, respectively. You can check the result for the deflection equation, $d\gamma^i/dt$, by making sure that the result satisfies $\sum_i \gamma^i d\gamma^i/dt = 0$. (You can derive this by differentiating the normalization condition, $\sum_{ij} \delta_{ij} \gamma^i \gamma^j = 1$, with respect to time.) Note that the total time derivative of a variable is related to the partial derivatives as, e.g.,

$$\frac{d\Phi}{dt} = \dot{\Phi} + \sum_i \frac{dx^i}{dt} \Phi_{,i} = \dot{\Phi} + \sum_i \frac{cp^i}{p^0} \Phi_{,i}. \tag{23}$$

2.2 Perturbed Conservation Equations For A Pressure-less Fluid

Consider the stress-energy tensor for a perfect fluid. We then take the limit that the pressure is much less than the energy density, which would be a good approximation for a non-relativistic fluid. The stress-energy tensor for such a pressure-less fluid is

$$T_{\mu\nu} = \rho \frac{(\sum_\alpha g_{\mu\alpha} u^\alpha)(\sum_\beta g_{\nu\beta} u^\beta)}{c^2}. \tag{24}$$

As usual, $u^\mu \equiv dx^\mu / d\tau$ is a four-dimensional velocity and τ is the proper time.

Suppose that the fluid is moving at a non-relativistic physical three-dimensional velocity of $V^i \ll c$. By "physical" velocity, we mean

$$V^i \equiv a u^i = a \frac{dx^i}{d\tau}. \tag{25}$$

We also expand the energy density into the mean, $\bar{\rho}$, and the fluctuation around the mean, δ:

$$\rho = \bar{\rho}(1 + \delta). \tag{26}$$

These perturbation variables, δ and V^i/c, are small in the same sense that Φ and Ψ are small. Therefore, we shall expand everything only up to the first order in Φ, Ψ, δ, and V^i/c. For example, T_{ij} is of order $(V/c)^2$, and thus can be ignored. On the other hand, T_{0i} is of order (V/c), and thus cannot be ignored unless it is multiplied by other perturbation variables.

Question 2.5: Expand the following conservation equations up to the first order in Φ, Ψ, δ, and V^i/c:

1. Energy conservation equation, $\sum_{\alpha\beta} g^{\alpha\beta} T_{0\alpha;\beta} = 0$
2. Momentum conservation equation, $\sum_{\alpha\beta} g^{\alpha\beta} T_{i\alpha;\beta} = 0$

Use the conservation equation for the mean density, $\dot{\bar{\rho}} + 3\frac{\dot{a}}{a}\bar{\rho} = 0$, to eliminate the mean contributions from the above equations, and then rewrite these equations in the following form:

$$\begin{aligned} \dot{\delta} &= \dots, \\ \frac{\dot{V}^i}{c} &= \dots. \end{aligned}$$

2.3 Large-scale Solutions of Einstein Equations During Matter Era

The energy and momentum conservation equations contain four unknown perturbation variables, δ, V^i, Ψ, and Φ. Therefore, we cannot find solutions unless we have (at least) two more equations. Such equations are provided by perturbed Einstein equations. Don't worry - you are not asked to derive them (though I would not stop you from deriving them). Here are the two equations that can be derived by combining perturbed Einstein equations:[2]

$$\frac{k^2}{a^2}\tilde{\Phi} = \frac{4\pi G}{c^4}\bar{\rho}\left(\tilde{\delta}+\frac{3\dot{a}\tilde{V}}{kc^2}\right), \tag{27}$$

$$\tilde{\Psi} = -\tilde{\Phi}. \tag{28}$$

Here, $\tilde{\Phi}$, $\tilde{\Psi}$, $\tilde{\delta}$, and $\tilde{V}$ are all in Fourier space, i.e., $\tilde{\Phi}=\tilde{\Phi}(\vec{k},t)$, $\tilde{\Psi}=\tilde{\Psi}(\vec{k},t)$, $\tilde{\delta}=\tilde{\delta}(\vec{k},t)$, and $\tilde{V}=\tilde{V}(\vec{k},t)$, and $\vec{k}$ is the **comoving wavenumber vector**. They are related to the original variables in position space by, e.g.,

$$\tilde{\Psi}(\vec{k},t) = \int d^3x\Psi(\vec{x},t)e^{-i\vec{k}\cdot\vec{x}}, \tag{29}$$

$$\Psi(\vec{x},t) = \int\frac{d^3k}{(2\pi)^3}\tilde{\Psi}(\vec{k},t)e^{i\vec{k}\cdot\vec{x}}. \tag{30}$$

