Some Basics of the Expansion of the Universe, Cosmic Microwave Background, and Large-scale Structure of the Universe

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AST396C/PHY396T

Elements of Cosmology

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Chapter 1

Expansion of the Universe

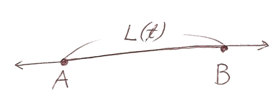
One of the main goals of cosmology is to figure out how the universe expands as a function of time.

1.1 Expansion and Conservation

To describe the evolution of the average universe, one needs only *two* kinds of equations:

1. The equation that relates the density and pressure of constituents of the universe (such as baryons, cold dark matter, photons, neutrinos, dark energy) to the expansion of the universe, and

2. The equation that describes the energy conservation of the constituents.



Consider a line connecting two arbitrary points in space (which is expanding), and call it L. As the universe expands, L changes with time. As you will derive in homework using General Relativity, the equation of motion for L is given by

$$\ddot{L}(t) = -\frac{4\pi G}{3} L(t) \sum_i \left[\rho_i(t) + 3P_i(t)\right], \tag{1.1}$$

where $\rho_i(t)$ and $P_i(t)$ are the energy and pressure of the ith component of the universe, respectively.

Here, note that the absolute value of L does not affect the equation of motion for L. Therefore, one may define a **dimensionless** "scale factor," $a(t)$, such that $L(t) \equiv a(t)x$, where x is a time-independent separation called a "comoving" separation, which is in units of length. In cosmology, we often encounter the **Hubble expansion rate**, $H(t)$, which is defined by

$$H(t) \equiv \frac{\dot{a}(t)}{a(t)}. \tag{1.2}$$

The dimension of this quantity is 1/(time). The age of the universe can be calculated from the above definition of H, which gives $H(t)dt = da/a$. Now, if we know H as a function of a instead of t, we obtain

$$t = \int \frac{da}{aH(a)}. \tag{1.3}$$

Another interpretation of H is found by writing $\dot{L}(t) = H(t)L(t)$, which tells us that $H(t)$ gives a relation between the distance, L, and the recession velocity, $\dot{L}$. For this reason, it is often convenient to write $H(t)$ in the following peculiar units:

$$H(t) = 100\ h(t)\ \text{km/s/Mpc},$$

where h is a dimensionless quantity. The current observations suggest that the present-day value of h is $h(t_{\text{today}}) \approx 0.7$.*

Dividing both sides of equation (1.1) by L and using $L(t) = a(t)x$, we find one of the key equations connecting the energy density and pressure to the expansion of the universe:

$$\boxed{\frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi G}{3}\sum_i [\rho_i(t) + 3P_i(t)]} \tag{1.4}$$

As expected, positive energy density and positive pressure slow down the expansion of the universe.†

This equation cannot be solved unless we know how ρ_i and P_i depend on time. How ρ_i depends on time is given by the energy conservation equation, while how P_i depends on time is usually given by the equation of state relating P_i to ρ_i and other quantities.

As you will derive in homework, the energy conservation equation is given by

$$\boxed{\sum_i \dot{\rho}_i(t) + 3\frac{\dot{a}(t)}{a(t)}\sum_i [\rho_i(t) + P_i(t)] = 0} \tag{1.5}$$

Equation (1.5) is general and does not assume presence or absence of possible interactions between different components. If we assume that each component is conserved separately, then we have

$$\dot{\rho}_i(t) + 3\frac{\dot{a}(t)}{a(t)}[\rho_i(t) + P_i(t)] = 0, \tag{1.6}$$

*The most precise value of $h(t_{\text{today}})$ to date from the direct measurement using low-z supernovae and Cepheid variable stars is $h(t_{\text{today}}) = 0.742 \pm 0.036$ (Riess, Macri, et al., ApJ, 699, 539 (2009)).

†If we ignore the effect of pressure relative to that of the energy density (which is always a good approximation for non-relativistic matter), and write $\rho(t)$ in terms of the total mass enclosed with a radius L, $\sum_i \rho_i(t) = \frac{3M}{4\pi L^3}$, then equation (1.1) becomes

$$\ddot{L} = -\frac{GM}{L^2},$$

which is the familiar Newtonian inverse-square law. Although one must not apply the Newtonian mechanics to describe the evolution of space (because Newtonian mechanism assumes static space), this is a convenient way to understand equations (1.1) and (1.4).

for each of the ith component. Note that the second term contains the pressure, and thus how the energy density evolves depends on the pressure.‡

Looking at equations (1.4) and (1.5), one might think that we cannot solve for $a(t)$ unless we have the equation of state giving $P_i(t)$ as a function of $\rho_i(t)$ etc. While in general that would be true, for these equations a little mathematical trick lets us combine equations (1.4) and (1.5) without knowing the evolution of $P(t)$!

First, rewrite equation (1.4) as

$$\frac{\ddot{a}(t)}{a(t)} = \frac{8\pi G}{3}\sum_i \rho_i(t) - 4\pi G \sum_i \left[\rho_i(t) + P_i(t)\right]. \tag{1.7}$$

Using equation (1.5) on the second term of the right hand side, we get

$$\begin{aligned}
\frac{\ddot{a}(t)}{a(t)} &= \frac{8\pi G}{3}\sum_i \rho_i(t) + \frac{4\pi G}{3}\frac{a(t)}{\dot{a}(t)}\sum_i \dot{\rho}_i(t) \\
\dot{a}(t)\ddot{a}(t) &= \frac{8\pi G a(t)\dot{a}(t)}{3}\sum_i \rho_i(t) + \frac{4\pi G a^2(t)}{3}\sum_i \dot{\rho}_i(t) \\
\frac{1}{2}(\dot{a}^2)^{\cdot} &= \frac{4\pi G (a^2)^{\cdot}}{3}\sum_i \rho_i(t) + \frac{4\pi G a^2(t)}{3}\sum_i \dot{\rho}_i(t). \qquad (1.8)
\end{aligned}$$

As this has the form of $\dot{A} = \dot{B}C + B\dot{C} = (BC)^{\cdot}$, it is easy to integrate and obtain:

$$\dot{a}^2(t) = \frac{8\pi G a^2(t)}{3}\sum_i \rho_i(t) - \kappa, \tag{1.9}$$

where κ is an integration constant, which is in units of $1/(\text{time})^2$. (A negative sign is for a historical reason.) Dividing both sides by $a^2(t)$, we finally arrive at the so-called **Friedmann equation**:

$$\frac{\dot{a}^2(t)}{a^2(t)} = \frac{8\pi G}{3}\sum_i \rho_i(t) - \frac{\kappa}{a^2(t)}. \tag{1.10}$$

‡While it is a wrong explanation, it is useful to compare this equation to the first law of thermodynamics:

$$TdS = dU + PdV,$$

where T, S, U, and V are the temperature, entropy, internal energy, and volume, respectively. To a very good accuracy, the entropy is conserved in the universe, $dS = 0$. The internal energy is $U \propto \rho a^3$ and the volume is $V \propto a^3$, and thus

$$d(\rho a^3) + P d(a^3) = 0,$$

which gives

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0.$$

This is a wrong explanation because it assumes that the pressure is doing work as a increases. However, in the average universe, the pressure is the same everywhere, and thus there is no under-pressure region against which the pressure can do work. Equation (1.5) must be derived using GR, which you will do in homework, but the above thermodynamic argument is an amusing way to arrive at the same equation. Also, this gives us some confidence that it is not crazy to think that the evolution of ρ depends on P.

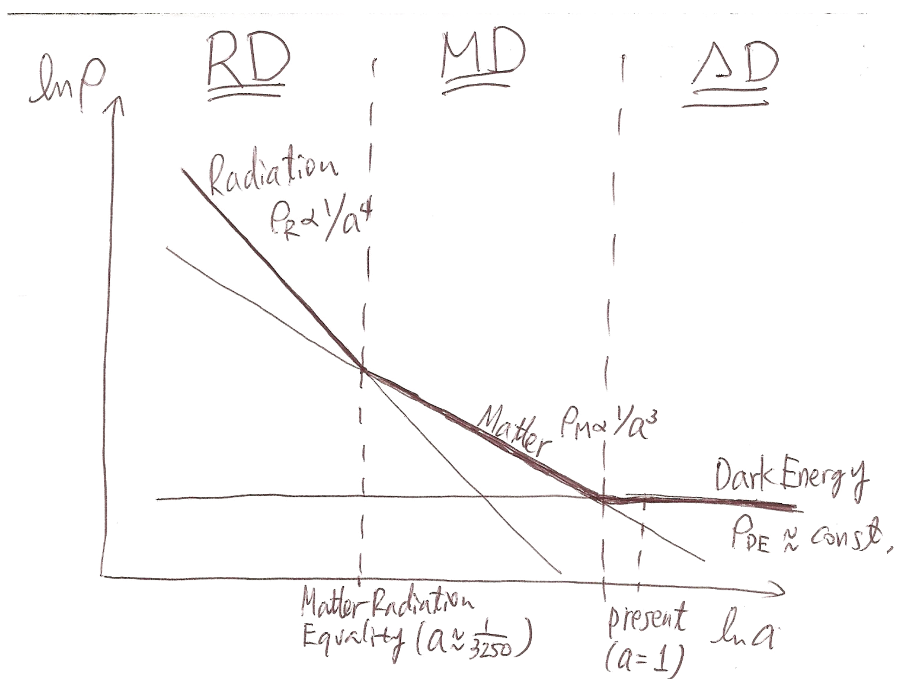
A beauty of this equation is that it is easy to solve, once a time dependence of $\rho_i(t)$ is known, which is usually the case.

General Relativity tells us that the integration constant, κ, is equal to $\pm c^2/R^2$ where R is the curvature radius of the universe (in units of length) and c the speed of light. When the geometry of the universe is flat (as suggested by observations), $R \to \infty$ (giving $\kappa \to 0$), and thus one can ignore this term. Since we have so much to learn, to save time we will not consider the curvature of the universe throughout (most of) this lecture:

$$\boxed{\frac{\dot{a}^2(t)}{a^2(t)} = \frac{8\pi G}{3} \sum_i \rho_i(t)} \tag{1.11}$$

1.2 Solutions of Friedmann Equation

In order to use solve equation (1.11) for $a(t)$, one must know how $\rho_i(t)$ depends on time.



To find solutions for $a(t)$, let us first assume that the universe is dominated by one energy component at a time, i.e.,

$$\frac{\dot{a}^2(t)}{a^2(t)} = \frac{8\pi G}{3} \sum_i \rho_i(t) \approx \frac{8\pi G}{3} \rho_i(t), \tag{1.12}$$

and further assume that ρ_i depends on $a(t)$ via a power-law:

$$\rho_i(t) \propto \frac{1}{a^{n_i}(t)}. \tag{1.13}$$

Finding the solution is straightforward:

$$a(t) \propto t^{2/n_i}. \tag{1.14}$$

This is usually an excellent approximation, except for the transition era where two energy components are equally important. There are 3 important cases:

1. **Radiation-dominated (RD) era**. A radiation component (photons, massless neutrinos, or any other massless particles) has a large pressure, $P_R = \rho_R/3$,[§] which gives $\rho_R(t) \propto 1/a^4(t)$, or $n_R = 4$. We thus obtain

$$a_{\rm RD}(t) \propto t^{1/2}. \tag{1.15}$$

The expansion of the universe decelerates. With this solution, we can relate the age of the universe to the Hubble expansion rate:

$$H(t) = \frac{\dot{a}(t)}{a(t)} = \frac{1}{2t}. \tag{1.16}$$

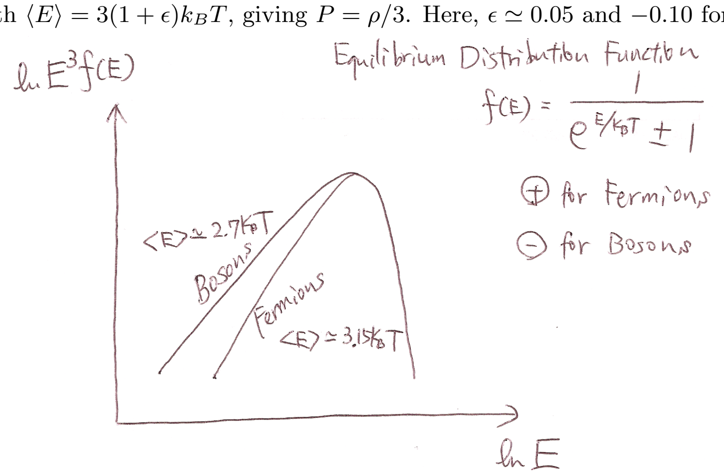
2. **Matter-dominated (MD) era**. A matter component (baryons, cold dark matter, or any other non-relativistic particles) has a negligible pressure compared to its energy density, $P_M \ll \rho_M$, which gives $\rho_M(t) \propto 1/a^3(t)$, or $n_M = 3$. We thus obtain

$$a_{\rm MD}(t) \propto t^{2/3}. \tag{1.17}$$

[§]Again, a "wrong" derivation, but there is an intuitive way to get this result using the equation of state for non-relativistic ideal gas (this is obviously a wrong derivation because we are about to apply non-relativistic equation of state to relativistic gas!):

$$P = nk_BT = \rho\frac{k_BT}{\langle E\rangle},$$

where n is the number density, T the temperature of gas, k_B the Boltzmann constant, and $\langle E\rangle$ the mean energy per particle. For relativistic particles in thermal equilibrium, $\langle E\rangle \approx 3k_BT$, which gives $P \approx \rho/3$. Now, actually, it turns out that the error we are making by using non-relativistic equation of state for relativistic gas cancels out precisely the error we are making by using an approximate relation $\langle E\rangle \approx 3k_BT$. This gives us the exact relation, $P = \rho/3$ for relativistic particles. More precisely, the equation of state for relativistic gas takes on the form $P = (1+\epsilon)\rho\frac{k_BT}{\langle E\rangle}$ with $\langle E\rangle = 3(1+\epsilon)k_BT$, giving $P = \rho/3$. Here, $\epsilon \simeq 0.05$ and -0.10 for Fermions and Bosons, respectively.



The expansion of the universe decelerates. With this solution, we can relate the age of the universe to the Hubble expansion rate:

$$H(t) = \frac{\dot{a}(t)}{a(t)} = \frac{2}{3t}. \tag{1.18}$$

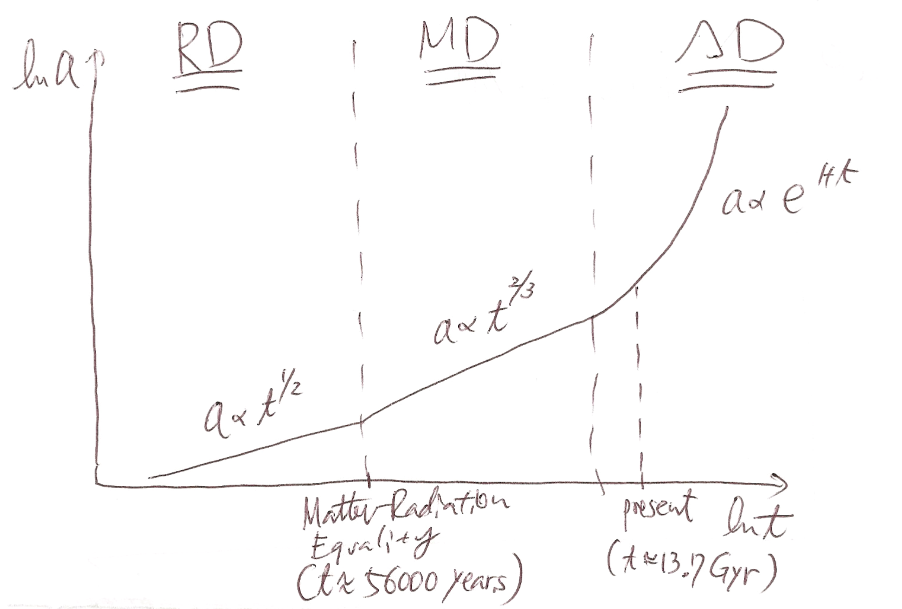
3. **Constant-energy-density-dominated (ΛD) era**. A hypothetical energy component (let's call it Λ) whose energy density is a constant over time, $n_\Lambda = 0$. In this case we cannot use equation (1.14). Going back to equation (1.12) and setting $\rho_\Lambda =$ constant, we get $\dot{a}/a =$ constant, whose solution is

$$a_{\Lambda \mathrm{D}}(t) \propto e^{Ht}, \tag{1.19}$$

where an integration constant, H, is the same as the Hubble expansion rate (which is a constant for this model). The expansion of the universe accelerates, which must mean that, according to the acceleration equation (1.4), the pressure of this energy component is *negative*. The conservation equation (1.5) tells us that such a component indeed has an enormous negative pressure given by

$$P_\Lambda = -\rho_\Lambda. \tag{1.20}$$

While this looks quite strange, we now know that something like this may actually exist in our universe, as the *current observations suggest that the present-day universe is indeed accelerating.*



1.3 Equation of State of "Dark Energy" and Density Parameters

The matter has $P_M \ll \rho_M$; the radiation has $P_R = \rho_R/3$; and Λ has $P_\Lambda = -\rho_\Lambda$. This motivates our writing the equation of state of the ith component in the following simple form:

$$P_i = w_i \rho_i. \tag{1.21}$$

Here, w_i is called the "equation of state parameter," and can depend on time (although it is usually taken to be constant).

Why this form? It is important to keep in mind that there is no fundamental reason why we should use this form. This form is often used either just for convenience, or simply for parametrizing something we do not know. At the very least, this form is exact for radiation, $w_R = 1/3$, and for Λ, $w_\Lambda = -1$. For matter, since $w_M \ll 1$, the exact value does not affect the results very much.

The equation of state parameter is almost exclusively used for parametrizing "dark energy," which is supposed to cause the observed acceleration of the universe. If we assume that w for dark energy, w_{DE}, is constant, then the current observations suggest that (Komatsu, et al., ApJS, 192, 18 (2011))

$$w_{DE} = -0.98 \pm 0.05 \text{ (68\% CL)}. \tag{1.22}$$

In other words, the energy density of dark energy is consistent with being a constant ($w_{DE} = w_\Lambda = -1$).

Determining w_{DE} with better accuracy may tell us something about the nature of dark energy, especially if $w_{DE} \neq 1$ is found with high statistical significance, as it would tell us that dark energy is something dynamical (time-dependent).

Ignoring a potential interaction between dark energy and other components in the universe (e.g., dark matter), the energy density of dark energy obeys (see equation (1.6))

$$\dot{\rho}_{DE}(t) + 3\frac{\dot{a}(t)}{a(t)}\left(1 + w_{\rm DE}\right)\rho_{DE}(t) = 0, \tag{1.23}$$

whose solution is $\rho_{DE}(t) \propto [a(t)]^{-3(1+w_{DE})}$. On the other hand, if we do *not* assume that w_{DE} is a constant, then the energy density of dark energy obeys

$$\dot{\rho}_{DE}(t) + 3\frac{\dot{a}(t)}{a(t)}\left[1 + w_{\rm DE}(t)\right]\rho_{\rm DE}(t) = 0, \tag{1.24}$$

whose solution is

$$\rho_{DE}(t) \propto e^{-3\int d\ln a[1+w_{DE}(a)]}. \tag{1.25}$$

Putting these results together, we obtain the Friedmann equation for our Universe containing radiation, matter, and dark energy (but not curvature) as

$$\frac{\dot{a}^2(t)}{a^2(t)} = H^2(t) = \frac{8\pi G}{3}\left[\rho_M(t_0)\frac{a^3(t_0)}{a^3(t)} + \rho_R(t_0)\frac{a^4(t_0)}{a^4(t)} + \rho_{DE}(t_0)e^{-3\int_{a(t_0)}^{a(t)} d\ln a[1+w_{DE}(a)]}\right], \tag{1.26}$$

where t_0 is some epoch, which is usually taken to be the present epoch.

Now, taking $t \to t_0$, we find the present-day expansion rate

$$H_0^2 \equiv H^2(t_0) = \frac{8\pi G}{3}\left[\rho_M(t_0) + \rho_R(t_0) + \rho_{DE}(t_0)\right] \equiv \frac{8\pi G}{3}\rho_c(t_0), \tag{1.27}$$

which has been determined to be $H_0 \approx 70$ km/s/Mpc. Here, $\rho_c(t_0)$ is the so-called "critical density" of the universe, which is equal to the total energy density of the universe when the universe is flat. The numerical value of the critical density is

$$\rho_c(t_0) \equiv \frac{3H_0^2}{8\pi G} = 2.775 \times 10^{11}\ h^2\ M_\odot\ \mathrm{Mpc}^{-3}. \tag{1.28}$$

The critical density provides a natural unit for the energy density of the universe, and thus it is convenient to measure all the energy densities in units of $\rho_c(t_0)$. Defining the so-called **density parameters**, Ω_i, as

$$\Omega_i \equiv \frac{\rho_i(t_0)}{\rho_c(t_0)}, \tag{1.29}$$

one can rewrite the Friedmann equation (1.26) in a compact form:

$$\boxed{\frac{H^2(t)}{H_0^2} = \Omega_M \frac{a^3(t_0)}{a^3(t)} + \Omega_R \frac{a^4(t_0)}{a^4(t)} + \Omega_{DE} e^{-3\int_{a(t_0)}^{a(t)} d\ln a[1+w_{DE}(a)]}} \tag{1.30}$$

Basically, most of the literature on cosmology (within the context of General Relativity) use this equation as the starting point.[¶] Taking $z = 0$, one finds that all the density parameters must sum to unity: $\sum_i \Omega_i = 1$.

In summary, the Friedmann equation is a combination of two key equations: (1) the equation describing how the universe decelerates/accelerates depending on the energy density and pressure of the constituents, and (2) the equation describing the energy conservation of the constituents. Once the Friedmann equation is given with the proper right hand side containing the energy densities of the relevant constituents of the universe, we can find $a(t)$ as a function of time easily.

[¶] An interesting possibility is that General Relativity may not be valid on cosmological scales. There are scenarios in which the form of the Friedmann equation is modified. One widely-explored example is the so-called Dvali-Gabadadze-Porrati (DGP) model (Dvali, Gabadadze & Porrati, Phys. Lett. B485, 208 (2000)). In this scenario, the Friedmann equation is modified to:

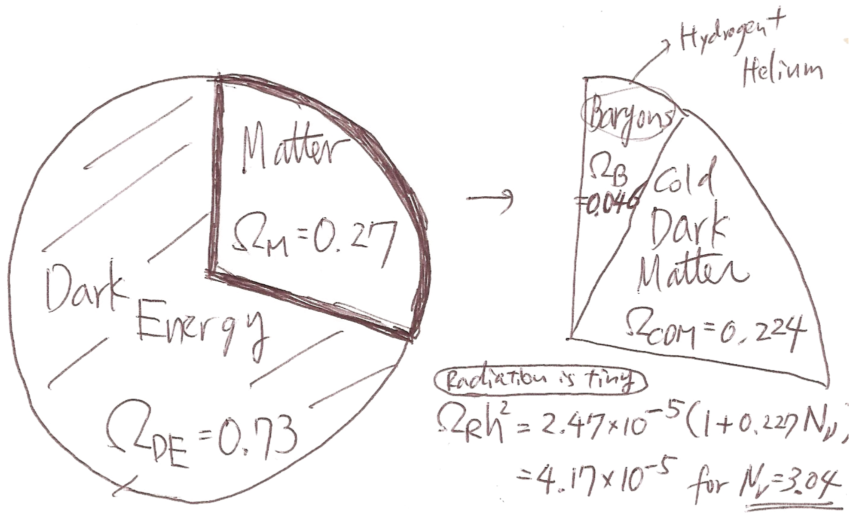
$$H^2(t) - \frac{H(t)}{r_c} = \frac{8\pi G}{3}\sum_i \rho_i(t),$$

where r_c is some length scale below which General Relativity is restored. (For $r \ll r_c$, the potential is given by $-G_N m/r$ where G_N is the ordinary Newtonian gravitational constant. For $r \gg r_c$, the potential is modified to $-G_5 m/r^2$ and decays faster. G_5 is the gravitational strength in the 5th dimension.) This model has attracted a huge attention of the cosmology community, as it was shown that this modified Friedmann equation gives an accelerating expansion without dark energy. Namely, even when the right hand side contains only matter, the solution for this equation can still exhibit an accelerating expansion. As this is a quadratic equation for $H(t)$, we can solve it and find

$$H(t) = \frac{1}{2}\left(\frac{1}{r_c} \pm \sqrt{\frac{1}{r_c^2} + \frac{32\pi G}{3}\rho_M(t)}\right).$$

At late times when $\rho(t)$ becomes negligible compared to the other term, one of the solutions is given by $a(t) \propto e^{t/r_c}$, i.e., an exponential, accelerated expansion.

At present, the radiation is totally negligible compared to matter, $\Omega_R/\Omega_M \simeq 1/3250$, and the dark energy density is about 3 times as large as the matter density, $\Omega_{DE}/\Omega_M \simeq 2.7$ (with $\Omega_M \simeq 0.27$ and $\Omega_{DE} \simeq 0.73$).



1.4 Redshift

As the universe expands, the wavelength of light, λ, is stretched linearly:

$$\lambda(t) \propto a(t), \tag{1.31}$$

which implies that photons lose energy as $E(t) \propto 1/a(t)$.

This is something one can observe, by comparing, for example, the observed wavelength of a hydrogen line to the rest-frame wavelength that we know from the laboratory experiment. We often use the **redshift**, z, to quantify the stretching of the wavelength:

$$1 + z \equiv \frac{\lambda(t_0)}{\lambda(t_{\text{emitted}})}. \tag{1.32}$$

The present-day corresponds to $z = 0$.

Using equation (1.31), we can relate the observed redshift to the ratio of the scale factors:

$$1 + z = \frac{a(t_0)}{a(t_{\text{emitted}})}. \tag{1.33}$$

Using this result in the Friedmann equation (1.30), we obtain the most-widely-used form of the Friedmann equation:

$$\boxed{\frac{H^2(z)}{H_0^2} = \Omega_M(1+z)^3 + \Omega_R(1+z)^4 + \Omega_{DE} e^{3\int_0^z d\ln(1+z)[1+w_{DE}(z)]}} \tag{1.34}$$

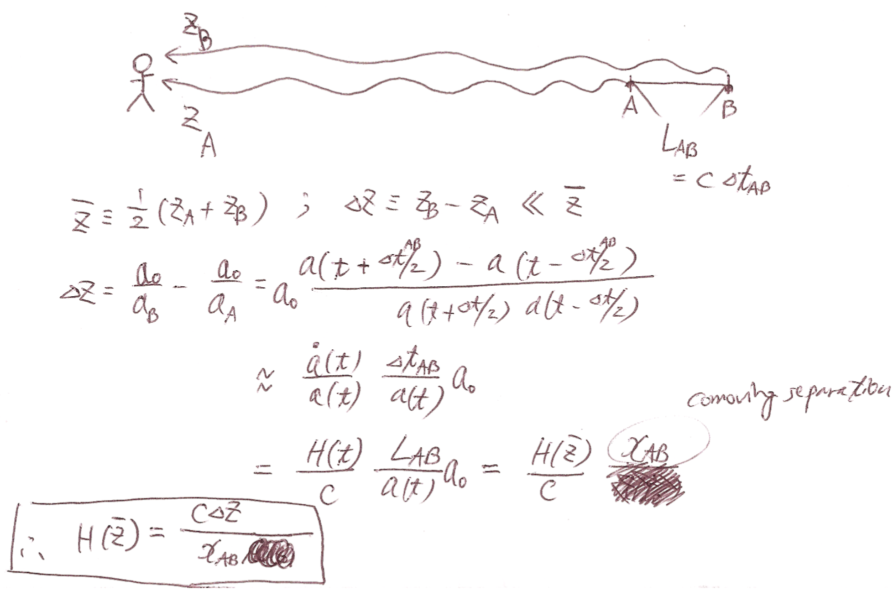
From this result, it follows that the **best way to determine the equation of state of dark energy is to measure $H(z)$ over a wide range of z.** If we can only measure the expansion rates at $z \ll 1$, then Taylor expansion of equation (1.34) with $\Omega_R \ll \Omega_M$ and $\Omega_{\rm DE} \simeq 1 - \Omega_M$ gives

$$\frac{H^2(z \ll 1)}{H_0^2} \approx 1 + 3\Omega_M z + 3(1 + w_{DE})(1 - \Omega_M)z. \tag{1.35}$$

As we know from observations that $|1 + w_{DE}|$ is small (of order 10^{-1} or less), the third term is tiny compared to other terms, making it difficult to measure w_{DE}. This is why we need to measure $H(z)$ over a wide redshift range.

1.5 Alcock-Paczyński Test

We have learned that, in order to determine w_{DE}, we need to measure $H(z)$ over a wide redshift range. But, how? *In principle*, one can measure $H(z)$ in the following way.



Consider two points A and B, which are separated by L_{AB} along the line of sight. Both points are on the Hubble flow. The tip (A) and tail (B) emit light, which we observe to be at redshifts of z_A and z_B, respectively. These are our observables. Now we show that the redshift difference, $\Delta z \equiv z_B - z_A$, is somehow related to the Hubble expansion rate at $\bar z = (z_A + z_B)/2$. Using $a(t_A) = a(\bar t + \Delta t_{AB}/2) \approx a(\bar t) + \dot a(\bar t)\Delta t_{AB}/2$ and similarly $a(t_B) = a(\bar t) - \dot a(\bar t)\Delta t_{AB}/2$, we find

$$\Delta z = z_B - z_A = \frac{a_0}{a(t_B)} - \frac{a_0}{a(t_A)} \approx a_0 \frac{\dot a(\bar t)}{a^2(\bar t)} \delta t_{AB},$$

where Δt_{AB} is the time the light takes to go from B to A, which is equal to L_{AB}/c. Therefore

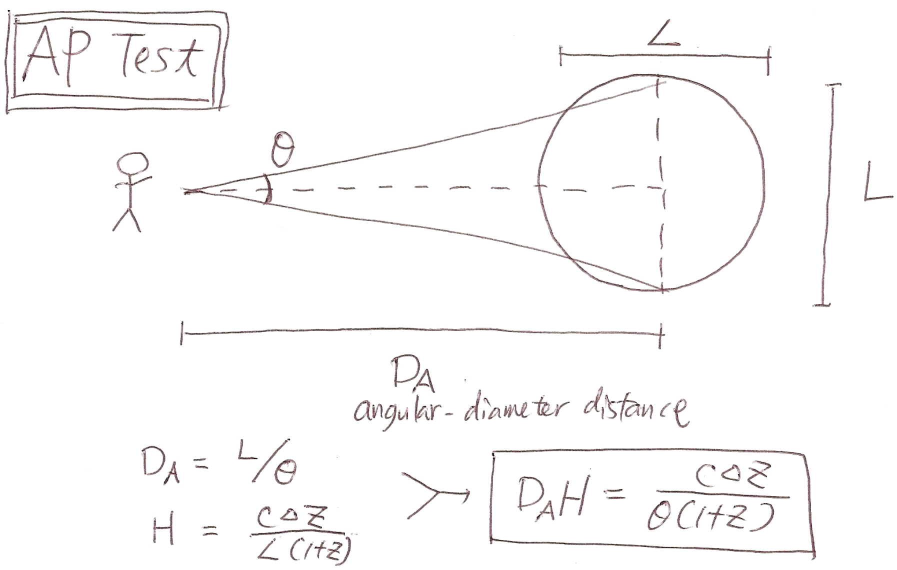
$$\Delta z = \frac{H(\bar z)}{c} \frac{a_0 L_{AB}}{a(\bar z)}. \tag{1.36}$$

Here, $a_0 L_{AB}/a(\bar{z}) = x_{AB}$ is a comoving separation (which is time-independent; a_0 is the scale factor at present). Rewriting the result in terms of $H(\bar{z})$ and x_{AB}, we finally find the relation between what we want to determine, $H(\bar{z})$, and the observable, Δz, as

$$\boxed{H(\bar{z}) = \frac{c\Delta z}{x_{AB}}} \tag{1.37}$$

This is a beautiful result, but has one problem. In order to use this method, we need to know the intrinsic comoving separation, x_{AB}, which is not always known. (As a matter of fact, x_{AB} is not known for most cases.) In other words, this method works if we have the *standard ruler*, for which the intrinsic size is known.

There is another way, which does not require the prior knowledge of the size. This was proposed first by Charles Alcock and Bohdan Paczyński in 1979 (Alcock & Paczyński, Nature, 281, 358 (1979)), and is known as the "Alcock-Paczyński test." While this method does not require the prior knowledge of the intrinsic size, it does still require an ideal situation: a collection of test particles (e.g., galaxies) which spatial distribution is spherically symmetric.



Consider a spherical distribution with a diameter of L. By measuring the redshift difference along the line of sight, we find $H(z) = c\Delta z/[L(1+z)]$. On the other hand, the angular extension of this spherical distribution of the sky, θ, is related to the intrinsic physical size, L, as

$$\theta = \frac{L}{D_A(z)}, \tag{1.38}$$

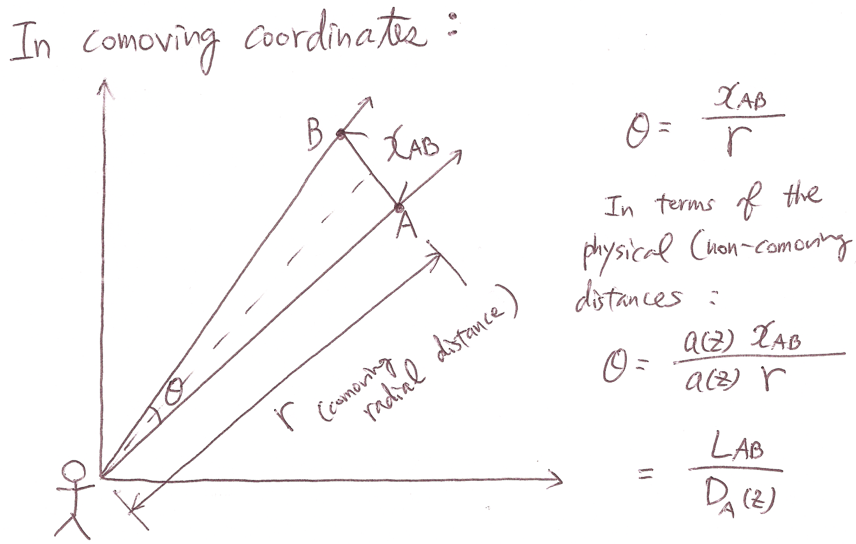
where $D_A(z)$ is the **angular diameter distance**. Therefore, by measuring the angular extension, θ, and the redshift difference, Δz, and combining them, we obtain

$$\boxed{D_A(z)H(z) = \frac{c\Delta z}{\theta(1+z)}} \tag{1.39}$$

The right hand side only contains the observables, and thus the Alcock-Paczyński test allows us to determine $D_A H$.

A challenge for this method is to find objects whose distribution is spherically symmetric. There is one known example, which is the distribution of the large-scale structure. We will come back to this later.

1.6 Angular Diameter Distance



In order to utilize the AP test (equation (1.39)), we need to relate the angular diameter distance, $D_A(z)$, to cosmological models. This can be done by realizing that the angular diameter distance is equal to the comoving radial distance times the scale factor:

$$D_A(z) = a(z)r = \frac{a_0 r}{1+z}. \tag{1.40}$$

Then, we can calculate $r(z)$ as follows. Along the path of photons coming toward us in a flat universe, we have $cdt = a(t)dr$.[‖] Therefore,

$$r = c\int_t^{t_0} \frac{dt'}{a(t')} = c\int_a^{a_0} \frac{da'}{(a')^2 H(a')} = c\int_0^z \frac{dz'}{a_0 H(z')}, \tag{1.41}$$

with $H(z)$ given by the Friedmann equation (1.34). The angular diameter distance is

$$\boxed{D_A(z) = \frac{c}{1+z}\int_0^z \frac{dz'}{H(z')}} \tag{1.42}$$

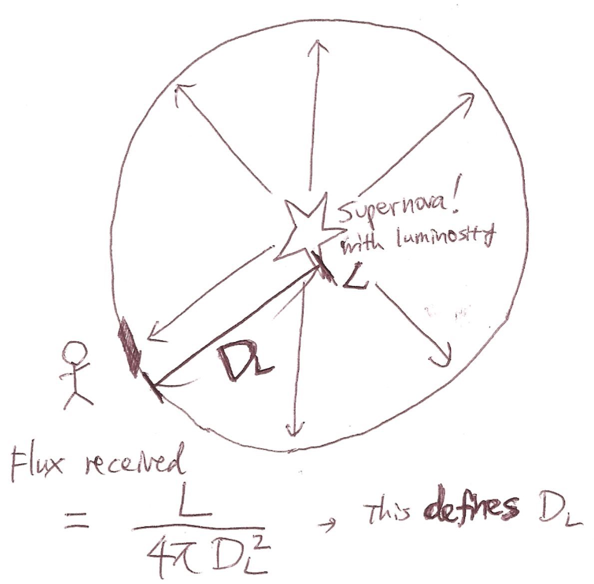
‖In a curved space, we have $cdt = a(t)\frac{dr}{\sqrt{1-Kr^2}}$ where $K = +1$ and -1 for positively and negatively curved spaces, respectively.

Using this result in equation (1.39), we find that the Alcock-Paczyński test provides (in a flat universe):

$$H(z)\int_0^z \frac{dz'}{H(z')} = \frac{\Delta z}{\theta}. \tag{1.43}$$

As the angular diameter distance is an integral of $1/H(z)$, it is less sensitive to the equation of state of dark energy. However, if we have many measurements of $D_A(z)$ at various redshifts, we can effectively differentiate $D_A(z)$ with respect to z, obtaining a measurement of $1/H(z)$. While we have not yet entered the era where we can do this with the angular diameter distance, we have been able to do this using the luminosity distances measured out to distant Type Ia supernovae, as described next.

1.7 Luminosity Distance



Perhaps the best known method for measuring distances in cosmology is the **luminosity distance**. This builds on a simple idea: the farther objects look dimmer. More specifically, the energy we receive per unit time per unit area, which is usually known as the "flux," is related to the intrinsic luminosity of the light source as $F = \frac{L}{4\pi D_L^2}$, where D_L is the luminosity distance. This equation defines D_L:

$$D_L \equiv \sqrt{\frac{L}{4\pi F}}. \tag{1.44}$$

The flux F is our observable; thus, in order to use this method, we need to have the light sources whose intrinsic luminosity is known, i.e., the *standard candles*.

Type Ia supernovae, which are believe to be thermonuclear explosion of white dwarf stars, are known to exhibit similar peak luminosities (after a few corrections), and have been used as the primary standard candles in the cosmology community. In fact, it was the observation of Type Ia supernovae which led to the discovery of the acceleration of the universe (Riess et al., AJ, 116, 1009 (1998); Perlmutter et al., ApJ, 517, 565 (1999)).