Here, $\vec{k}\cdot\vec{x}\equiv\sum_{ij}\delta_{ij}k^ix^j$. For example, the left hand side of the first perturbed Einstein equation, $(k^2/a^2)\Phi$, came from the Laplacian of Φ:

$$\begin{aligned}\frac{1}{a^2}\nabla^2\Phi(\vec{x},t) &= \frac{1}{a^2}\int\frac{d^3k}{(2\pi)^3}\tilde{\Phi}(\vec{k},t)\left(\nabla^2e^{i\vec{k}\cdot\vec{x}}\right)\\ &= \frac{1}{a^2}\int\frac{d^3k}{(2\pi)^3}\tilde{\Phi}(\vec{k},t)\left(-k^2e^{i\vec{k}\cdot\vec{x}}\right),\end{aligned} \tag{31}$$

where $\nabla^2\equiv\sum_{ij}\delta^{ij}\frac{\partial^2}{\partial x^i\partial x^j}$, and $k^2\equiv\sum_{ij}\delta_{ij}k^ik^j$. Also, $\tilde{V}$ in the right hand side of the first perturbed Einstein equation is defined as $\tilde{V}(\vec{k},t)\equiv i\hat{k}\cdot\tilde{\vec{V}}(\vec{k},t)$ (where $\hat{k}\equiv\vec{k}/k$ is a unit vector), i.e.,

$$\begin{aligned}\vec{\nabla}\cdot\vec{V}(\vec{x},t) &= \int\frac{d^3k}{(2\pi)^3}\tilde{\vec{V}}(\vec{k},t)\cdot\left(\vec{\nabla}e^{i\vec{k}\cdot\vec{x}}\right)\\ &= \int\frac{d^3k}{(2\pi)^3}\tilde{\vec{V}}(\vec{k},t)\cdot\left(i\vec{k}e^{i\vec{k}\cdot\vec{x}}\right) \qquad (32)\\ &\equiv \int\frac{d^3k}{(2\pi)^3}k\tilde{V}(\vec{k},t)e^{i\vec{k}\cdot\vec{x}}. \qquad (33)\end{aligned}$$

Here, $\vec{\nabla}\cdot\vec{V}\equiv\sum_k V^k_{,k}$.

Now, let us use the above four equations to find solutions for Ψ, Φ, V, and δ. From now on, we shall drop the tildes on variables in Fourier space for simplicity. It is convenient to change the independent variable from t to the scale factor, a. Finally, let us define the following variable:

$$\epsilon(a)\equiv\frac{ck}{\dot{a}}, \tag{34}$$

[2]To those who wish to derive these results: the first equation can be obtained by combining perturbed $G_{00}=(8\pi G/c^4)T_{00}$ and $G_{0i}=(8\pi G/c^4)T_{0i}$, while the second equation can be obtained from the traceless part of $G_{ij}=(8\pi G/c^4)T_{ij}$.

which goes as $\epsilon \propto \sqrt{a}$ during the matter-dominated era. This quantity is useful, as it is much less than unity for fluctuations whose wavelength is longer than the Hubble length ($\approx$horizon size):

$$\epsilon \ll 1 \quad \text{for super-horizon fluctuations, } k \ll aH/c,$$

where $H = \dot{a}/a$ is the Hubble expansion rate. Therefore, we can find large-scale (long-wavelength; super-horizon) solutions by consistently ignoring higher-order terms of ϵ.

Question 2.6: Using the Fourier-space variables and ϵ, show that the energy- and momentum-conservation equations $\boxed{\text{can be re-written as follows}}$:

$$\delta' = -\frac{\epsilon}{a}\frac{V}{c} - 3\Phi', \tag{35}$$

$$\frac{V'}{c} = -\frac{1}{a}\frac{V}{c} + \frac{\epsilon}{a}\Psi, \tag{36}$$

where the primes denote derivatives with respect to a.

Question 2.7: Using $\Phi = -\Psi$, we now have the following three equations for three unknown variables:

$$\delta' = -\frac{\epsilon}{a}\frac{V}{c} - 3\Phi', \tag{37}$$

$$\frac{V'}{c} = -\frac{1}{a}\frac{V}{c} - \frac{\epsilon}{a}\Phi, \tag{38}$$

$$\epsilon^2\Phi = \frac{3}{2}\left(\delta + \frac{3V}{\epsilon c}\right). \tag{39}$$

Once again, during the matter era, $\epsilon \propto \sqrt{a}$. Solve these equations on super-horizon scales, $\epsilon \ll 1$, and $\boxed{\text{show that non-decaying solutions are given by}}$

$$\delta = 2\Phi, \tag{40}$$

$$\frac{V}{c} = -\frac{2}{3}\epsilon\Phi. \tag{41}$$

By "non-decaying solutions" we mean the solutions that go as $\propto a^n$ where $n \geq 0$. Finally, $\boxed{\text{show that } \Phi \text{ (and hence } \Psi\text{) is a constant and does not depend on } a \text{ in the super-horizon limit}}$.