Now, we must relate D_L to cosmological models. To do this, we first note that the energy emitted by a supernova is diluted by the surface area, which is $4\pi r^2 a_0^2$. Second, each photon emitted by a supernova loses energy as $E \propto a/a_0 = 1/(1+z)$. Third, the rate at which photons are received per unit time is dilated by a factor of $a/a_0 = 1/(1+z)$ compared to the rate at which the light was emitted by a supernova. (I.e., we receive fewer photons per second at our location, relative to the number of photons emitted per second at the source). This leads to the cosmological inverse-square-law formula:

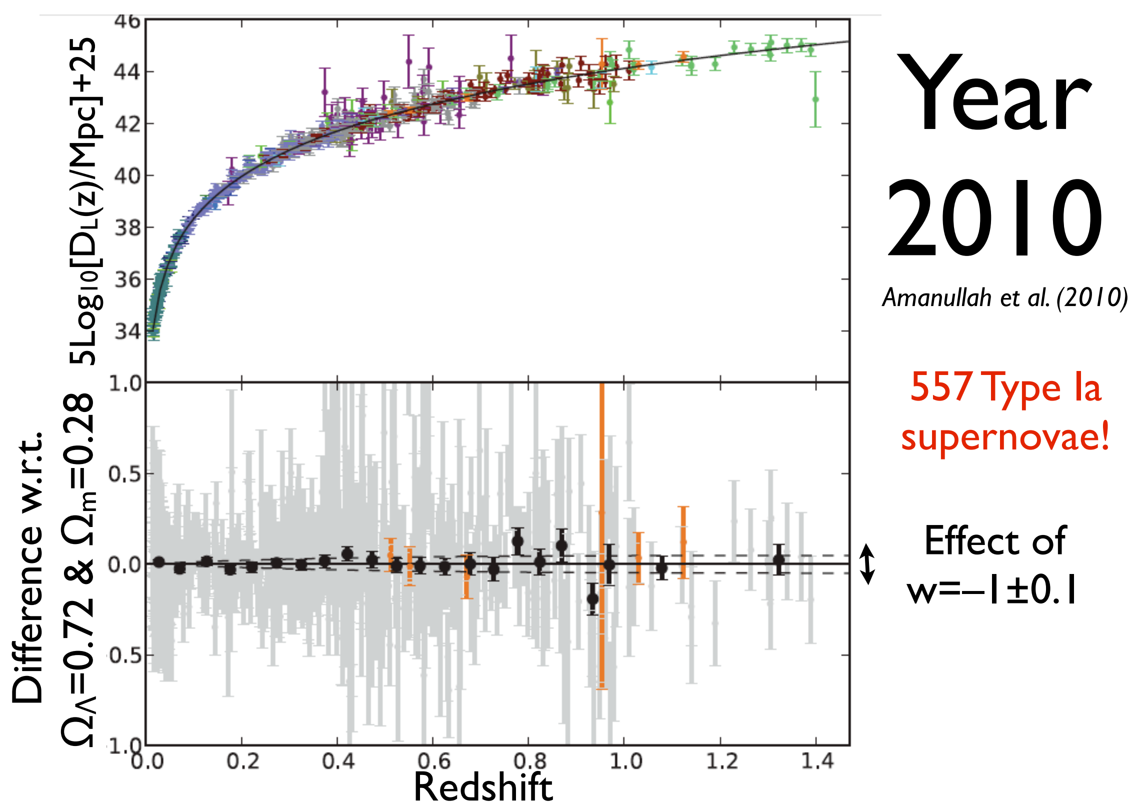
$$F = \frac{L/(1+z)^2}{4\pi r^2 a_0^2}. \tag{1.45}$$

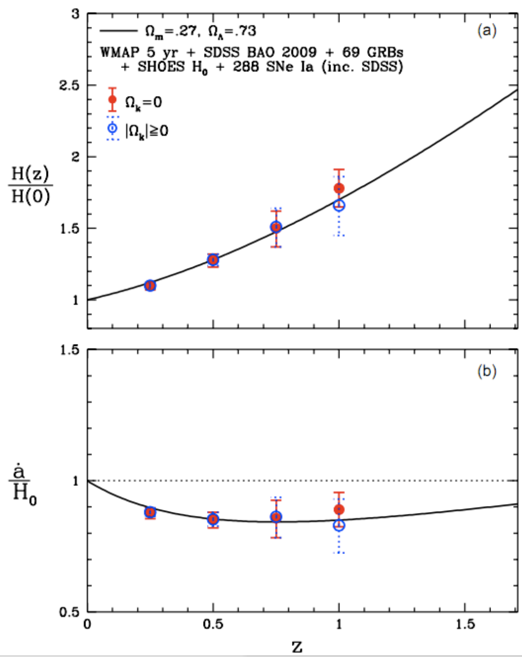
Comparing this formula to the definition of D_L above, we conclude that

$$\boxed{D_L(z) = a_0(1+z)r = (1+z)^2 D_A(z)} \tag{1.46}$$

This relation, $D_L(z) = (1+z)^2 D_A(z)$, is exact, and does not depend on cosmological models.

As of today, hundreds of distant Type Ia supernovae have been observed, and $D_L(z)$ has been determined out to $z = 1.7$. One can fit the data to $D_L = c(1+z)\int dz/H(z)$ and constrain the cosmological parameters such as Ω_M and w_{DE}.





- Constraints on H(z) as determined by differentiating the luminosity distance data of 288 Type Ia supernovae and 69 Gamma-ray Bursts
  - Ref: Yun Wang, Phys. Rev. D, 80, 123525 (2009)

One may also differentiate the D_L data with respect to z, and see if one can measure $1/H(z)$.

1.8 Effects of Changing Effective Relativistic Degrees of Freedom

The expansion rate during the matter era (well after the matter-radiation equality, but well before the dark energy domination) is given by $H(z)/H_0 = \sqrt{\Omega_M}(1+z)^{3/2}$. However, in general, we should be careful about applying this formula blindly to arbitrarily high redshifts, as some "matter" would start behaving as if they were radiation (massless particles) when the kinetic energy of the particles exceeds the rest mass energy. This can happen because the universe was hotter when it was younger.

1.8.1 Neutrinos

A good example is the effect of massive neutrinos on the expansion rate of the universe. When the mass of neutrinos, m_ν, is larger than roughly $3k_BT_\nu$ (T_ν is the neutrino temperature, which is equal to $(4/11)^{1/3}$ of the photon temperature in the standard scenario for $T_\nu \ll 1$ MeV), neutrinos behave as non-relativistic particles. In the opposite limit, they behave as relativistic particles.

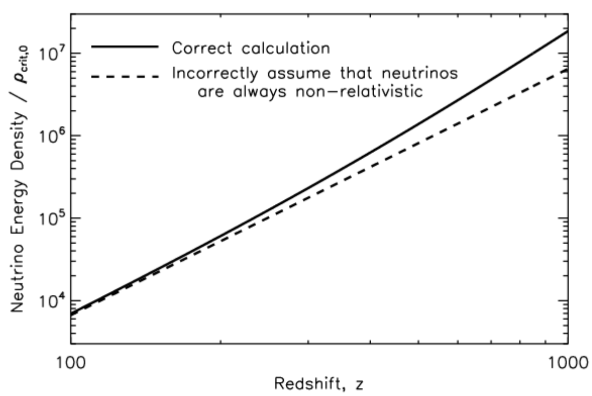
While we do know that neutrinos have finite masses, we do not know the actual values of the masses. The current limit suggests that the sum of the masses of 3 neutrino species is greater than 0.05 eV, but smaller than 0.6 eV (or $m_\nu < 0.2$ eV for each of the 3 species if we assume that all neutrino species have equal masses). As 1 eV corresponds to 1.16×10^4 K, neutrinos could have become non-relativistic when the neutrino temperature fell below 770 K, or the redshift less than 400. At the very least, one of the neutrino species must have become non-relativistic when the neutrino temperature fell below 190 K (or $z < 100$).

As the expansion rate is solely determined by the energy density of the constituents (in a flat universe), all we need to calculate is the energy density of neutrinos. As neutrinos are Fermions and were in thermal equilibrium in the early universe, their distribution function is given by the Fermi-Dirac distribution. Also, as they decoupled from the plasma when neutrinos were still highly relativistic (when the temperature of the universe was about 2 MeV$\sim$ 20 billion K), their distribution function will remain the Fermi-Dirac distribution for **massless** particles, **even after neutrinos became non-relativistic**.

With this information, we calculate the energy density of neutrinos (in natural units) by integrating the distribution function times energy per particle:**

$$\rho_\nu(z) = (1+z)^4 \int \frac{q^2 dq}{\pi^2} \frac{\sum_i \sqrt{q^2 + m_{\nu,i}^2/(1+z)^2}}{e^{q/T_{\nu 0}}+1}. \tag{1.47}$$

This can be evaluated numerically, and the result is shown for $m_\nu = 0.2$ below.



**This is derived as follows. The energy density of 1 neutrino species is given (in natural units) by

$$\rho_{\nu,i} = 2\int \frac{d^3p}{(2\pi)^3}\frac{E_i(p)}{e^{p/T_\nu}+1} = \int \frac{p^2 dp}{\pi^2}\frac{\sqrt{p^2+m_{\nu,i}^2}}{e^{p/T_\nu}+1}.$$

Defining $q \equiv p/(1+z)$, we rewrite this equation as

$$\rho_{\nu,i} = (1+z)^4 \int \frac{q^2 dq}{\pi^2}\frac{\sqrt{q^2+m_{\nu,i}^2/(1+z)^2}}{e^{q(1+z)/T_\nu}+1}.$$

Finally, using $(1+z)/T_\nu = 1/T_{\nu 0}$, where $T_{\nu 0}$ is the present-day temperature of neutrino, we obtain

$$\rho_{\nu,i} = (1+z)^4 \int \frac{q^2 dq}{\pi^2}\frac{\sqrt{q^2+m_{\nu,i}^2/(1+z)^2}}{e^{q/T_{\nu 0}}+1}.$$

1.8.2 General Consideration

As we go farther back in time, various other particles, such as electrons and positrons, become relativistic, and these effects must be taken into account when calculating the expansion rate.

More specifically, when the temperature of the universe was higher than above 1 MeV, but lower than 2 times the muon mass (105.7 MeV), the relativistic particles included photons, 3 neutrino species, electrons, and positrons. And, they all shared the same temperature, T. The energy density can be found easily by integrating the corresponding distribution functions times energy per particle. In natural units, we find

$$\rho_\gamma = 2\int \frac{p^3 dp}{2\pi^2}\frac{1}{e^{p/T}-1} = \frac{\pi^2}{15}T^4, \tag{1.48}$$

$$\rho_\nu = 6\int \frac{p^3 dp}{2\pi^2}\frac{1}{e^{p/T}+1} = \frac{7\pi^2}{40}T^4, \tag{1.49}$$

$$\rho_{e^\pm} = 4\int \frac{p^3 dp}{2\pi^2}\frac{1}{e^{p/T}+1} = \frac{7\pi^2}{60}T^4. \tag{1.50}$$

Here, "2" for photons is the number of helicity states (i.e., left and right circular polarization states); "6" for neutrinos is the number of helicity state (1; just left-handed neutrinos) times the number of neutrino species (3) times 2 because we count both neutrinos and anti-neutrinos; and "4" for electrons/positrons is the number of spin states (2; up and down) times 2 because we count both electrons and positrons.

It is more common to define the "effective number of relativistic degrees of freedom" by writing the total radiation energy as

$$\rho_R = \rho_\gamma + \rho_\nu + \rho_{e^\pm} = \frac{\pi^2}{30}g_* T^4, \tag{1.51}$$

where

$$g_* = 2 + \frac{7}{8}(6+4) = \frac{43}{4}. \tag{1.52}$$

With this, the expansion rate during the radiation era is given by

$$H^2 = \frac{8\pi G}{3}\rho_R = \frac{4\pi^3 G}{45}g_* T^4. \tag{1.53}$$

Therefore, when we calculate the expansion rate during the radiation era, we must be careful about how many relativistic degrees of freedom we have in the universe at a given time. For $g_* = 43/4$, we obtain

$$\frac{1}{H(T)} = 1.48\left(\frac{1\text{ MeV}}{T}\right)^2 \text{ sec.} \tag{1.54}$$

As the age of the universe during the radiation era is $t = 1/(2H)$, we also have

$$\boxed{t = \frac{1}{2H(T)} = 0.74\left(\frac{1\text{ MeV}}{T}\right)^2 \text{ sec}} \tag{1.55}$$

Again, this formula is valid only for 1 MeV $< T \ll 200$ MeV. Above this temperature, we will need to count muons as relativistic particles, etc.

PROBLEM SET 1

1.1 Expansion of the Universe

In this section, we will use Einstein's General Relativity to derive the equations that describe the expanding universe. Einstein's General Relativity describes the evolution of gravitational fields for a given source of energy density, momentum, and stress (e.g., pressure). Schematically,

$$[\text{Curvature of Space-time}] = \frac{8\pi G}{c^4}[\text{Energy density, Momentum, and Stress}]$$

Here, the dimension of "curvature of space-time" is $1/(\text{length})^2$, as the curvature is usually defined as the second derivative of a function with respect to independent variables, and for our application the independent variables are space-time coordinates: $x^\mu = (ct, x^1, x^2, x^3)$ for $\mu = 0$, 1, 2, 3.

1.1.1 Space-time Curvature: Left Hand Side of Einstein's Equation

The coefficient on the right hand side, $8\pi G/c^4$, is chosen such that Einstein's gravitational field equations reduce to the familiar Poisson equation when gravitational fields are weak and static, and the space is not expanding: $\nabla^2\phi_N = 4\pi G\rho_M$, where ϕ_N is the usual Newtonian potential, and ρ_M is the mass density. Let us rewrite it in the following suggestive form:

$$\nabla^2\left(2\frac{\phi_N}{c^2}\right) = \frac{8\pi G}{c^4}(\rho_M c^2).$$

Here, as ϕ_N/c^2 is dimensionless, and thus the left hand side has the dimension of curvature, i.e., $1/(\text{length})^2$. The right hand side contains $\rho_M c^2$, which is energy density; thus, G/c^4 correctly converts energy density into curvature. Now, this equation tells us something Newton did not know but Einstein finally figured out: the second derivative of the dimensionless Newtonian potential times 2 with respect to space coordinates is the curvature of space, and mass deforms space.

In order to calculate curvature of space-time, we need to know how to calculate a distance between two points. Of course, everyone knows that, in Cartesian coordinates, the distance between two points in flat space separated by $dx^i = (dx^1, dx^2, dx^3)$ is given by $dl = \sqrt{(dx^1)^2 + (dx^2)^2 + (dx^3)^2}$, or

$$dl^2 = \sum_{i=1}^{3}\sum_{j=1}^{3}\delta_{ij}dx^i dx^j, \tag{1.56}$$

where $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$. Since space is flat, the curvature of this space is zero. This is a consequence of the coefficients of $dx^i dx^j$ on the right hand side of equation (1.56) being independent of coordinates. In general, when space is not flat but curved, the distance between two points can be written as

$$dl^2 = \sum_{i=1}^{3}\sum_{j=1}^{3}g_{ij}(x)dx^i dx^j, \tag{1.57}$$

where $g_{ij}(x)$ is known as the **metric tensor**. Schematically, the curvature of space is given by the second derivatives of the metric tensor with respect to space coordinates:

$$\text{Curvature of Space} \sim \frac{\partial^2 g_{ij}}{\partial x^k \partial x^l}.$$

In General Relativity, we extend this to the curvature of *space-time*. The distance between two points in space and time separated by $dx^\mu = (cdt, dx^1, dx^2, dx^3)$ is given by

$$ds^2 = \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g_{\mu\nu}(x) dx^\mu dx^\nu, \tag{1.58}$$

and

$$\text{Curvature of Space-time} \sim \frac{\partial^2 g_{\mu\nu}}{\partial x^\mu \partial x^\nu}.$$

Now, let us get into the gory details! The precise definition of space-time curvature, known as the **Riemann curvature tensor**, is given by††

$$R^\mu_{\nu\rho\sigma} \equiv \frac{\partial \Gamma^\mu_{\nu\sigma}}{\partial x^\rho} - \frac{\partial \Gamma^\mu_{\nu\rho}}{\partial x^\sigma} + \sum_\alpha \Gamma^\alpha_{\nu\sigma}\Gamma^\mu_{\alpha\rho} - \sum_\alpha \Gamma^\alpha_{\nu\rho}\Gamma^\mu_{\alpha\sigma}, \tag{1.59}$$

where Γ is the so-called Christoffel symbol, also known as the **affine connection**:

$$\Gamma^\mu_{\nu\rho} \equiv \frac{1}{2}\sum_\alpha g^{\mu\alpha}\left(\frac{\partial g_{\alpha\rho}}{\partial x^\nu} + \frac{\partial g_{\nu\alpha}}{\partial x^\rho} - \frac{\partial g_{\nu\rho}}{\partial x^\alpha}\right). \tag{1.60}$$

The metric tensor with the superscripts, $g^{\mu\alpha}$, is the inverse of the metric tensor, in the sense that

$$\sum_\alpha g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu,$$

where $\delta^\mu_\nu = 1$ for $\mu = \nu$ and zero otherwise.

Question 1.1: For an expanding universe with flat space, the distance between two points in space is given by, perhaps not surprisingly,

$$dl^2 = a^2(t)\sum_{i=1}^{3}\sum_{j=1}^{3} \delta_{ij} dx^i dx^j, \tag{1.61}$$

where x denotes **comoving coordinates**. The scale factor, $a(t)$, depends only on time t. Then, the distance between two points in *space-time* is given by

$$\begin{aligned} ds^2 &= -c^2dt^2 + dl^2 \\ &= -c^2dt^2 + a^2(t)\sum_{i=1}^{3}\sum_{j=1}^{3} \delta_{ij} dx^i dx^j. \end{aligned} \tag{1.62}$$

††Different definitions of curvature are used in the literature. Here, we follow the definition used by Misner, Thorne, and Wheeler, "*Gravitation*" (1973). Steven Weinberg's recent textbook, "*Cosmology*," uses the opposite sign.

Non-zero components of the metric tensor are

$$g_{00} = -1; \quad g_{ii} = a^2(t) \text{ for } i = 1, 2, 3,$$

and those of the corresponding inverse are

$$g^{00} = -1; \quad g^{ii} = \frac{1}{a^2(t)} \text{ for } i = 1, 2, 3.$$

This metric is known as the **Robertson-Walker metric** (for flat space), and describes the distance between two points in space-time of a homogeneous, isotropic, and expanding universe. For this metric, non-zero components of the affine connection are Γ^i_{j0} and Γ^0_{ij}. Calculate Γ^i_{j0} and Γ^0_{ij}. The answers will contain a, $\dot{a}/c$, and δ_{ij}. Once again, our space-time coordinates are $x^\mu = (ct, x^1, x^2, x^3)$.

Question 1.2: Einstein's field equations do not use all the components of the Riemann tensor, but only use a part of it. Specifically, they will use the so-called **Ricci tensor**:

$$\begin{aligned} R_{\mu\nu} &\equiv \sum_\alpha R^\alpha_{\mu\alpha\nu} \\ &= \sum_\alpha \left(\frac{\partial \Gamma^\alpha_{\mu\nu}}{\partial x^\alpha} - \frac{\partial \Gamma^\alpha_{\mu\alpha}}{\partial x^\nu} \right) + \sum_{\alpha\beta} \left(\Gamma^\beta_{\mu\nu} \Gamma^\alpha_{\beta\alpha} - \Gamma^\beta_{\mu\alpha} \Gamma^\alpha_{\beta\nu} \right), \end{aligned} \tag{1.63}$$

and the **Ricci scalar**:

$$R \equiv \sum_{\mu\nu} g^{\mu\nu} R_{\mu\nu}. \tag{1.64}$$

For the above flat Robertson-Walker metric, non-zero components of the Ricci tensor are R_{00} and R_{ij}. Calculate R_{00}, R_{ij}, and R. The answers will contain a, $\dot{a}/c$, $\ddot{a}/c^2$, and/or δ_{ij}.

Question 1.3: The left hand side of Einstein's equation is called the **Einstein tensor**, denoted by $G_{\mu\nu}$, and is defined as

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \tag{1.65}$$

Calculate G_{00} and G_{ij}.

1.1.2 Stress-Energy Tensor: Right Hand Side of Einstein's Equation

The precise form of Einstein's field equation is

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{1.66}$$

where $T_{\mu\nu}$ is called the **stress-energy tensor** (also sometimes called "energy-momentum tensor"). As the name suggests, the components of $T_{\mu\nu}$ represent the following quantities:

- T_{00}: Energy density,

- T_{0i}: Momentum, and
- T_{ij}: Stress (which includes pressure, viscosity, and heat conduction).

For a perfect fluid, the stress-energy tensor takes on the following specific form:

$$T_{\mu\nu} = Pg_{\mu\nu} + (\rho + P)\,\frac{(\sum_\alpha g_{\mu\alpha}u^\alpha)(\sum_\beta g_{\nu\beta}u^\beta)}{c^2}, \tag{1.67}$$

where ρ and P are the energy density and pressure, respectively, and u^μ is a four-dimensional velocity of a fluid element. The spatial components of a four velocity, u^i, represent the usual 3-dimensional velocity of a fluid element, while the temporal component, u^0, is determined by the normalization condition of u^μ:

$$g_{\mu\nu}u^\mu u^\nu = -c^2. \tag{1.68}$$

Note that the 3-dimensional velocity, u^i, does not contain the apparent motion due to the expansion of the universe, but only contains the true motion of fluid elements.

Question 1.4: In a homogeneous, isotropic, and expanding universe, fluid elements simply move along the expansion of the universe, and the 3-dimensional velocity vanishes. (In other words, fluids are comoving with expansion.) Therefore, such a fluid element has $u^i = 0$, and the normalization condition gives $u^0 = c$. Non-zero components of the stress-energy tensor are T_{00} and T_{ij}. Calculate T_{00} and T_{ij} for the flat Robertson-Walker metric and comoving fluid.

Question 1.5: Now, we are ready to obtain Einstein's equations. First, write down $G_{00} = (8\pi G/c^4)T_{00}$ and $G_{ij} = (8\pi G/c^4)T_{ij}$ for the flat Robertson-Walker metric and comoving fluid in terms of a, $\dot{a}/c$, $\ddot{a}/c^2$, and/or δ_{ij}. Then, by combining these equations, obtain the right hand side of

$$\frac{\dot{a}^2}{a^2} =$$

$$\frac{\ddot{a}}{a} =$$

The first equation is the Friedmann equation, and the second one is the acceleration equation that we have learned in class (with $c = 1$).