Hint: you cannot ignore ϵ when two different variables are involved, e.g., $A + \epsilon B \neq A$, because you do not know a priori how A compares with B. You can ignore the terms of order ϵ only when you are sure that ϵ is compared to order unity, e.g., $A' + \frac{A}{a} + \epsilon\frac{A}{a} \approx A' + \frac{A}{a}$.

$\boxed{\text{Do not use Mathematica to solve these coupled differential equations! Use your brain, please.}}$

3 Cosmic Microwave Background - II

While the speed of light is kept for completeness below, you may set $c = 1$ if you wish.

3.1 Temperature Anisotropy From Gravitational Waves

Gravitational waves stretch space as they propagate through space. This deformation of space is characterized by the following metric:

$$ds^2 = -c^2 dt^2 + a^2(t) \sum_{ij} (\delta_{ij} + h_{ij}) dx^i dx^j,$$

where h_{ij} is the so-called **tensor metric perturbation**. (On the other hand, Φ and Ψ that we have dealt with before are called "scalar metric perturbations".) The tensor metric perturbation is symmetric ($h_{ij} = h_{ji}$), traceless ($\sum_{i=1}^{3} h_{ii} = 0$), and transverse ($\sum_{j=1}^{3} \frac{\partial h_{ij}}{\partial x^j} = 0$).

At the first-order of perturbations, scalar and tensor perturbations are decoupled, and thus we can ignore the scalar perturbations when analyzing the tensor perturbations.

Question 3.1: Write down the geodesic equation for $p \equiv (\sum_{ij} g_{ij} p^i p^j)^{1/2}$ with the metric given above, up to the first order in h_{ij}. Then, by integrating the geodesic equation over time, derive the formula for the observed temperature anisotropy from gravitational waves as

$$\left.\frac{\delta T}{\bar{T}}\right|_{\mathcal{O}} = \left.\frac{\delta T}{\bar{T}}\right|_{\mathcal{E}} + \int_{t_{\mathcal{E}}}^{t_{\mathcal{O}}} dt\ (\dots)$$

where $(\dots)$ should contain only $\dot{h}_{ij}$ and γ^i (where γ^i is the unit vector of the direction of photons, satisfying $\sum_{ij} \delta_{ij} \gamma^i \gamma^j = 1$). *Hint*: you should check the result by making sure that you can recover a part of the scalar integrated Sachs–Wolfe effect, $-\dot{\Phi}$, by using the scalar metric perturbation, $h_{ij} = 2\Phi\delta_{ij}$. (You cannot recover the terms containing Ψ because $g_{00} = -1$ for the above metric.)

From now on, set $\left.\frac{\delta T}{\bar{T}}\right|_{\mathcal{E}} = 0$.

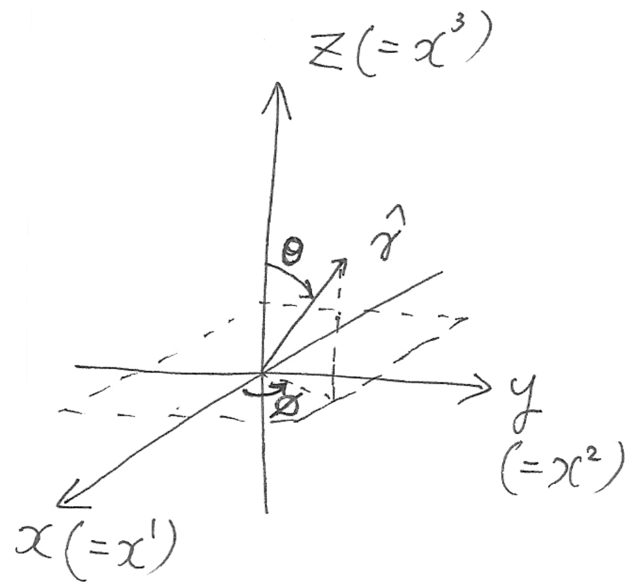
Question 3.2: Consider a gravitational wave propagating in the z ($= x^3$) direction. For this special case, the components of the tensor metric perturbation are given by