1.1.3 Energy Conservation

Combining the above equations for $\dot{a}/a$ and $\ddot{a}/a$ will yield the energy conservation equation, $\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0$. In other words, the energy conservation is already built into Einstein's equations.

Question 1.6: Alternatively, one can derive the energy conservation equation directly from the conservation of the stress-energy tensor. In General Relativity, the "conservation" means that the **covariant derivative** (rather than the partial derivative) of the stress-energy tensor vanishes.

$$0 = \sum_{\alpha\beta} g^{\alpha\beta}T_{\mu\alpha;\beta} \equiv \sum_{\alpha\beta} g^{\alpha\beta}\left(\frac{\partial T_{\mu\alpha}}{\partial x^\beta} - \sum_\lambda \Gamma^\lambda_{\alpha\beta}T_{\mu\lambda} - \sum_\lambda \Gamma^\lambda_{\mu\beta}T_{\lambda\alpha}\right). \tag{1.69}$$

The energy conservation equation is $\sum_{\alpha\beta} g^{\alpha\beta} T_{0\alpha;\beta} = 0$, while the momentum conservation equation is $\sum_{\alpha\beta} g^{\alpha\beta} T_{i\alpha;\beta} = 0$. Reproduce $\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0$ from $\sum_{\alpha\beta} g^{\alpha\beta} T_{0\alpha;\beta} = 0$.

1.1.4 Cosmological Redshift

Consider a non-relativistic particle, which is moving in a gravitational field with a 3-dimensional velocity of $u^i \ll c$. The other external forces (such as the electromagnetic force) are absent. According to General Relativity, the equation of motion of such a particle is

$$\frac{du^i}{d\tau} + \sum_{\alpha\beta} \Gamma^i_{\alpha\beta} u^\alpha u^\beta = 0, \tag{1.70}$$

where $d\tau \equiv \sqrt{-ds^2}/c$ is called the **proper time**. The four-dimensional velocity is given by $u^\mu = dx^\mu/d\tau$; thus, $u^0 = cdt/d\tau$ and $u^i = dx^i/d\tau$.

Question 1.7: Using the affine connection for the flat Robertson-Walker metric, rewrite the equation of motion in terms of $\dot{u}^i = du^i/dt$, $\dot{a}/a$ and u^i. Show how u^i changes with the scale factor, $a(t)$.

Chapter 2

Cosmic Microwave Background

2.1 Basic Properties

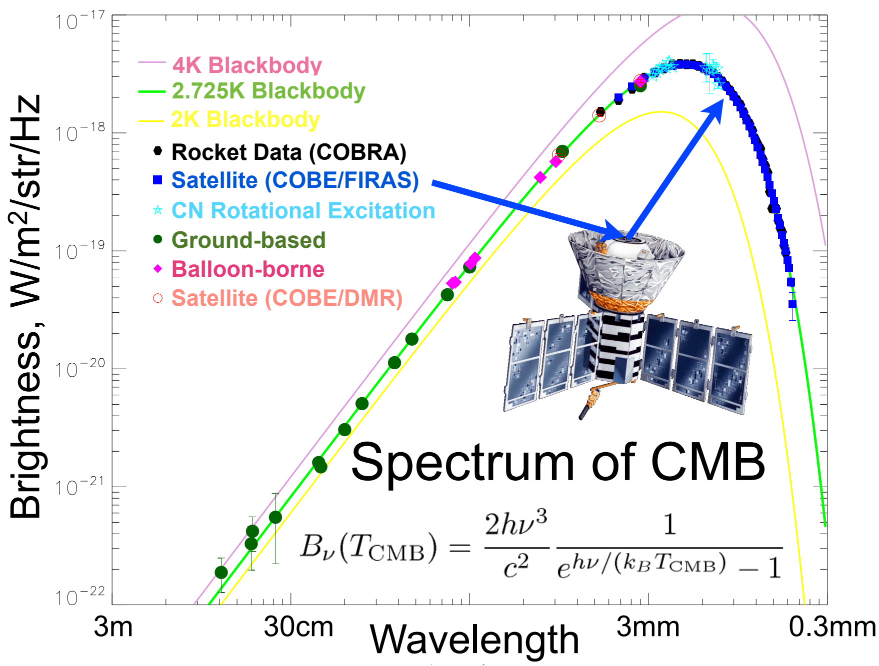
The cosmic microwave background is the oldest light that one can ever hope to measure directly. This light delivers the direct information of the physics condition of the universe when the universe was only 380,000 years old (which is $z = 1090$).

The important characteristics of the cosmic microwave background are:

- The spectrum of the microwave background is a **blackbody**:

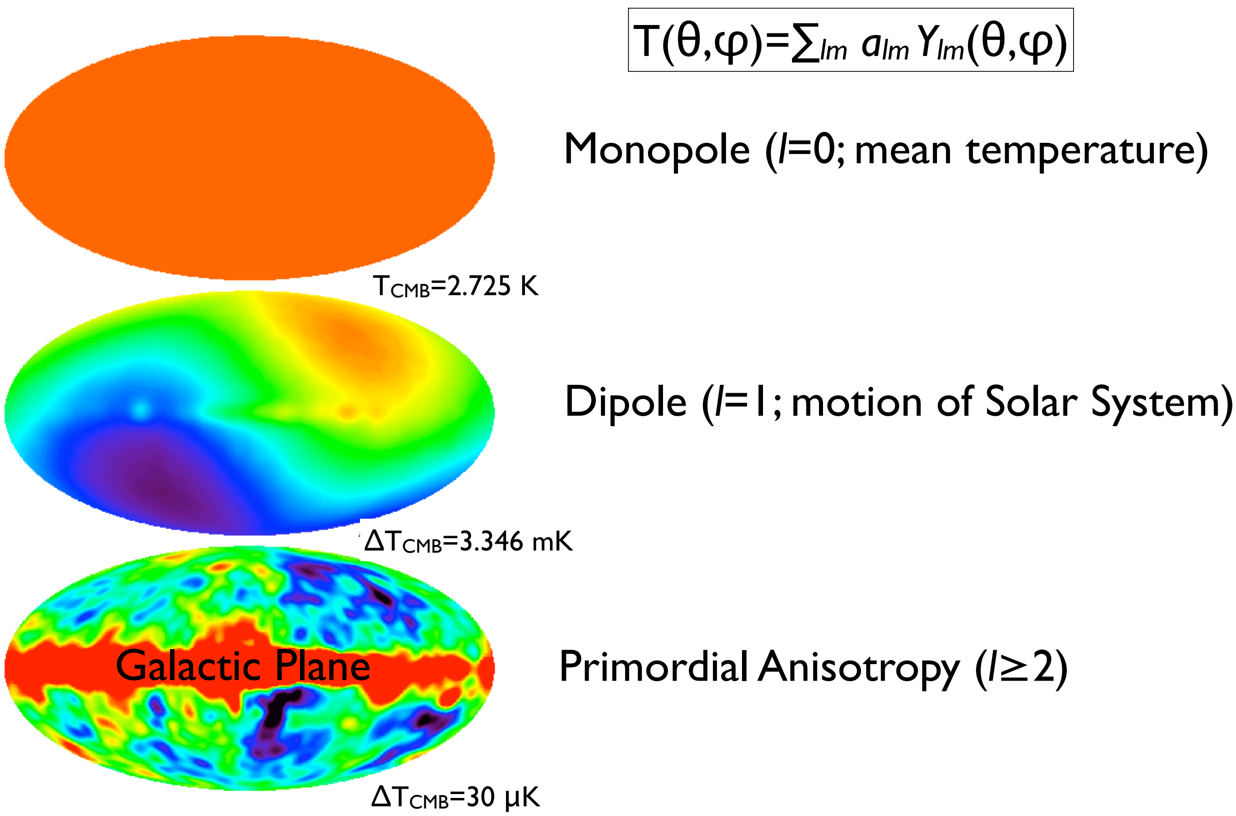
$$B_\nu(T_{\rm CMB}) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/(k_B T_{\rm CMB})-1}}, \tag{2.1}$$

  with the temperature of $T_{\rm CMB} = 2.725$ K.



- The photons of the microwave background are **numerous**: their number density is $n_{\rm CMB} = 410~{\rm cm}^{-3}$,* which is about 2 *billion* times the number density of baryons. We do not quite know why baryons are so few compared to photons.

- The distribution of the microwave background on the sky is isotropic to the precision of 10^{-3}. Most of the residual anisotropy, at the level of a few mK, is due to the motion of our Solar system with respect to the rest frame of the cosmic microwave background, and is called the *dipole anisotropy*. After removing the dipole component, we are left with the **primordial anisotropy** at the level of 10^{-5}: $\delta T_{\rm CMB} \approx 30~\mu{\rm K}$.

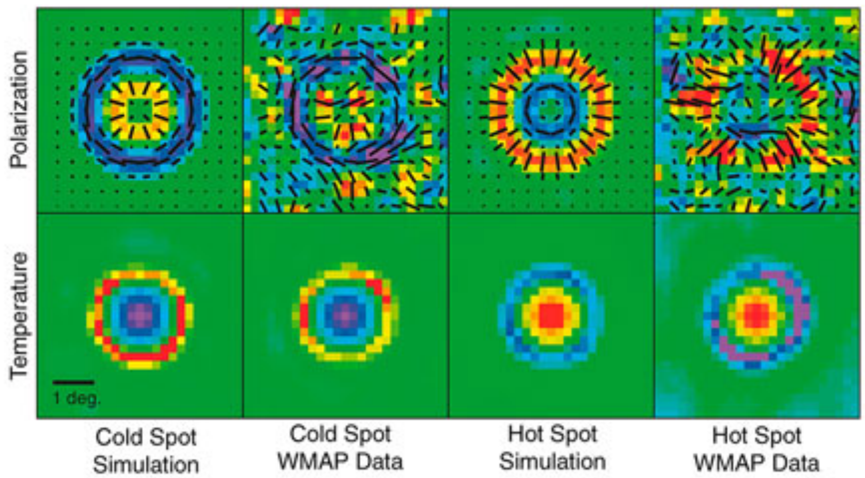


*This number can be obtained by integrating the distribution function:

$$
\begin{aligned}
n_{\rm CMB} &= 2\int \frac{d^3p}{(2\pi)^3\hbar^3}\frac{1}{e^{pc/(k_BT_{\rm CMB})}-1} \\
&= \frac{2\zeta(3)}{\pi^2}\left(\frac{k_BT_{\rm CMB}}{c\hbar}\right)^3,
\end{aligned}
$$

where $\zeta(3) \simeq 1.202$, $T_{\rm CMB} = 2.725$ K, and $k_B/(c\hbar) = 4.367~{\rm cm}^{-1}~{\rm K}^{-1}$.

- The cosmic microwave background is **polarized**, but only very weakly. The dominant polarization pattern is radial/tangential around temperature spots.



2.2 Evolution of Temperature and Entropy Conservation

2.2.1 Naive Consideration

How does $T_{\rm CMB}$ change with time? There are several ways of getting this.

1. The cosmological redshift reduces the energy of photons as $E \propto 1/a(t)$. The mean energy per particle, $\langle E \rangle$, is[†]

$$\langle E \rangle = \frac{\rho_{\rm CMB}}{n_{\rm CMB}} = \frac{2\int \frac{d^3p}{(2\pi)^3\hbar^3}\frac{pc}{e^{pc/(k_BT_{\rm CMB})}-1}}{2\int \frac{d^3p}{(2\pi)^3\hbar^3}\frac{1}{e^{pc/(k_BT_{\rm CMB})}-1}} = \frac{\frac{\pi^2}{15}}{\frac{2\zeta(3)}{\pi^2}}(k_BT_{\rm CMB}) \simeq 2.70(k_BT_{\rm CMB}). \tag{2.2}$$

   Therefore, we obtain $T_{\rm CMB} \propto 1/a(t)$.

2. Use the conservation of the number of photons, $n_{\rm CMB}V \propto n_{\rm CMB}a^3 = $ constant. This gives $n_{\rm CMB}a^3 = \frac{2\zeta(3)}{\pi^2(c\hbar)^3}(k_BT_{\rm CMB})^3a^3 = $ constant, giving $T_{\rm CMB} \propto 1/a(t)$.

3. Use the energy conservation, $\rho_{\rm CMB} = \frac{\pi^2}{15(c\hbar)^3}(k_BT_{\rm CMB})^4 \propto 1/a^4(t)$, giving $T_{\rm CMB} \propto 1/a(t)$.

These results are valid as long as there is no net creation or destruction of photons.

[†]The mean particle energy can also be found from the blackbody formula:

$$\langle E \rangle = \frac{\int_0^\infty d\nu B_\nu(T_{\rm CMB})}{\int_0^\infty d\nu \frac{B_\nu(T_{\rm CMB})}{h\nu}}.$$

2.2.2 Entropy Conservation

Is there a general formula that we can use for calculating the evolution of temperature, even when there *is* net creation or destruction of photons? The conservation of entropy provides such a formula. Roughly speaking, the entropy is proportional to the number of particles, i.e., $S \approx k_B nV = \text{constant}$. Because photons are much more numerous than matter particles, the entropy of the universe is completely dominated by that of photons (and neutrinos, whose number density is similar to the photon number density).

Let us calculate entropy. We begin with the first-law of thermodynamics, $TdS = dU + PdV$ (where U is the internal energy), and another thermodynamic equation, $VdP = HdT/T = (U + PV)dT/T$ (where $H = U + PV$ is the enthalpy). By combining these equations, we obtain

$$dS = d\left(\frac{U + PV}{T}\right). \tag{2.3}$$

Integrating, we get

$$\boxed{S = \frac{U + PV}{T} + \text{constant}} \tag{2.4}$$

The integration constant should be chosen such that $S = 0$ for the absolute zero temperature, $T = 0$. We set the integration constant to be zero. Here, both U and P contain all the particles in the universe, including both radiation and matter: $U = U_R + U_M$ and $P = P_R + P_M$.

- **Radiation**. For radiation, we have $U_R = \rho_R V$ and $P_R = \rho_R/3$. We find

  $$S_R = \frac{4\rho_R V}{3T}. \tag{2.5}$$

  Using the mean particle energy, $\langle E_R \rangle = \rho_R/n_R$, one may rewrite this result as

  $$S_R = k_B n_R V \times \frac{4\langle E_R \rangle}{3k_B T} \approx 4k_B n_R V. \tag{2.6}$$

  Therefore, indeed the entropy is given by the number of particles (times k_B). More precisely, by writing the radiation energy density as

  $$\rho_R = \frac{\pi^2}{30} g_* \frac{(k_B T)^4}{(c\hbar)^3}, \tag{2.7}$$

  we obtain

  $$S_R = k_B \left[\frac{\pi^2}{30} g_* \left(\frac{k_B T}{c\hbar}\right)^3\right] V. \tag{2.8}$$

  As the effective number of relativistic degrees of freedom, g_*, can change with time, the entropy conservation, $S_R = \text{constant}$, with $V \propto a^3(t)$ gives

  $$\boxed{T \propto \frac{1}{g_*^{1/3} a(t)}} \tag{2.9}$$

  Therefore, a simple relation such as $T \propto 1/a(t)$ holds only when the effective number of relativistic species does not change, $g_* = \text{constant}$.

- **Matter**. For matter, we have $U_M = \frac{3}{2}k_B n_M V T$[‡] and $P_M = n_M k_B T$. We find

$$S_M = \frac{5}{2}k_B n_M V. \tag{2.10}$$

Again, indeed the entropy is given by the number of particles (times k_B)[§], and $n_R \gg n_M$ (e.g., the number density of photons is 2 billion times that of baryons) guarantees that we can safely ignore the matter contribution to entropy.

2.2.3 Photon Heating due to Electron-Positron Annihilation

A good example for the temperature change due to the change in g_* is the electron-positron annihilation:

$$e^+ + e^- \to \gamma + \gamma.$$

When the temperature of the universe was greater than the rest mass energy of an electron, 0.511 MeV, the pair-creation,

$$\gamma + \gamma \to e^+ + e^-,$$

also occurred; however, when the universe cooled down below 0.511 MeV, the pair creation no longer occurred.

In addition, particles behave as if they were relativistic when the temperature is greater than $\approx m/3$; thus, electrons and positrons were sufficiently relativistic when the temperature of the universe was greater than their rest mass energy.

Now, let us apply the entropy conservation:

$$T_2 = T_1 \left(\frac{g_{*,1}}{g_{*,2}}\right)^{1/3}, \tag{2.11}$$

where T_1 and T_2 are the photon temperatures before and after the annihilation, respectively. The effective numbers of relativistic degrees of freedom are

$$\begin{aligned} g_{*,1} &= 2 + \frac{7}{8} \times 4 = \frac{11}{2}, \\ g_{*,2} &= 2, \end{aligned}$$

‡Here, we do not include the mass energy in the internal energy.

§This expression, derived from thermodynamics of ideal gas, is only approximate. More rigorous derivation using the famous Boltzmann's entropy formula, $S = k_B \ln W$, where W is the number of possible states, gives the so-called **Sackur–Tetrode equation** for non-relativistic, monatomic ideal gas:

$$S_M = k_B n_B V \left[\frac{5}{2} + \ln\left(\frac{1}{n_M \Lambda^3}\right)\right],$$

where $\Lambda \equiv \hbar\sqrt{2\pi/(m k_B T)}$ is known as the *thermal de Broglie length* and m is the particle mass. Note that this formula is valid only when $n_M \Lambda^3 \ll 1$ (which means that quantum effects are negligible).

before and after the annihilation, respectively. Therefore, we conclude that the annihilation increases the photon temperature by a factor of $(11/4)^{1/3}$:

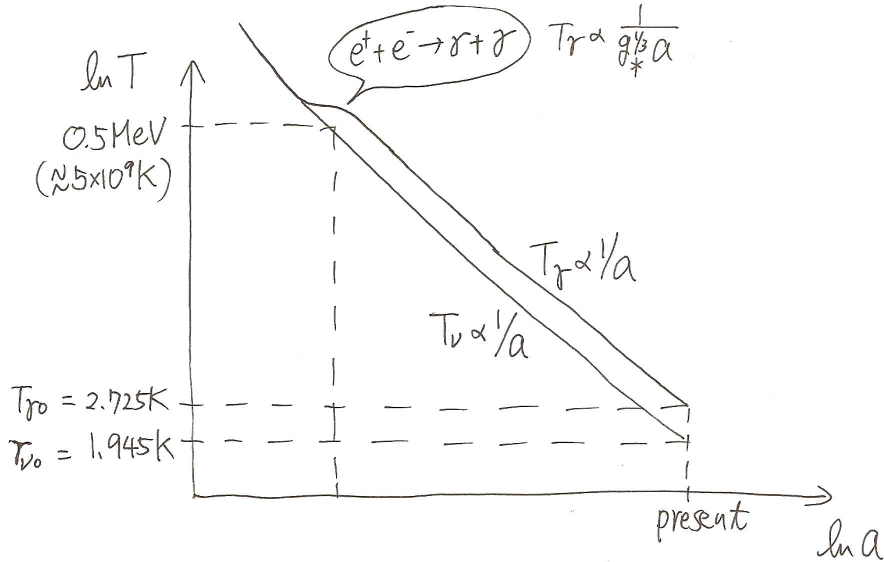
$$T_2 = T_1 \left(\frac{11}{4}\right)^{1/3}. \tag{2.12}$$

After this, the photon temperature decreased as $T \propto 1/a(t)$.

As neutrinos decoupled from the plasma before the electron-positron annihilation epoch, the annihilation did not heat neutrinos. As a result, the annihilation creates a mismatch between the neutrino temperature and photon temperature, and the mismatch is given by the above factor. Specifically, the neutrino temperature, T_ν, is **lower** than the photon temperature, T_γ, by a factor of $(4/11)^{1/3}$:

$$\boxed{T_\nu = T_\gamma \left(\frac{4}{11}\right)^{1/3}} \tag{2.13}$$

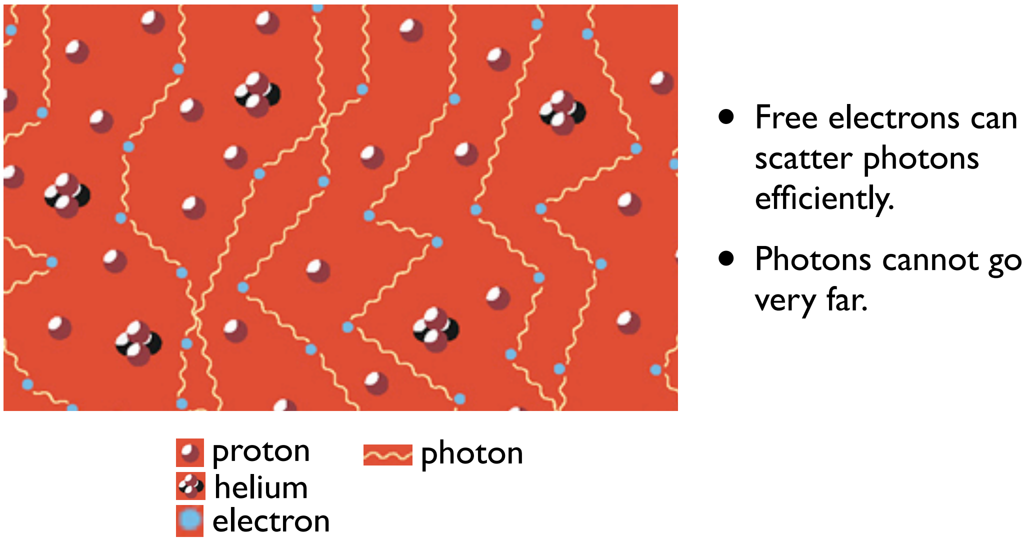
The present-day neutrino temperature is $2.725 \times (4/11)^{1/3} = 1.945$ K.



2.3 Recombination and Decoupling

2.3.1 Opaque Universe

While there were about equal numbers of electrons, positrons, and photons before the annihilation epoch, the number of electrons after the annihilation epoch is about 2 billion times smaller than that of photons, as most of electrons annihilated with positrons. (Why there was a tiny excess of electrons over positrons is still a mystery.) However, this tiny amount of electrons is enough to keep the universe "opaque," as they efficiently scatter photons.



Whether the scattering is efficient or not can be quantified by the ratio of the **mean free time** of photons, $1/(\sigma_T n_e c)$, and the Hubble time, $1/H$. Here, $\sigma_T = 6.65 \times 10^{-25}$ cm^2 is the Thomson scattering cross section, and n_e is the number density of free electrons. The scattering is efficient enough to keep the universe opaque if the mean free time is short compared to the Hubble time, i.e., $H/(\sigma_T n_e c) < 1$. In fact, the scattering is so efficient that the universe remains opaque when the universe is matter-dominated, for which the Hubble rate is given by $H = H_0\sqrt{\Omega_m(1+z)^3}$. Let us calculate

$$\frac{H}{\sigma_T n_e c} = \frac{H_0}{c}\frac{\sqrt{\Omega_m(1+z)^3}}{\sigma_T n_{\rm CMB}}\frac{n_{\rm CMB}}{n_e}. \tag{2.14}$$

Using $n_{\rm CMB} = 410(1+z)^3$ cm^{-3}, $n_{\rm CMB}/n_e \approx 2\times10^9$, $c/H_0 = 2998\ h^{-1}$ Mpc $= 9.25\ h^{-1}\ \times 10^{27}$ cm, and $\Omega_m h^2 = 0.13$, we obtain

$$\frac{H}{\sigma_T n_e c} \simeq 0.9\times10^{-2}\left(\frac{1000}{1+z}\right)^{3/2}\left(\frac{n_{\rm CMB}/n_e}{2\times10^9}\right). \tag{2.15}$$

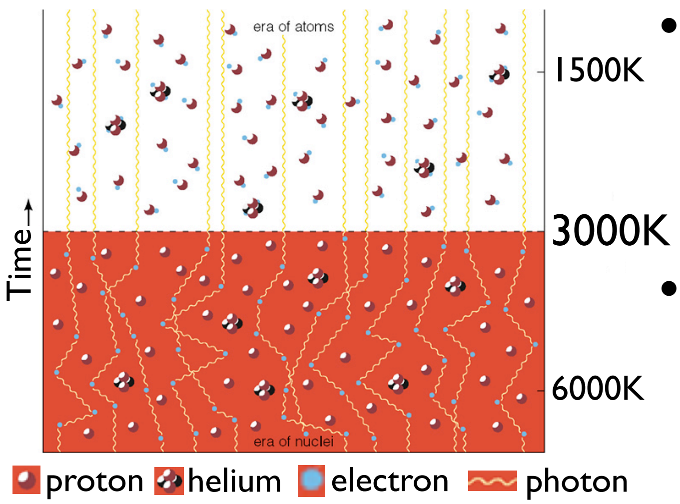
Therefore, at $z \approx 10^3$, the mean free time of photons was still only 1% of the Hubble time, and the universe was still quite opaque.

2.3.2 Neutral Hydrogen Formation and Decoupling

However, at around this epoch ($z \approx 10^3$, or $T_{\rm CMB} \approx 3000$ K), the electron number density rapidly fell relative to $n_{\rm CMB}$, resulting in the **decoupling** of photons from the electron scattering. What happened? At this temperature, the universe was cool enough for electrons to be captured by protons, forming neutral hydrogen atoms:

$$p + e^- \to H + \gamma.$$

Once started, this process rapidly eats electrons, reducing their number density and thus allowing for photons to propagate freely.



- [***recombination***] When the temperature falls below 3000 K, almost all electrons are captured by protons and helium nuclei.
- [***decoupling***] Photons are no longer scattered. I.e., photons and electrons are no longer coupled.

Why 3000 K?

As the ionization energy of hydrogen atoms is 13.6 eV, one might think that the neutral hydrogen begins to form when the temperature of photons falls below 13.6 eV $\simeq 1.6 \times 10^5$ K. However, in reality, the formation of hydrogen atoms is delayed until $T \approx 3700$ K.

When the temperature is $T = 1.6 \times 10^5$ K, only 15% of photons have energies lower than 13.6 eV. When the temperature drops to $T = 70,000$ K, about a half of photons have energies lower than 13.6 eV. Still, there are so many photons *per hydrogen atom* to begin with, and thus, roughly speaking, the ratio of the number of photons to the number of electrons, which is about a billion, gives a logarithmic correction to the temperature of the hydrogen formation epoch as $T \approx 70,000 \text{ K}/\ln(10^9) \approx 3400$ K. Finally, while a significant amount of hydrogen atoms are formed at this temperature, photons do not decouple from the plasma until the universe cools down to $T \approx 3000$ K.

The first approximation would be to assume that protons, electrons, and hydrogen atoms are in thermal equilibrium. At this temperature all of these species are non-relativistic, and their equilibrium densities are given by the non-relativistic limits of the Fermi-Dirac distribution:

$$n_p = 2\int \frac{d^3p}{(2\pi)^3\hbar^3} \exp\left(-\frac{m_pc^2 + \frac{p^2}{2m_p} - \mu_p}{k_BT}\right) = 2e^{(\mu_p - m_pc^2)/(k_BT)}\left(\frac{m_pk_BT}{2\pi\hbar^2}\right)^{3/2}, \tag{2.16}$$

$$n_e = 2\int \frac{d^3p}{(2\pi)^3\hbar^3} \exp\left(-\frac{m_ec^2 + \frac{p^2}{2m_e} - \mu_e}{k_BT}\right) = 2e^{(\mu_e - m_ec^2)/(k_BT)}\left(\frac{m_ek_BT}{2\pi\hbar^2}\right)^{3/2}, \tag{2.17}$$

$$n_H = 4\int \frac{d^3p}{(2\pi)^3\hbar^3} \exp\left(-\frac{m_Hc^2 + \frac{p^2}{2m_H} - \mu_H}{k_BT}\right) = 4e^{(\mu_H - m_pc^2)/(k_BT)}\left(\frac{m_Hk_BT}{2\pi\hbar^2}\right)^{3/2}. \tag{2.18}$$

Now, we also assume that protons, electrons, and hydrogen atoms are in *ionization equilibrium*, by which we mean that the reaction $p + e^- \leftrightarrow H + \gamma$ occurs fast enough to reach the chemical equilibrium:

$$\mu_p + \mu_e = \mu_H. \tag{2.19}$$

(Note that photon's chemical potential is zero.) This condition lets us combine the above 3 number densities to obtain the so-called **Saha equation**:

$$\frac{n_p n_e}{n_H} = \left(\frac{m_p}{m_H}\frac{m_e k_B T}{2\pi\hbar^2}\right)^{3/2} e^{-(m_p+m_e-m_H)c^2/(k_BT)}. \tag{2.20}$$

Here, the mass difference in the exponential is the binding energy of an hydrogen atom, which is of course equal to its ionization energy:

$$B_H \equiv (m_p + m_e - m_H)c^2 = 13.6 \text{ eV}. \tag{2.21}$$

Since $m_e \approx m_p/2000$ and $m_p \approx 1$ GeV, we can set $m_p \approx m_H$ in the parenthesis in front of the exponential factor. Finally, the charge neutrality demands $n_e = n_p$. We thus obtain

$$\frac{n_p^2}{n_H} \approx \left(\frac{m_e k_B T}{2\pi\hbar^2}\right)^{3/2} e^{-B_H/(k_BT)}. \tag{2.22}$$

Now, define the **ionization fraction**:

$$X \equiv \frac{n_p}{n_p + n_H}, \tag{2.23}$$

which goes from 1 (fully ionized hydrogen) to 0 (fully neutral hydrogen). The Saha equation now reads:

$$\frac{X^2}{1-X} = \frac{1}{n_p + n_H}\left(\frac{m_e k_B T}{2\pi\hbar^2}\right)^{3/2} e^{-B_H/(k_BT)}. \tag{2.24}$$

The goal here is to solve this equation for X as a function of the temperature, T. For this purpose, it is convenient to relate $n_p + n_H$ to the baryon mass density of the universe. We use the result from the Big Bang Nucleosynthesis (BBN): 76% of the baryonic mass in the universe after BBN is contained in protons (and the rest in helium nuclei). Therefore, $m_p(n_p + n_H) = 0.76\rho_b$. We then define the *time-independent* **baryon-to-photon ratio**:

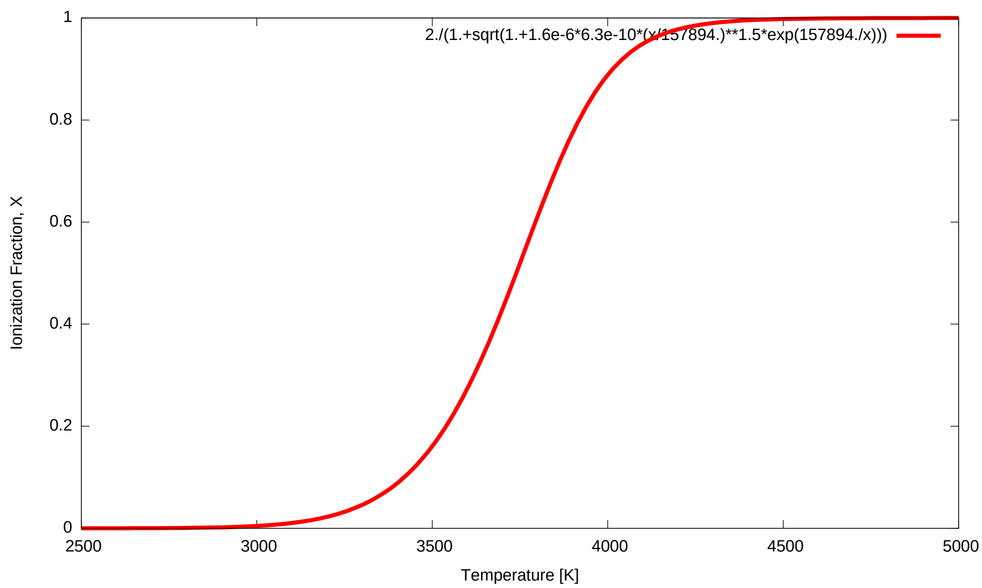
$$\boxed{\eta \equiv \frac{\rho_b}{m_p n_{\rm CMB}} = 273.9(\Omega_b h^2) \times 10^{-10}} \tag{2.25}$$

which takes on the value $\eta = 6.30 \times 10^{-10}$ for $\Omega_b h^2 = 0.023$. Therefore, there are 1.6 billion photons per baryon. Note that we have used $n_{\rm CMB} = 410 \text{ cm}^{-3}(T/T_0)^3$ with $T_0 = 2.725$ K for computing the numerical value of η. Putting all the numerical values in, we finally arrive at the following dimensionless form of the Saha equation:

$$\boxed{\frac{X^2}{1-X} = \frac{2.50 \times 10^6}{\eta}\tilde{T}^{-3/2} e^{-1/\tilde{T}}} \tag{2.26}$$

where $\tilde{T} \equiv k_B T/B_H = T/(157894\text{ K})$. This is a simple quadratic equation, which can be easily solved for X. The solution is

$$\boxed{X(T) = \frac{2}{1+\sqrt{1+(1.6\times 10^{-6}\eta)\tilde{T}^{3/2}e^{1/\tilde{T}}}}} \tag{2.27}$$



Let us find an approximate temperature, $T_{\rm rec}$, at which the universe is half neutral, $X = 1/2$. Then we have $\tilde{T}_{\rm rec}^{3/2} e^{1/\tilde{T}_{\rm rec}} = 5\times 10^6/\eta$, whose numerical solution is $\tilde{T}_{\rm rec} = 0.0237$, or $\boxed{T_{\rm rec} = 3740\text{ K}}$.[¶] It may be illustrative to find $T_{\rm rec}$ for $\eta = 1$ (i.e., equal numbers of baryons and photons). We find $T_{\rm rec} \approx 7900$ K; thus, even in the situation where there is one photon per baryon, the temperature of the universe at the hydrogen formation epoch (where the universe is half neutral) is significantly lower than the temperature corresponding to the hydrogen ionization energy, 1.6×10^5 K.

Now, with the ionization history calculated, we can re-calculate the ratio of the mean free time to the Hubble time to find the temperature of the epoch at which photons decouple from the

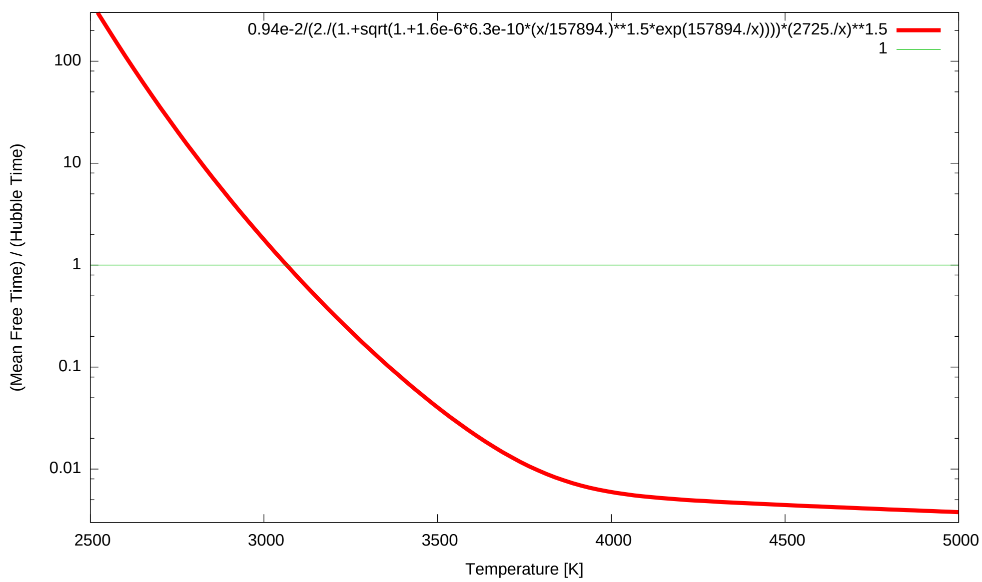
[¶]While it is not very accurate, we may solve this equation iteratively. Taking the logarithm of both sides, we get

$$\frac{3}{2}\ln\tilde{T}_{\rm rec} + \frac{1}{\tilde{T}_{\rm rec}} = \ln\left(\frac{5\times 10^6}{\eta}\right).$$

The zeroth-order iterative solution would then be obtained by ignoring the first term on the left hand side: $\tilde{T}_{\rm rec} = 1/\ln(5\times 10^6/\eta)$, which gives $T_{\rm rec} \simeq 4300$ K for $\eta = 6.3\times 10^{-10}$. One may improve accuracy of the solution by inserting this zeroth-order solution into the first term on the left hand side, and resolving for $\tilde{T}_{\rm rec}$. In any case, this analysis shows that the recombination temperature is reduced by a factor of $\ln(1/\eta)$.

electron scattering. We rewrite Eq. (2.14) as

$$\begin{aligned}
\frac{H}{\sigma_T n_e c} &= \frac{H_0}{c}\frac{\sqrt{\Omega_m(1+z)^3}}{\sigma_T n_{\rm CMB}}\frac{1}{0.76\eta X(z)} \\
&= \frac{0.94\times 10^{-2}}{X(z)}\left(\frac{1000}{1+z}\right)^{3/2}\left(\frac{6.3\times 10^{-10}}{\eta}\right) \\
&= \frac{0.94\times 10^{-2}}{X(T)}\left(\frac{2725\ \rm K}{T}\right)^{3/2}\left(\frac{6.3\times 10^{-10}}{\eta}\right).
\end{aligned}$$



Indeed, the mean free time becomes comparable to the Hubble time when the temperature of the universe is $\boxed{T_{\rm dec}\approx 3000\ \rm K}$, or $\boxed{z_{\rm dec}\approx 1100}$. (The solution of $H/(\sigma_T n_e c) = 1$ from the above equation gives $T_{\rm dec} = 3065$ K.) This is the epoch at which the universe became transparent, and photons began to propagate freely in space. We are detecting photons coming from this epoch as the cosmic microwave background. This epoch of often called the "decoupling epoch," or the "last scattering surface."

Freeze-out of Recombination

The above equilibrium calculation shows that all of electrons will eventually be captured by protons, leaving no free electrons at low temperatures. However, as the recombination rate is proportional to $n_e n_p$, the rate falls rapidly as the number densities go down due to the expansion of the universe. Eventually the recombination time becomes comparable to the Hubble time, and the recombination stops. This is the epoch of recombination **freeze-out**.

The recombination time per proton is given by $1/(\langle\sigma_{\rm rec}v\rangle n_e)$, where $\langle\sigma_{\rm rec}v\rangle$ is given by

$$\langle\sigma_{\rm rec}v\rangle = 2.33\times 10^{-14}\ \frac{\ln(1/\tilde{T})}{\tilde{T}^{1/2}}\ {\rm cm}^3\ {\rm s}^{-1}. \tag{2.28}$$

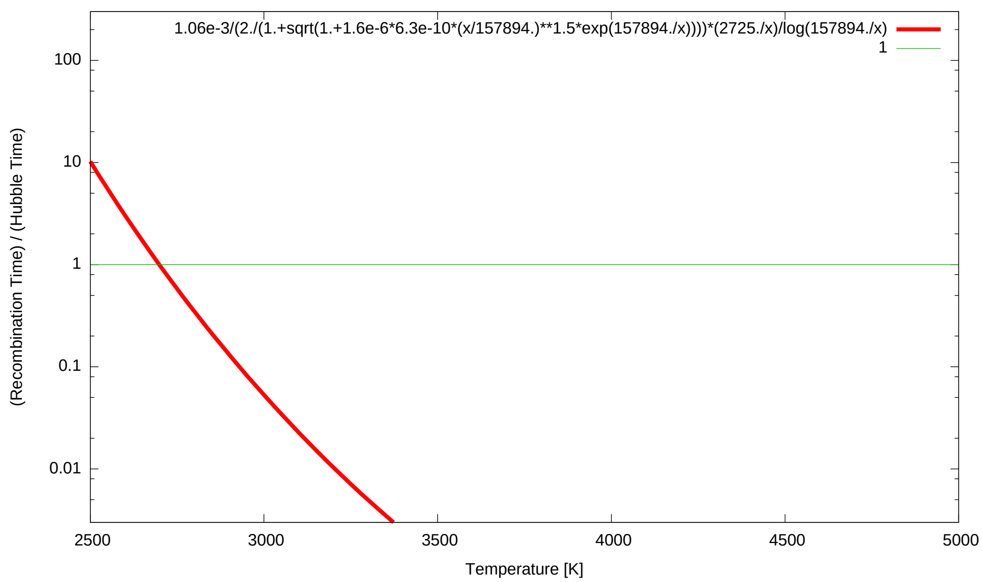
It is convenient to divide this by the speed of light:

$$\langle\sigma_{\rm rec}v\rangle/c = 7.77\times 10^{-25}\,\frac{\ln(1/\tilde{T})}{\tilde{T}^{1/2}}\ {\rm cm}^2, \tag{2.29}$$

which is the same order of magnitude as the Thomson-scattering cross section, $\sigma_T = 6.65 \times 10^{-25}$ cm^2. Then, the ratio of the recombination time to the Hubble time is given by

$$\begin{aligned}\frac{H}{\langle\sigma_{\rm rec}v\rangle n_e} &= \frac{H_0}{c}\frac{\sqrt{\Omega_m(1+z)^3}}{(\langle\sigma_{\rm rec}v\rangle/c)n_{\rm CMB}}\frac{1}{0.76\eta X(z)}\\ &= \frac{1.06\times 10^{-3}}{X(T)\ln(157894/T)}\left(\frac{2725\ {\rm K}}{T}\right)\left(\frac{6.3\times 10^{-10}}{\eta}\right).\end{aligned}$$

As this ratio is smaller than that for the decoupling (Eq. (2.28)) by a factor of ten, the **recombination freeze-out occurs after photons decouple from the plasma**. The above ratio (Eq. (2.30)) crosses unity at $\boxed{T_{\rm freeze-out} = 2700\ {\rm K}}$, which is lower than the decoupling temperature, ≈ 3000 K.