$$h_{ij} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

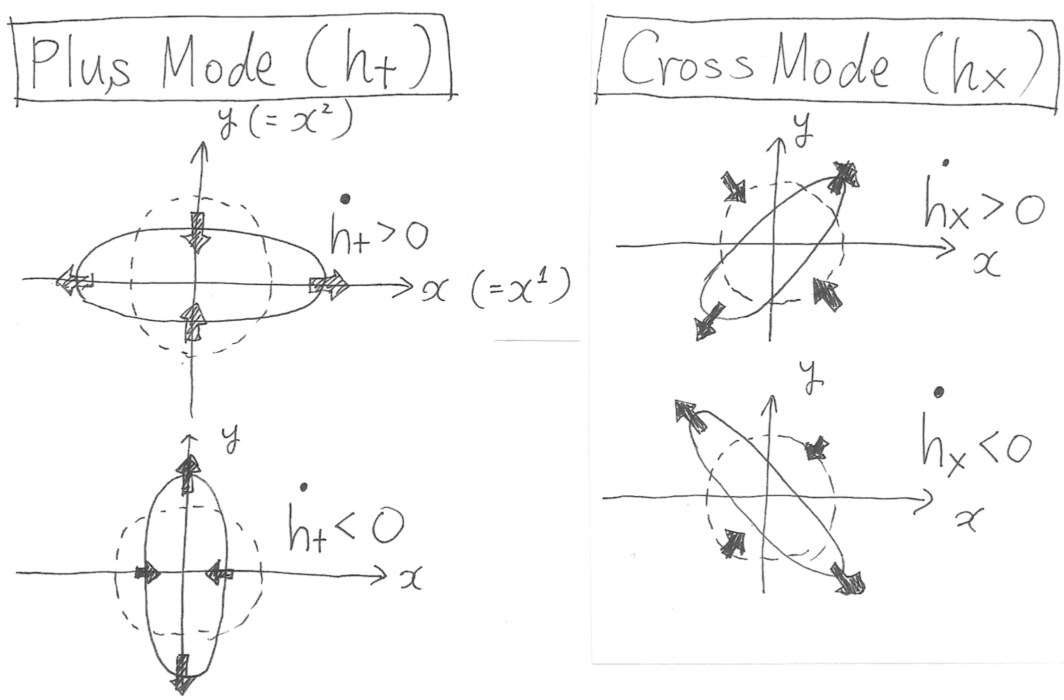
where h_+ and $h_\times$ denote two *linear* polarization states of a gravitational wave. Using polar coordinates for the propagation direction of photons with respect to the gravitational wave:

$$\gamma^i = (\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta),$$

$\boxed{\text{rewrite the equation for } \frac{\delta T}{T}\big|_{\mathcal{O}} \text{ in terms of } \int \dot{h}_+ dt,\ \int \dot{h}_\times dt, \text{ and trigonometric functions.}}$



Question 3.3: A gravitational wave with $\dot{h}_+ > 0$ stretches space in x direction, while that with $\dot{h}_\times > 0$ stretches space in 45° direction (see the figure below). This stretching of space causes gravitational redshifts and blueshifts in the corresponding directions. Using this picture, give **physical** explanations for the result obtained in Question 3.2. (In other words, now that you have an equation, how much physical interpretation can you get out of this equation?) *For example*: in which cases do you find hot ($\Delta T > 0$) or cold ($\Delta T < 0$), and why?; compare the results for $\theta = 0$ and $\theta = \pi/2$, and give a physical explanation for the difference; compare the results for $\phi = 0$, $\pi/4$, $\pi/2$, and $3\pi/4$, and give a physical explanation for the difference. Use graphics as needed. It is easier to think about this from a point of view of photons: if you were a photon, how would you experience redshift or blueshift, depending on the angle between your propagation direction and the direction of the gravitational wave, or depending on the azimuthal angle?



Question 3.4: As it is evident from the above figure, a gravitational wave produces a quadrupolar ($l = 2$) temperature anisotropy. To see this more clearly, it is convenient to define the following *circular* polarization amplitudes, h_R (right-handed) and h_L (left-handed), as

$$h_+ = \frac{1}{\sqrt{2}}(h_R + h_L), \tag{42}$$

$$h_\times = \frac{i}{\sqrt{2}}(h_R - h_L). \tag{43}$$

Using h_R and h_L, and the definitions for spherical harmonics, Y_l^m, with $l = 2$:

$$Y_2^{\pm 2}(\theta, \phi) = \sqrt{\frac{15}{32\pi}} \sin^2\theta e^{\pm 2i\phi}, \tag{44}$$

$$Y_2^{\pm 1}(\theta, \phi) = (\pm 1)\sqrt{\frac{15}{8\pi}} \sin\theta\cos\theta e^{\pm i\phi}, \tag{45}$$

$$Y_2^{0}(\theta, \phi) = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1), \tag{46}$$

rewrite the equation for $\frac{\delta T}{T}\Big\vert_{\mathcal{O}}$ in terms of $\int \dot{h}_R dt$, $\int \dot{h}_L dt$, and Y_2^m.

3.2 Polarization From Gravitational Waves

Thomson scattering of a quadrupolar temperature anisotropy by an electron can produce linear polarization. In terms of the Stokes parameters produced by a scattering, $Q(\theta, \phi)$ and $U(\theta, \phi)$, there is a formula relating the temperature quadrupole to polarization by a single scattering:

$$Q + iU = -\frac{\sqrt{6}}{10} \sum_{m=\pm 2} {}_2Y_2^m(\theta,\phi) \int d\tilde{\Omega}\, \left.\frac{\delta T}{\bar{T}}\right|_{\mathcal{O}} (\tilde{\theta},\tilde{\phi}) Y_2^{m*}(\tilde{\theta},\tilde{\phi}), \tag{47}$$

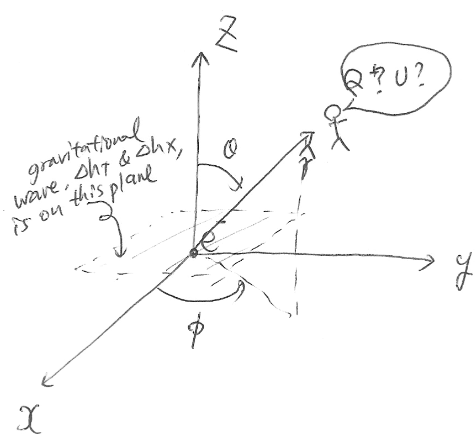
$$Q - iU = -\frac{\sqrt{6}}{10} \sum_{m=\pm 2} {}_{-2}Y_2^m(\theta,\phi) \int d\tilde{\Omega}\, \left.\frac{\delta T}{\bar{T}}\right|_{\mathcal{O}} (\tilde{\theta},\tilde{\phi}) Y_2^{m*}(\tilde{\theta},\tilde{\phi}), \tag{48}$$

where $d\tilde{\Omega} = d\cos\tilde{\theta} d\tilde{\phi}$, and ${}_2Y_l^m$ is a spin-2 harmonics given by

$${}_2Y_2^{\pm 2} = \sqrt{\frac{5}{64\pi}} (1 \mp \cos\theta)^2 e^{\pm 2i\phi}, \tag{49}$$

$${}_{-2}Y_2^{\pm 2} = \sqrt{\frac{5}{64\pi}} (1 \pm \cos\theta)^2 e^{\pm 2i\phi}. \tag{50}$$

Note that an electron is at the origin, and photons are scattered by this electron at the origin into various directions, (θ,ϕ). In other words, these are the Stokes parameters of polarization that would be observed by observers at various directions from this electron.



Now, to simplify the analysis, let us assume that we have $\Delta h_R \equiv \int \dot{h}_R dt$ and $\Delta h_L \equiv \int \dot{h}_L dt$ at the origin, and similarly define the linear polarization amplitudes of gravitational waves:

$$\Delta h_+ \equiv \frac{1}{\sqrt{2}} (\Delta h_R + \Delta h_L), \tag{51}$$

$$\Delta h_\times \equiv \frac{i}{\sqrt{2}} (\Delta h_R - \Delta h_L). \tag{52}$$

Question 3.5: $\boxed{\text{Calculate } Q(\theta,\phi) \text{ and } U(\theta,\phi) \text{ in terms of } \Delta h_{+,\times} \text{ and trigonometric functions.}}$

Question 3.6: $\boxed{\text{Give physical explanations for the results obtained in Question 3.5.}}$ *For example*: compare Q and U at $\theta = \pi/2$ and $\phi = 0$, and explain the origin of the difference; compare the results at different ϕ, and give a physical explanation for the behavior. Use graphics as needed.

For this problem, it is easier to think about this from a point of view of an electron at the origin: if you were an electron scattering photons into various directions, what polarization would you produce depending on the scattering direction and the direction of the gravitational wave, or depending on the azimuthal angle?