We can also calculate the **residual ionization fraction** of the recombination, i.e., the ionization fraction left after the recombination freeze-out, by evaluating $X(T)$ at $T = 2700$ K. We find

$$\boxed{X(2700\ {\rm K}) = 2.7\times 10^{-4}} \tag{2.30}$$

In other words, after the recombination freeze-out, there remains one free electron per about 4000 hydrogen atoms. This seems like a small amount: however, this small amount of residual electrons is necessary for forming hydrogen molecules via $H + e^- \to H^- + \gamma$ followed by $H^- + H \to H_2 + e^-$. The hydrogen molecules formed in this way are expected to play an important role in cooling gas and forming the first generation of stars (Galli and Palla, A&A, 335, 403 (1998)).

2.4 Temperature Anisotropy

2.4.1 Dipole Anisotropy

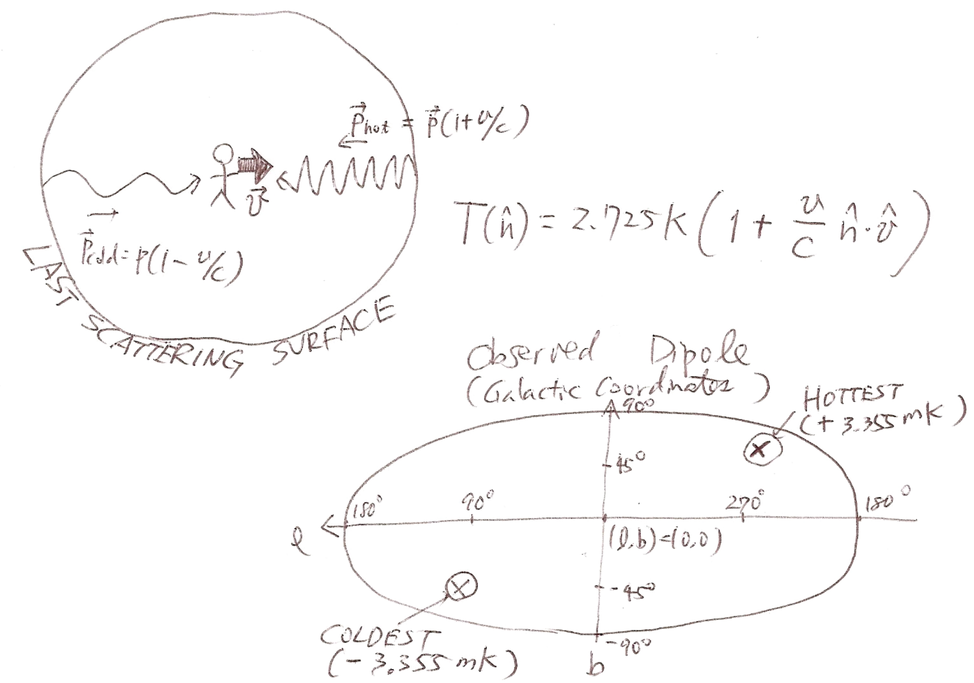
The dipole anisotropy at the level of mK is caused by the motion of Solar System with respect to the rest frame of the cosmic microwave background. Due to the Doppler effect, momentum of photons of the microwave background appears to be larger in the direction of our motion:

$$p_{\mathcal{O}}(\hat{n}) = \frac{p}{\gamma(1 - \hat{n} \cdot \frac{\vec{v}}{c})}, \tag{2.31}$$

where $p_{\mathcal{O}}(\hat{n})$ is the observed momentum of photons coming from a direction $\hat{n}$,[‖] p is the momentum in the rest frame of the cosmic microwave background, and $\gamma \equiv (1 - v^2/c^2)^{-1/2}$ is the Lorentz factor. Expanding this expression to the first order in v/c, we obtain

$$p_{\mathcal{O}}(\hat{n}) \approx p\left(1 + \hat{n} \cdot \frac{\vec{v}}{c}\right). \tag{2.32}$$

As expected, when photons are coming from the direction of our motion, i.e., $\hat{n} \cdot \hat{v} = 1$, the observed momentum takes on the maximum value, $p_{\mathcal{O}} = p(1 + v/c)$.



Now, as the cosmic microwave background is a blackbody, we can relate the change in the momentum of photons to the change in the temperature as

$$\frac{\delta T}{T} \equiv \frac{T_{\mathcal{O}}(\hat{n}) - T}{T} = \frac{p_{\mathcal{O}}(\hat{n}) - p}{p} = \hat{n} \cdot \frac{\vec{v}}{c}. \tag{2.33}$$

‖ Since photons are coming toward us, the propagation direction of photons, $\hat{p}$, is opposite of the line of sight direction, i.e., $\hat{p} = -\hat{n}$.

This is the formula for the dipole anisotropy. The measured value of dipole in the direction of motion is $\delta T = 3.355 \pm 0.008$ mK (Table 6 of Jarosik et al., ApJS, 192, 14 (2011)). The direction of motion in Galactic coordinates is $(l, b) = (263.99 \pm 0.14, 48.26 \pm 0.03)$ (in degrees). This gives $\delta T/T = 3.355 \times 10^{-3}/2.725 = 1.23 \times 10^{-3}$. By equating this to v/c, we find**

$$\boxed{v = 368 \text{ km/s}} \tag{2.34}$$

This velocity should be the vector sum of various components:

$$\vec{v} = (\vec{v}_{\rm Sun} - \vec{v}_{\rm MW}) + (\vec{v}_{\rm MW} - \vec{v}_{\rm LG}) + \vec{v}_{\rm LG}, \tag{2.35}$$

where

1. $\vec{v}_{\rm Sun} - \vec{v}_{\rm MW}$ is the orbiting velocity of Solar System with respect to the center of our Galaxy (Milky Way). This component is known (222.0 ± 5.0 km/s in the direction of $(l, b) = (91.1, 0)$ degrees), and thus can be subtracted.

2. $\vec{v}_{\rm MW} - \vec{v}_{\rm LG}$ is the velocity of our Galaxy (Milky Way) with respect to the center-of-mass of Local Group of galaxies. As the dominant masses of Local Group are given by Milky Way and Andromeda Galaxy (M31), which is a nearby galaxy, this component is small (≈ 80 km/s).

3. $\vec{v}_{\rm LG}$ is the velocity of the center-of-mass of Local Group with respect to the rest frame of the cosmic microwave background. This component represents the cosmological velocity flow (called the "bulk flow").

It turns out that the sum of the first two components, i.e., motion of Sun relative to the center-of-mass of Local Group, has a magnitude (307 km/s) comparable to the measured velocity, but is in nearly the opposite direction ($(l, b) = (105 \pm 5, -7 \pm 4)$ degrees; Yahil, Tammann & Sandage, ApJ, 217, 903 (1997)). As a result, the inferred bulk flow component has a large velocity:

$$v_{\rm LG} = 626 \pm 30 \text{ km/s}, \tag{2.36}$$

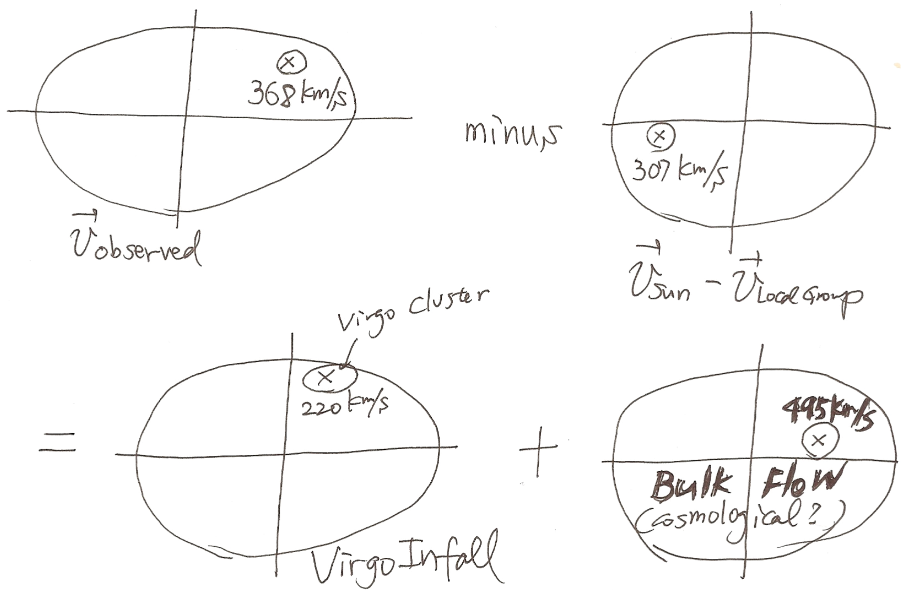
in the direction of $(l, b) = (276 \pm 2, 30 \pm 2)$ degrees (Sandage, Reindl & Tammann, ApJ, 714, 1441 (2010)).

Who is pulling Local Group? One obvious nearby mass concentration is the Virgo clusters of galaxies (at 16.5 Mpc). After subtracting an estimate of the infall velocity to Virgo (220 km/s) in the direction of $(l, b) = (283.8, 74.5)$ degrees, the velocity of Local Group corrected for the Virgo infall is

$$\boxed{v_{\rm LG} = 495 \pm 25 \text{ km/s}} \quad \text{Corrected for Virgo infall} \tag{2.37}$$

in the direction of $(l, b) = (275 \pm 2, 12 \pm 4)$ degrees (Sandage, Reindl & Tammann, ApJ, 714, 1441 (2010)). Therefore, Virgo cannot be solely responsible for the motion of Local Group. We still do not know who is responsible for this velocity.

**Rotation velocity of Earth around Sun, 30 km/s, has been removed from this value, as this component varies annually.



Recently, Nusser and David (arXiv:1101.1650) show that the measurements of peculiar velocities of nearby spiral galaxies within 100 h^{-1} Mpc give the velocity of 333 ± 38 km/s in the direction of $(l, b) = (276 \pm 3, 14 \pm 3)$ degrees after correcting for the Virgo infall. This measurement accounts for most of the velocity inferred from the cosmic microwave background, but is still lower. This implies that mass concentrations on $> 100\ h^{-1}$ Mpc are partially responsible for the bulk flow of Local Group. It is encouraging that the directions inferred from both methods are in an excellent agreement.

2.4.2 Sachs–Wolfe Effect

After removing the dipole anisotropy, what remains is the **primordial anisotropy**. It exhibits much richer angular distributions than dipole. This component can be divided into 2 contributions:

1. Gravitational effect (called the **Sachs–Wolfe effect**), and
2. Scattering effect.

This problem can be dealt with most intuitively by following the evolution of momentum of photons in a clumpy universe. In a homogeneous universe, we know that the momentum just redshifts away as $p \propto 1/a$; thus, the evolution equation would simply be:

$$\frac{1}{p}\frac{dp}{dt} = -\frac{1}{a}\frac{da}{dt}. \tag{2.38}$$

However, in a clumpy universe, photons receive gravitational blue/redshifts. The evolution equation, which you will derive in the homework question, is (with $c = 1$)

$$\frac{1}{p}\frac{dp}{dt} = -\frac{1}{a}\frac{da}{dt} - \sum_i \frac{\gamma^i}{a}\frac{\partial\Psi}{\partial x^i} - \frac{\partial\Phi}{\partial t}. \tag{2.39}$$

Note that we carefully distinguish between the total derivatives and partial derivatives here. The γ^i is a unit vector satisfying

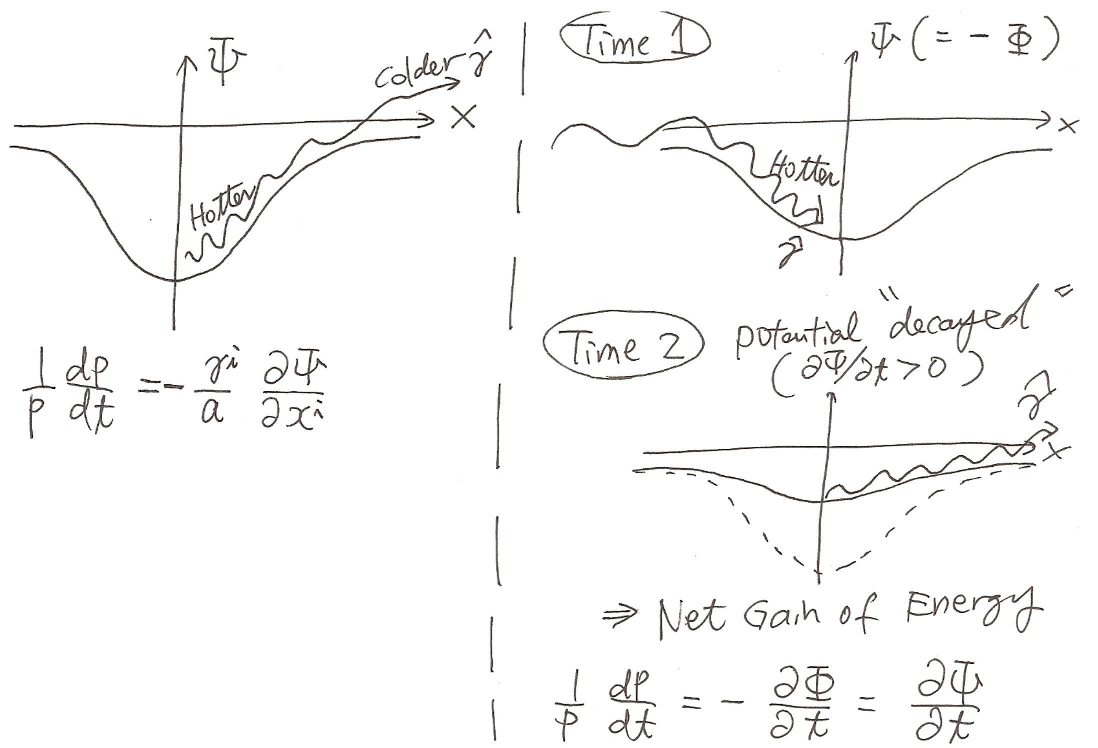
$$\sum_{ij} \delta_{ij}\gamma^i\gamma^j = 1, \tag{2.40}$$

which gives a direction of momentum. There is a factor of $1/a$ in the second term because x^i denotes the comoving coordinates.

The perturbation variables, Ψ and Φ, are the Newtonian potential and the so-called **curvature perturbation**, respectively. They are defined by the following perturbed metric:

$$ds^2 = -[1 + 2\Psi(x^i, t)]dt^2 + a^2(t)[1 + 2\Phi(x^i, t)]\sum_{ij} \delta_{ij} dx^i dx^j. \tag{2.41}$$

For example, for a point mass with mass M, these variables reduce to the familiar forms: $\Psi = -GM/r$ and $\Phi = -\Psi = GM/r$ (with $c = 1$).



The magnitude of momentum, p, is defined by

$$p^2 = \sum_{ij} g_{ij}p^ip^j = -g_{00}(p^0)^2. \tag{2.42}$$

The last equality follows from the normalization condition of momentum of massless particles, $\sum_{\mu\nu} g_{\mu\nu}p^\mu p^\nu = 0$. From this, one finds that $p = \sqrt{(1 + 2\Psi)}p^0 \simeq (1 + \Psi)p^0$. Note that it is p, rather than p^0, that is directly related to the temperature, i.e., $T \propto p$. Finally, from the above definition of p and the normalization condition for the unit vector γ^i ($\sum_{ij} \delta_{ij}\gamma^i\gamma^j = 1$), one can derive the relation between p^i and γ^i to the first order in perturbation:

$$p^i = \frac{p\gamma^i}{a}(1 - \Phi). \tag{2.43}$$

It is convenient to rewrite equation (2.39) using

$$\frac{\gamma^i}{a}\frac{\partial \Psi}{\partial x^i} = \left(\frac{\partial \Psi}{\partial t} + \frac{\gamma^i}{a}\frac{\partial \Psi}{\partial x^i}\right) - \frac{\partial \Psi}{\partial t} = \frac{d\Psi}{dt} - \frac{\partial \Psi}{\partial t}. \tag{2.44}$$

Then, we obtain

$$\frac{1}{p}\frac{dp}{dt} = -\frac{1}{a}\frac{da}{dt} - \frac{d\Psi}{dt} + \frac{\partial \Psi}{\partial t} - \frac{\partial \Phi}{\partial t}, \tag{2.45}$$

which can be readily integrated to give

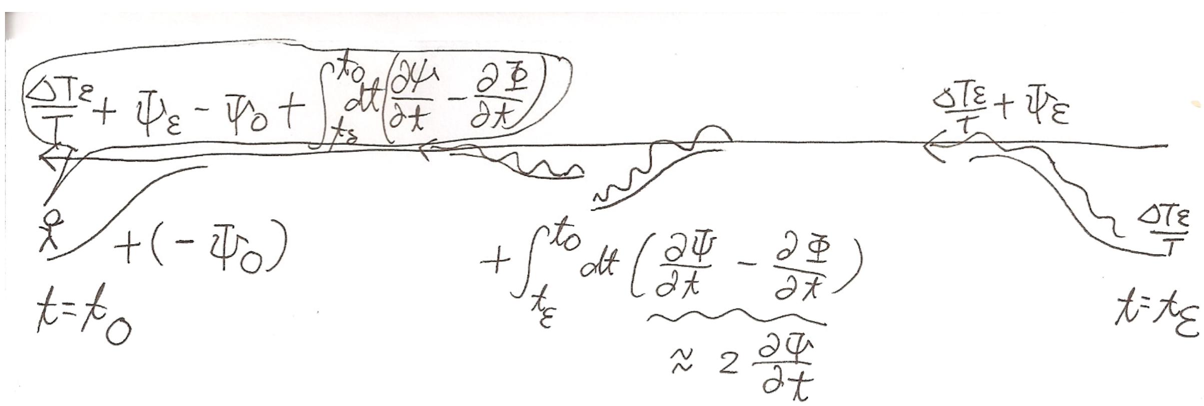
$$\ln(ap)_{\mathcal{O}} = \ln(ap)_{\mathcal{E}} + (\Psi_{\mathcal{E}} - \Psi_{\mathcal{O}}) + \int_{t_{\mathcal{E}}}^{t_{\mathcal{O}}} dt\, \frac{\partial}{\partial t}(\Psi - \Phi), \tag{2.46}$$

where "$\mathcal{O}$" and "$\mathcal{E}$" denote the "observed epoch" and "emitted epoch," respectively. Finally, we rewrite this result using the temperature anisotropy:

$$ap \propto a\bar{T}\left(1 + \frac{\delta T}{\bar{T}}\right). \tag{2.47}$$

Here, $\bar{T}$ is the mean temperature and depends only on time. Taylor-expanding the logarithm to the first order in $\delta T/\bar{T}$, and recalling $a_{\mathcal{O}}\bar{T}_{\mathcal{O}} = a_{\mathcal{E}}\bar{T}_{\mathcal{E}}$ for the mean temperature, we finally obtain:

$$\left.\frac{\delta T}{\bar{T}}\right|_{\mathcal{O}} = \left.\frac{\delta T}{\bar{T}}\right|_{\mathcal{E}} + (\Psi_{\mathcal{E}} - \Psi_{\mathcal{O}}) + \int_{t_{\mathcal{E}}}^{t_{\mathcal{O}}} dt\, \frac{\partial}{\partial t}(\Psi - \Phi). \tag{2.48}$$



To this, we must add the Doppler terms due to the velocity at emission and observed location:

$$\boxed{\left.\frac{\delta T}{\bar{T}}\right|_{\mathcal{O}} = \left.\frac{\delta T}{\bar{T}}\right|_{\mathcal{E}} + (\Psi_{\mathcal{E}} - \Psi_{\mathcal{O}}) + \int_{t_{\mathcal{E}}}^{t_{\mathcal{O}}} dt\, \frac{\partial}{\partial t}(\Psi - \Phi) + \sum_i \gamma^i(v^i_{\mathcal{E}} - v^i_{\mathcal{O}})} \tag{2.49}$$

The last term, $-\sum \gamma^i v^i_{\mathcal{O}}$, is the dipole anisotropy discussed in the previous section.

This result has a simple interpretation.

1. There was an initial temperature anisotropy at the last scattering surface, $\delta T/\bar{T}|_{\mathcal{E}}$ (which remains to be calculated), as well as the Doppler effect, $\sum \gamma^i v^i_{\mathcal{E}}$.

2. After the last scattering, photons escape from a potential well, losing energy: $\delta T/\bar{T}|_{\mathcal{E}} + \Psi_{\mathcal{E}} + \sum \gamma^i v^i_{\mathcal{E}}$.

3. While photons are propagating toward us, photons gain or lose energy depending on how $\Psi - \Phi$ ($\approx 2\Psi$) changes with time, giving $\delta T/\bar{T}|_{\mathcal{E}} + \Psi_{\mathcal{E}} + \int_{t_{\mathcal{E}}}^{t_{\mathcal{O}}} dt\ \frac{\partial}{\partial t}(\Psi - \Phi) + \sum \gamma^i v^i_{\mathcal{E}}$.

4. Finally, photons enter a potential well at our location, $\Psi_{\mathcal{O}}$, gaining energy. Also, they receive the Doppler shift due to our local motion, giving $\delta T/\bar{T}|_{\mathcal{E}} + \Psi_{\mathcal{E}} - \Psi_{\mathcal{O}} + \int_{t_{\mathcal{E}}}^{t_{\mathcal{O}}} dt\ \frac{\partial}{\partial t}(\Psi - \Phi) + \sum \gamma^i (v^i_{\mathcal{E}} - v^i_{\mathcal{O}})$.

In particular, $\delta T/\bar{T}|_{\mathcal{E}} + \Psi_{\mathcal{E}} - \Psi_{\mathcal{O}}$ is usually called the **Sachs–Wolfe effect**, and $\int_{t_{\mathcal{E}}}^{t_{\mathcal{O}}} dt\ \frac{\partial}{\partial t}(\Psi - \Phi)$ is called the **integrated Sachs–Wolfe effect**. All of these terms were derived by Sachs and Wolfe in 1967 (Sachs and Wolfe, ApJ, 147, 73 (1967)).

Adiabatic Initial Condition

How do we calculate the initial temperature fluctuation at the last scattering surface, $\delta T/\bar{T}|_{\mathcal{E}}$? To calculate this, we must specify the initial condition for perturbations. In principle, this cannot be known a priori without using the observational data. There are two widely explored initial conditions:

- Adiabatic initial condition

- Non-adiabatic initial condition

The current observational data favor the adiabatic initial condition, and we have not yet found any evidence for non-adiabatic initial condition. Therefore, we shall focus on the adiabatic initial condition.

What is it? This is the initial condition in which radiation and matter are perturbed in a similar way. It is called *adiabatic*, as the entropy density per matter particle is constant (unperturbed):

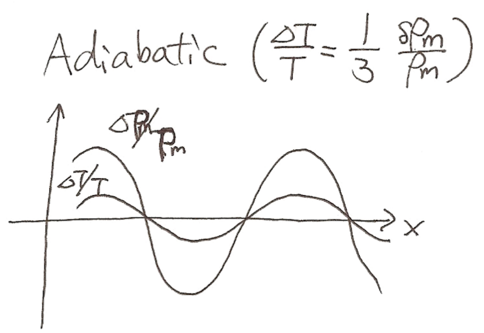
$$\frac{S/a^3}{n_M} \propto \frac{T^3}{n_M} = \text{constant}, \tag{2.50}$$

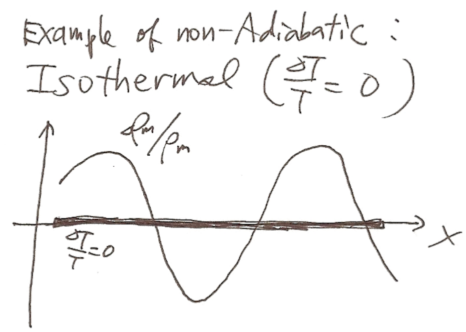
whose variation gives

$$\frac{\bar{T}^3}{\bar{n}_M}\left(3\frac{\delta T}{\bar{T}} - \frac{\delta n_M}{\bar{n}_M}\right) = 0. \tag{2.51}$$

Therefore, the adiabatic initial condition corresponds to

$$\frac{\delta T}{\bar{T}} = \frac{1}{3}\frac{\delta n_M}{\bar{n}_M} = \frac{1}{3}\frac{\delta \rho_M}{\bar{\rho}_M}. \tag{2.52}$$





"Non-adiabatic initial conditions" would have $\delta T/\bar{T} \neq \delta\rho_M/(3\bar{\rho}_M)$.

As this is the initial condition, it holds only on very large scales, much larger than the horizon size at the last scattering surface. While it is not obvious or intuitive, on such large scales, as you derive in the homework question, the density fluctuation during the matter-dominated era is related to the Newtonian potential as

$$\frac{\delta\rho_M}{\bar{\rho}_M} = -2\Psi \quad \text{(Matter-dominated \& super-horizon).} \tag{2.53}$$

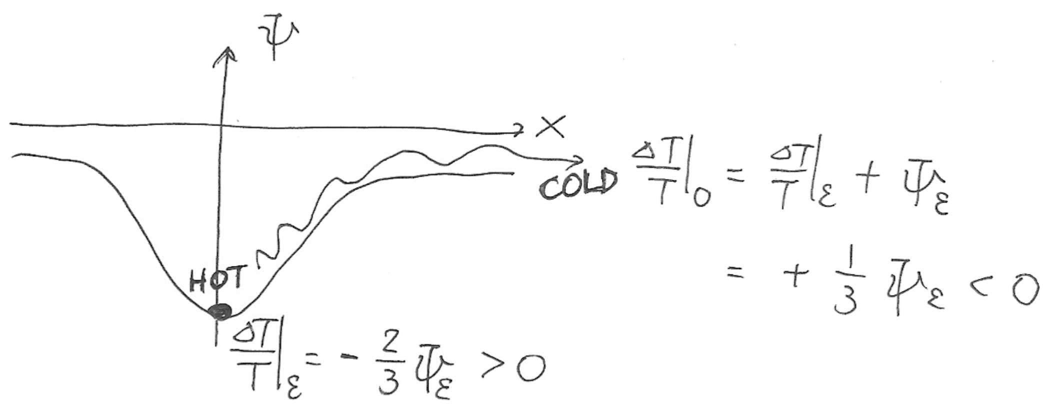
This gives, on large scales, the initial temperature fluctuation of

$$\left.\frac{\delta T}{\bar{T}}\right|_{\mathcal{E}} = \frac{1}{3}\left.\frac{\delta\rho_M}{\bar{\rho}_M}\right|_{\mathcal{E}} = -\frac{2}{3}\Psi_{\mathcal{E}}. \tag{2.54}$$

Then, the Sachs–Wolfe formula gives

$$\boxed{\left.\frac{\delta T}{\bar{T}}\right|_{\mathcal{O}} = \frac{1}{3}\Psi_{\mathcal{E}} + \dots} \tag{2.55}$$

Therefore, on large scales, an over-density region (i.e., a potential well) appears as a cold spot on the sky. While the temperature at the bottom of the potential well is hotter than the average ($-\frac{2}{3}\Psi$), photons lose more energy (Ψ) as they climb up the potential well, resulting in a cold spot ($-\frac{2}{3}\Psi + \Psi = \frac{1}{3}\Psi$).



Observing the primordial perturbation via the Sachs–Wolfe effect

As you have seen from the homework problem, $\Psi = -\Phi$ during the matter era, and both Ψ and Φ remain constant during the matter era. Therefore, after the decoupling, but before the dark energy dominated era, the integrated Sachs–Wolfe effect vanishes. In this case the observed temperature anisotropy toward a direction $\hat{n}^i$ is given by

$$\left.\frac{\delta T}{\bar{T}}\right|_{\mathcal{O}}(\hat{n}^i) = \frac{1}{3}\Psi_{\mathcal{E}} - \Psi_{\mathcal{O}} - \sum_i \hat{n}^i(v^i_{\mathcal{E}} - v^i_{\mathcal{O}}) \quad \text{during the matter era.} \tag{2.56}$$

Here, we have used the fact that the direction of photon, γ^i, is equal to $-\hat{n}^i$.

Now, let us consider temperature anisotropy on very large angular scales - the angular scale that is greater than the Hubble length at the decoupling epoch. As you have seen from the homework problem, the velocity perturbation vanishes in the large-scale limit, as it is proportional to ϵ, which is given by (with $c = 1$)

$$\epsilon \equiv \frac{k}{\dot{a}} = \frac{k}{aH}, \tag{2.57}$$

where k is the comoving wavenumber. Let us calculate the angular size of the Hubble length at $z = 1090$. The comoving wavenumber, λ, is related to k as $\lambda = 2\pi/k$. The angular size that corresponds to the half wavelength is then

$$\theta = \frac{\lambda/2}{d_A}, \tag{2.58}$$

where d_A is the *comoving* angular diameter distance:

$$d_A \equiv (1+z)D_A = c\int_0^z \frac{dz'}{H(z')} = 14 \text{ Gpc} \quad \text{for } z = 1090. \tag{2.59}$$

The numerator is the half-wavelength corresponding to the Hubble size:

$$\frac{\lambda_H}{2} = \frac{\pi}{k_H} = \frac{\pi}{aH}. \tag{2.60}$$

For $\Omega_M h^2 = 0.13$ and $\Omega_R h^2 = 4.17 \times 10^{-5}$, we find

$$aH = \frac{\sqrt{\Omega_M h^2(1+z) + \Omega_R h^2(1+z)^2}}{3 \text{ Gpc}} = 4.6 \text{ Gpc}^{-1} \quad \text{for } z = 1090. \tag{2.61}$$

Therefore, the angular size that corresponds to the Hubble length at $z = 1090$ is

$$\theta = \frac{\pi}{aHd_A} = \frac{180^\circ}{4.6 \times 14} = 2.8^\circ. \tag{2.62}$$

This means that, for angular scales much greater than 3°, we can ignore the contribution from the velocity perturbation at $z = 1090$, i.e., $v_{\mathcal{E}}$, and obtain

$$\left.\frac{\delta T}{\bar{T}}\right|_{\mathcal{O}}(\hat{n}^i) = \frac{1}{3}\Psi_{\mathcal{E}}(r_*\hat{n}^i) - \Psi_{\mathcal{O}} + \sum_i \hat{n}^i v^i_{\mathcal{O}} \quad \text{on large angular scales.} \tag{2.63}$$

Here, we explicitly show that $\Psi_{\mathcal{E}}$ is a three-dimensional quantity, $\Psi_{\mathcal{E}} = \Psi_{\mathcal{E}}(x^i)$, and what we observe is the potential at the last scattering surface whose comoving distance is $r_* = d_A(z = 1090) = 14$ Gpc. On the other hand, the second term, $\Psi_{\mathcal{O}}$, is the value of Ψ at our location, which is just a number, and merely adds a constant to the value of $\delta T/T$ over all sky; thus, this is a monopole term ($l = 0$). The third term, $\hat{n} \cdot \vec{v}_{\mathcal{O}}$, is the dipole anisotropy ($l = 1$) due to our local motion, which we have studied in the previous section.

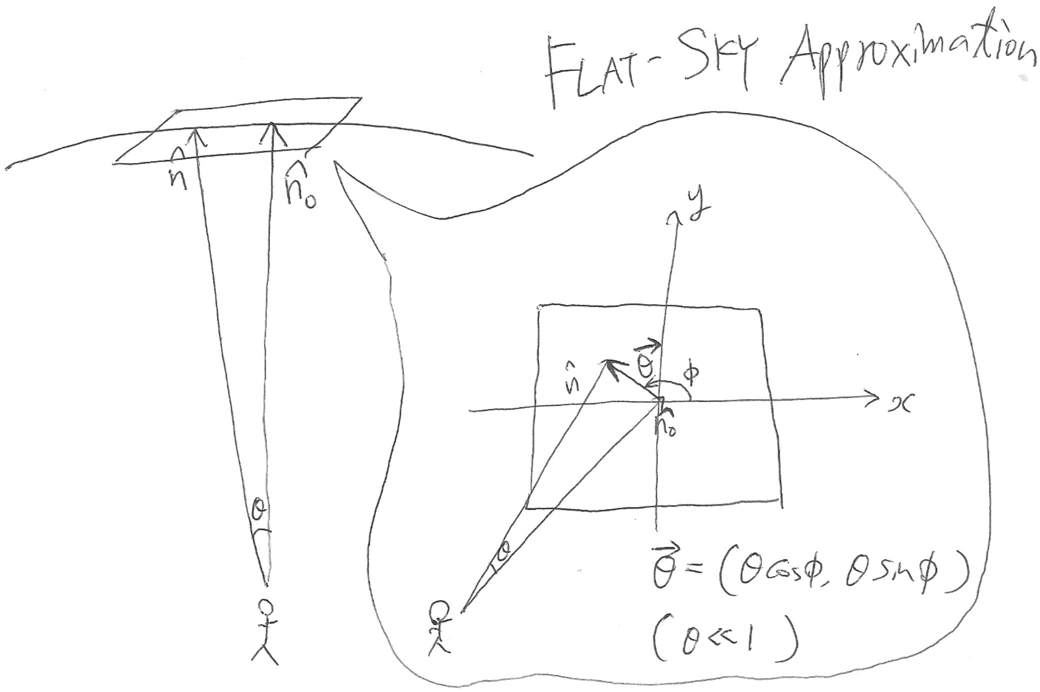
Therefore, if we ignore the monopole and dipole and focus on the primordial anisotropy with $l \geq 2$, we are left with the Sachs–Wolfe term:

$$\left.\frac{\delta T}{\bar{T}}\right|_{\mathcal{O}}(\hat{n}^i) = \frac{1}{3}\Psi_{\mathcal{E}}(r_*\hat{n}^i) \quad \text{on large angular scales and } l \geq 2. \tag{2.64}$$

This is an important result - since the angular size is greater than that of the Hubble length at $z = 1090$, the temperature anisotropy we observe on this scale is not altered by the physics at $z > 1090$. In other words, what we observe on large angular scales must reflect the initial, **primordial perturbation** (except for the integrated Sachs-Wolfe effect which we ignore here).

In order to characterize the observed temperature anisotropy, let us consider a patch of the sky whose center has the direction vector $\hat{n}^i_0$, and introduce the angular coordinates on this patch, $\vec{\theta} = (\sin\theta\cos\phi, \sin\theta\sin\phi)$. Furthermore, let us assume that the angular size is greater than 3°, but is much less than 60°, which corresponds to 1 radian. In this case, $\theta \ll 1$, and thus the angular coordinates become

$$\vec{\theta} = (\theta\cos\phi, \theta\sin\phi). \tag{2.65}$$



We then Fourier-transform the temperature anisotropy on this patch:

$$\left.\frac{\delta T}{\bar{T}}\right|_{\mathcal{O}}(\hat{n}^i) = \frac{1}{\bar{T}}\int \frac{d^2l}{(2\pi)^2}\widetilde{\delta T}(\vec{l})e^{i\vec{l}\cdot\vec{\theta}}. \tag{2.66}$$

We also Fourier-transform Ψ in 3-d space:

$$\Psi_{\mathcal{E}}(x^i) = \int \frac{d^3k}{(2\pi)^3}\tilde{\Psi}(\vec{k})e^{i\vec{k}\cdot\vec{x}}. \tag{2.67}$$

Remember that Ψ is constant during the matter era.

Now, we wish to find the relation between $\widetilde{\delta T}(\vec{l})$ and $\tilde{\Psi}(\vec{k})$:

$$\begin{aligned}\frac{\widetilde{\delta T}(\vec{l})}{\bar{T}} &= \int d^2\theta e^{-i\vec{l}\cdot\vec{\theta}}\frac{1}{3}\int \frac{d^3k}{(2\pi)^3}\tilde{\Psi}(\vec{k})e^{i\vec{k}\cdot(r_*\hat{n})} \\ &= \int d^2\theta e^{-i\vec{l}\cdot\vec{\theta}}\frac{1}{3}\int \frac{d^3k}{(2\pi)^3}\tilde{\Psi}(\vec{k})e^{i\vec{k}_\perp\cdot(r_*\vec{\theta})}e^{ik_\parallel r_*\cos\theta} \\ &= \int d^2\theta\frac{1}{3}\int \frac{d^3k}{(2\pi)^3}\tilde{\Psi}(\vec{k})e^{i(\vec{k}_\perp r_* - \vec{l})\cdot\vec{\theta}}e^{ik_\parallel r_*\cos\theta}\end{aligned} \tag{2.68}$$

Here, we have defined $\vec{k}_\perp$ and $k_\parallel$ such that

$$\vec{k} = (\vec{k}_\perp, k_\parallel), \tag{2.69}$$

where $\vec{k}_\perp$ is the wavenumber vector on the patch, and $k_\parallel$ is the wavenumber along the line of sight.

To proceed further, we use the fact that we consider the region in which $\theta \ll 1$ (so that we can treat a section of the sky as a flat surface):

$$
\begin{aligned}
\frac{\widetilde{\delta T}(\vec{l})}{\bar{T}} &\approx \int d^2\theta \frac{1}{3} \int \frac{d^3k}{(2\pi)^3} \tilde{\Psi}(\vec{k}) e^{i(\vec{k}_\perp r_* - \vec{l})\cdot\vec{\theta}} e^{ik_\parallel r_*} \quad (\cos\theta \approx 1) \\
&= \frac{1}{3} \int \frac{d^3k}{(2\pi)^3} \tilde{\Psi}(\vec{k}) \left[\int d^2\theta e^{i(\vec{k}_\perp r_* - \vec{l})\cdot\vec{\theta}} \right] e^{ik_\parallel r_*} \\
&= \frac{1}{3} \int \frac{d^3k}{(2\pi)^3} \tilde{\Psi}(\vec{k}) \left[(2\pi)^2 \delta_D^{(2)}(\vec{k}_\perp r_* - \vec{l}) \right] e^{ik_\parallel r_*} \\
&= \frac{1}{3} \int \frac{d^2k_\perp dk_\parallel}{(2\pi)^3} \tilde{\Psi}(\vec{k}) \left[(2\pi)^2 \frac{\delta_D^{(2)}(\vec{k}_\perp - \vec{l}/r_*)}{r_*^2} \right] e^{ik_\parallel r_*} \\
&= \frac{1}{3r_*^2} \int \frac{dk_\parallel}{2\pi} \tilde{\Psi}\left(\vec{k}_\perp = \frac{\vec{l}}{r_*}, k_\parallel \right) e^{ik_\parallel r_*}.
\end{aligned} \tag{2.70}
$$

Finally, there is no way to predict the value of $\tilde{\Psi}(\vec{k})$ for any given value of $\vec{k}$ because Ψ is a random (stochastic) variable. However, what we *can* do it to calculate its variance, which is called the **power spectrum**:

$$
\langle \tilde{\Psi}(\vec{k}) \tilde{\Psi}^*(\vec{k}) \rangle = (2\pi)^3 P_\Psi(k) \delta_D^{(3)}(\vec{k} - \vec{k}'), \tag{2.71}
$$

$$
\frac{1}{\bar{T}^2} \langle \widetilde{\delta T}(\vec{l}) \widetilde{\delta T}^*(\vec{l}') \rangle = (2\pi)^2 C_l \delta_D^{(2)}(\vec{l} - \vec{l}'). \tag{2.72}
$$

The **angular power spectrum** of the temperature anisotropy, C_l, is an observable quantity. Therefore, the remaining task is to relate the observable, C_l, to the power spectrum of Φ, $P_\Psi(k)$.

$$
\begin{aligned}
\frac{1}{\bar{T}^2} \langle \widetilde{\delta T}(\vec{l}) \widetilde{\delta T}^*(\vec{l}') \rangle &= \frac{1}{9r_*^4} \int \frac{dk_\parallel}{2\pi} \int \frac{dk'_\parallel}{2\pi} \left\langle \tilde{\Psi}\left(\frac{\vec{l}}{r_*}, k_\parallel\right) \tilde{\Psi}^*\left(\frac{\vec{l}'}{r_*}, k'_\parallel\right) \right\rangle \\
&\quad \times e^{i(k_\parallel - k'_\parallel) r_*} \\
&= \frac{1}{9r_*^4} \int \frac{dk_\parallel}{2\pi} \int \frac{dk'_\parallel}{2\pi} P_\Psi\left(\sqrt{\frac{l^2}{r_*^2} + k_\parallel^2} \right) e^{i(k_\parallel - k'_\parallel) r_*} \\
&\quad \times \delta_D^{(2)}\left(\frac{\vec{l}}{r_*} - \frac{\vec{l}'}{r_*} \right) \delta_D^{(1)}(k_\parallel - k'_\parallel) (2\pi)^3 \\
&= \frac{1}{9r_*^2} (2\pi)^2 \delta_D^{(2)}(\vec{l} - \vec{l}') \int \frac{dk_\parallel}{2\pi} P_\Psi\left(\sqrt{\frac{l^2}{r_*^2} + k_\parallel^2} \right)
\end{aligned}
$$

$$
\therefore C_l = \frac{1}{9r_*^2} \int \frac{dk_\parallel}{2\pi} P_\Psi\left(\sqrt{\frac{l^2}{r_*^2} + k_\parallel^2} \right)
$$

The result is

$$C_l = \frac{1}{9r_*^2}\int \frac{dk_\parallel}{2\pi} P_\Psi\left(\sqrt{\frac{l^2}{r_*^2}+k_\parallel^2}\right). \tag{2.73}$$

Note that the small-angle approximation, $\theta \ll 1$, corresponds to $l \gg 1$, as these are related via

$$l = \frac{\pi}{\theta}. \tag{2.74}$$

In terms of k and r_*, we have

$$l = kr_* = 14\left(\frac{k}{1\ \mathrm{Gpc}^{-1}}\right). \tag{2.75}$$

For example, the multipole that corresponds to the wavenumber of the Hubble horizon size at $z = 1090$, $aH = 4.6\ \mathrm{Gpc}^{-1}$, is

$$l_H = 64. \tag{2.76}$$

Therefore, the argument given here is valid only for $1 \ll l \ll 64$.

Now, we must make an assumption about the form of $P_\Psi(k)$. We now believe that primordial fluctuations were generated during the period of **inflation** - an exponential expansion of the universe during a tiny fraction of a second after the birth of the universe. As you will learn from Bhaskar toward the end of this course, inflation predicts the following power-law form of the initial power spectrum:

$$P_\Psi(k) \propto k^{n_s-4}, \tag{2.77}$$

where n_s is called the **spectral tilt**. The current data give (Komatsu et al., ApJS, 192, 18 (2011)) $n_s = 0.968 \pm 0.012$ (68% CL). As for the normalization of $P_\Psi(k)$, we usually parametrize it as

$$P_\Psi(k) = \frac{2\pi^2}{k^3}\Delta_\Psi^2(k_0)\left(\frac{k}{k_0}\right)^{n_s-1}, \tag{2.78}$$

where k_0 is some arbitrary pivot wavenumber which is often taken to be $k_0 = 2\ \mathrm{Gpc}^{-1} = 0.002\ \mathrm{Mpc}^{-1}$, and $\Delta_\Phi^2(k_0)$ is the normalization.

The special case is $n_s = 1$ (which is called the **Harrison-Zel'dovich-Peebles spectrum**, and is close to the observed value, $n_s = 0.968 \pm 0.012$), for which

$$C_l = \frac{2\pi}{l^2}\frac{1}{9}\Delta_\Psi^2(k_0). \tag{2.79}$$

This motivates our writing

$$\boxed{\frac{l^2 C_l}{2\pi} = \frac{1}{9}\Delta_\Psi^2(k_0) = \frac{1}{9}\frac{k^3 P_\Psi(k)}{2\pi^2} \quad \text{for } n_s = 1} \tag{2.80}$$

Since this quantity does not depend on l, this spectrum (with $n_s = 1$) is called the **scale-invariant spectrum**. Note also that $k^3 P_\Psi(k)$ does not depend on k. For $n_s \neq 1$, we have [††]

$$\frac{l^2 C_l}{2\pi} = \frac{1}{9}\Delta_\Psi^2(k_0)\left(\frac{l}{k_0 r_*}\right)^{n_s-1}\frac{\sqrt{\pi}}{2}\frac{\Gamma[(3-n_s)/2]}{\Gamma[(4-n_s)/2]}. \tag{2.84}$$

[††] This formula cannot be used for small l (such as $l = 2$) because we have treated our patch of the sky as a flat surface, which allowed us to use the familiar Fourier transform. For the full-sky treatment, we must take into account

$$C_\ell = \frac{1}{9 r_*^2}\int \frac{dk_\parallel}{2\pi}\, P_\Psi\left(\sqrt{\frac{\ell^2}{r_*^2}+k_\parallel^2}\right) = \frac{1}{9 r_*^2}\,\frac{\pi}{k_0^{n_s-1}}\,\Delta_\Psi^2(k_0)\int dk_\parallel \times \left(\sqrt{\frac{\ell^2}{r_*^2}+k_\parallel^2}\right)^{n_s-4}$$

$$= \frac{1}{9 r_*^2}\,\frac{\pi^{3/2}}{k_0^{n_s-1}}\,\Delta_\Psi^2(k_0)\,\frac{\Gamma\left(\frac{3-n_s}{2}\right)}{\Gamma\left(\frac{4-n_s}{2}\right)}\left(\frac{\ell}{r_*}\right)^{n_s-3}$$

$$= \frac{\pi^{3/2}}{9}\,\Delta_\Psi^2(k_0)\,\frac{\Gamma\left(\frac{3-n_s}{2}\right)}{\Gamma\left(\frac{4-n_s}{2}\right)}\left(\frac{\ell}{r_* k_0}\right)^{n_s-1}\frac{1}{\ell^2}$$

$$\boxed{\therefore\ \frac{\ell^2 C_\ell}{2\pi} = \frac{\sqrt{\pi}}{18}\,\Delta_\Psi^2(k_0)\,\frac{\Gamma\left(\frac{3-n_s}{2}\right)}{\Gamma\left(\frac{4-n_s}{2}\right)}\left(\frac{\ell}{r_* k_0}\right)^{n_s-1} = \underbrace{\frac{1}{9}\,\Delta_\Psi^2(k_0)}_{\text{For } n_s=1}}$$

We are not quite done yet. While Ψ (and hence Φ) is constant during the matter era, they change as the universe transitions from radiation-dominated to matter-dominated. They also change as the universe exits the inflationary period and becomes radiation dominated. Therefore, $P_\Psi(k)$ that we determine from the observation of the microwave background, which is $P_\Psi(k)$ during the matter era, cannot be directly compared with the prediction from inflation.

Fortunately, there is an easy solution for this problem. On very large scales, $k \ll aH$, there exists a **conserved quantity** called ζ, which is defined as

$$\boxed{\zeta \equiv \Phi - \frac{aH}{k}V = \Phi - \frac{V}{\epsilon}} \tag{2.85}$$

---

the fact that the sky is a sphere. For this purpose, we must use spherical harmonics decomposition rather than the Fourier transform. In any case, the exact result in the Sachs–Wolfe limit is

$$\begin{aligned} C_l &= \frac{2}{9\pi}\int k^2 dk P_\Psi(k) j_l^2(kr_*) \\ &= \frac{2\pi}{9}\Delta_\Psi^2(k_0)\frac{1}{(k_0 r_*)^{n_s-1}}\frac{\sqrt{\pi}}{2}\frac{\Gamma[(3-n_s)/2]}{\Gamma[(4-n_s)/2]}\frac{\Gamma[l+(n_s-1)/2]}{\Gamma[l+(5-n_s)/2]}. \end{aligned} \tag{2.81}$$

For $n_s = 1$,

$$C_l = \frac{2\pi}{l(l+1)}\frac{1}{9}\Delta_\Psi^2(k_0), \tag{2.82}$$

or

$$\boxed{\frac{l(l+1)C_l}{2\pi} = \frac{1}{9}\Delta_\Psi^2(k_0) \quad \text{for } n_s = 1} \tag{2.83}$$

For $l \gg 1$, we indeed recover the flat-sky result, $l^2 C_l/(2\pi) = \Delta_\Psi^2(k_0)/9$. This result explains why people tend to plot $l(l+1)C_l/(2\pi)$ against l.

This quantity remains constant on $k \ll aH$ regardless of the contents of the universe. **It is ζ that is predicted by theories of inflation.** Now, if we plug in the large-scale solution for the velocity perturbation during the matter era that you find from the homework problem, $V = -\frac{2}{3}\epsilon\Phi$, we find

$$\zeta = \Phi + \frac{2}{3}\Phi = \frac{5}{3}\Phi = -\frac{5}{3}\Psi \quad \text{during the matter era.} \tag{2.86}$$

Therefore, the Sachs–Wolfe formula is modified to

$$\boxed{\left.\frac{\delta T}{\bar{T}}\right|_{\mathcal{O}}(\hat{n}^i) = -\frac{1}{5}\zeta(r_*\hat{n}^i)} \tag{2.87}$$

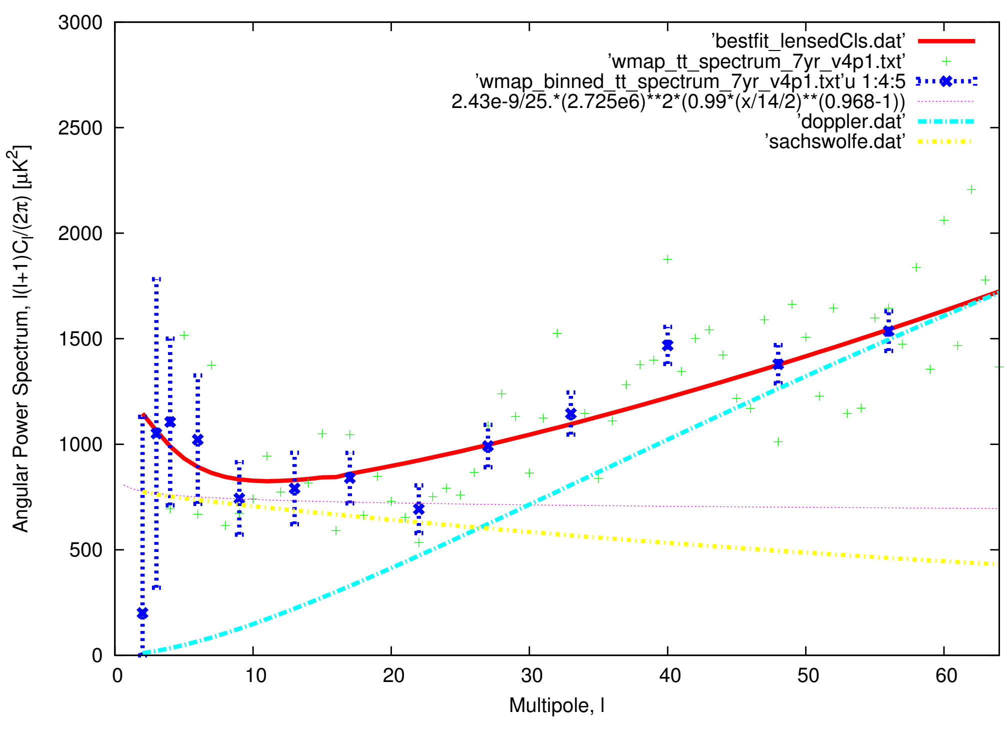
and

$$\frac{l^2 C_l}{2\pi} = \frac{1}{25}\Delta^2_\zeta(k_0)\left(\frac{l}{k_0 r_*}\right)^{n_s-1}\frac{\sqrt{\pi}}{2}\frac{\Gamma[(3-n_s)/2]}{\Gamma[(4-n_s)/2]}. \tag{2.88}$$

The current data give (Komatsu et al., ApJS, 192, 18 (2011))

$$\Delta^2_\zeta(k_0) = (2.43 \pm 0.09) \times 10^{-9}. \tag{2.89}$$

One should be impressed by these results! **Using the observation of the cosmic microwave background, we were able to measure the amplitude and the scale-dependence of the initial perturbations generated during inflation.** Studying the high-energy world before the Big Bang became a real science!



2.4.3 Gravitational Waves

So far, we have studied how gravitational potential, Ψ, produces anisotropy in the cosmic microwave background. However, this is not the only source. Another source of temperature anisotropy is the gravitational waves. Gravitational waves stretch space, causing photons to redshift or blueshift depending on the phase of gravitational waves. This stretching of space is described by the *tensor metric perturbation*, $h_{ij}(x^i, t)$. The metric that includes $h_{ij}(x^i, t)$ on top of the Friedmann-Robertson-Walker background is given by (with $c = 1$):

$$ds^2 = -dt^2 + a^2(t) \sum_{ij} [\delta_{ij} + h_{ij}(x^i, t)] dx^i dx^j. \tag{2.90}$$

This metric perturbation, $h_{ij}(x^i, t)$, is the gravitational wave itself. In other words, it is $h_{ij}(x^i, t)$ that propagates as a wave. Gravitational waves have the following properties:

1. Gravitational waves are *transverse* (just like electromagnetic waves). Therefore, they do not distort space along their propagation direction, but only distort space in the direction perpendicular to their propagation.

2. Gravitational waves have *two* polarization states (just like electromagnetic waves).

3. Gravitational waves (gravitons) are *spin-2*. (For comparison, electromagnetic waves (photons) are spin-1.) Therefore, h_{ij} is a rank-2 tensor field, whereas electromagnetic waves are described by a vector potential, A^i.

As an example, let us take a single plane wave as a gravitational wave propagating in z direction ($z \equiv x^3$). We have $h_{ij} \propto e^{ik_3 x^3}$. Because h_{ij} is transverse, we must have

$$\sum_j k^j h_{ij} = k^3 h_{i3} = 0. \tag{2.91}$$

As the metric is a symmetric tensor, h_{ij} is also symmetric, i.e., $h_{ij} = h_{ji}$. Using this information, we can write:

$$h_{ij} = \begin{pmatrix} h_{11} & h_{12} & 0 \\ h_{12} & h_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.92}$$

However, since h_{ij} has only two polarization states, h_{11} and h_{22} must be related somehow. This relation can be found by noting that gravitational waves change the *shape* of space, but do not change the *size*. In other words, it shears space, but does not expand or contract it. This means that the determinant of $\delta_{ij} + h_{ij}$ is unity:

$$\det(\delta_{ij} + h_{ij}) = (1 + h_{11})(1 + h_{22}) - h_{12}^2 = 1. \tag{2.93}$$

To first order in h_{ij}, this condition gives

$$h_{11} + h_{22} = 0. \tag{2.94}$$

Therefore, h_{ij} is a traceless tensor, $\sum_i h_{ii} = 0$.

We conventionally write the components of h_{ij} propagating in z direction as

$$h_{ij} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.95}$$

where h_+ and $h_\times$ represent two linear polarization states of gravitational waves. As one may guess from this matrix, a wave would distort space along x and y directions when $h_+ \neq 0$ and $h_\times = 0$, and it would distort space along 45° degrees when $h_+ = 0$ and $h_\times \neq 0$. This means that h_+ and $h_\times$ are not invariant under coordinate rotation. On the contrary, for **clock-wise** rotation of coordinates by an angle ϕ, h_+ and $h_\times$ transform as

$$\begin{pmatrix} h_+ \\ h_\times \end{pmatrix} \to \begin{pmatrix} h'_+ \\ h'_\times \end{pmatrix} = \begin{pmatrix} \cos 2\phi & -\sin 2\phi \\ \sin 2\phi & \cos 2\phi \end{pmatrix} \begin{pmatrix} h_+ \\ h_\times \end{pmatrix}, \tag{2.96}$$

or equivalently $h_+ \pm i h_\times \to h'_+ \pm i h'_\times = e^{\pm 2i\phi}(h_+ \pm i h_\times)$. Therefore, gravitational waves are indeed a spin-2 field.

How would gravitational waves produce temperature anisotropy? Suppose that we have a gravitational wave propagating in z direction.

1. If photons are propagating in the same direction (i.e., z direction), then there would be no change in temperature, as a gravitational wave does not distort space along its propagation direction.

2. If photons are propagating in x direction, then there would be redshift ($\delta T < 0$) if $\dot{h}_+ > 0$ (because space is stretching in x direction), and blueshift ($\delta T > 0$) if $\dot{h}_+ < 0$ (because space is contracting in x direction).

3. If photons are propagating in y direction, then there would be blueshift ($\delta T > 0$) if $\dot{h}_+ > 0$ (because space is contracting in y direction), and redshift ($\delta T < 0$) if $\dot{h}_+ < 0$ (because space is stretching in y direction).

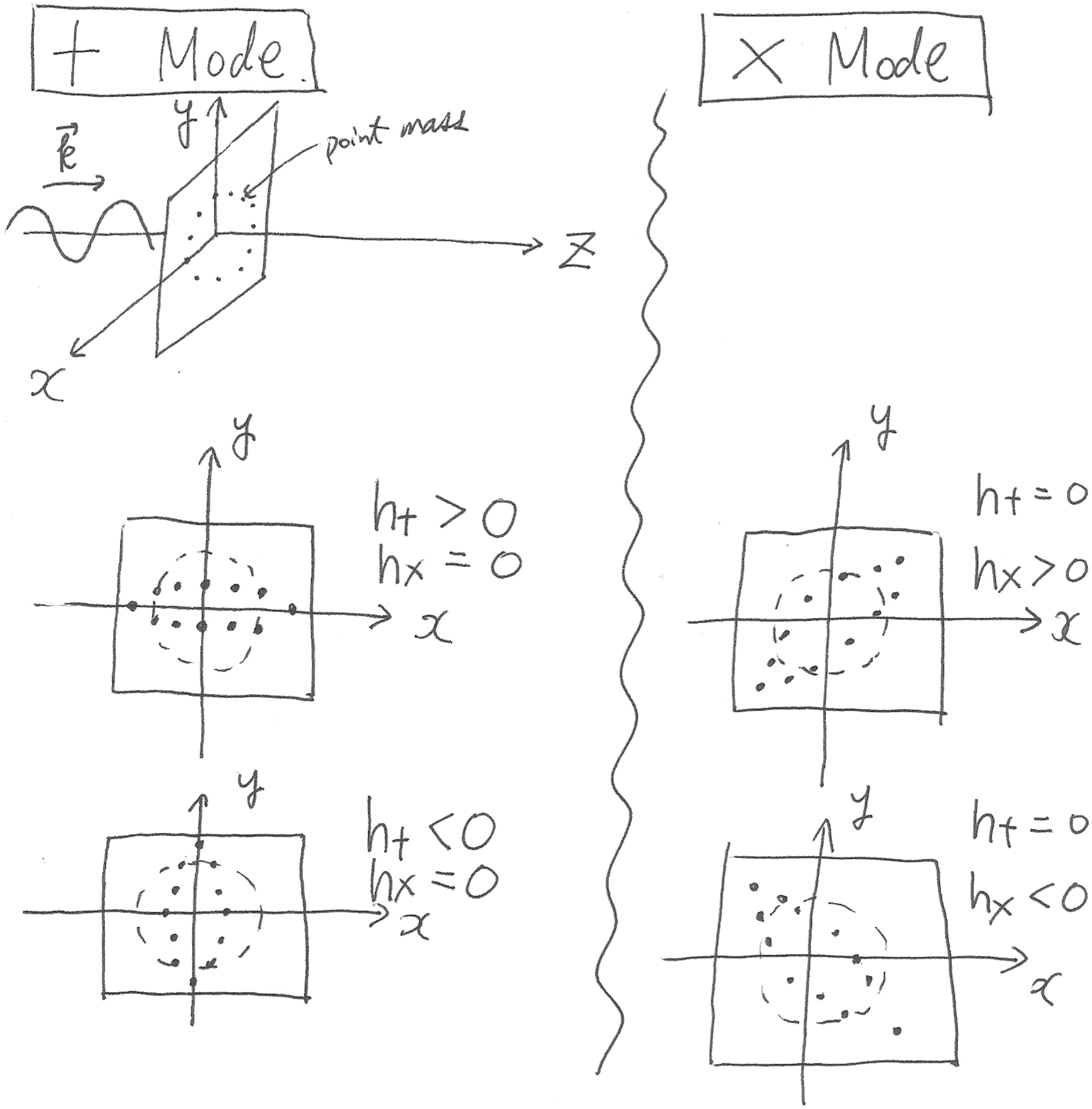
4. If photons are propagating in 45° direction, then there would be redshift ($\delta T < 0$) if $\dot{h}_\times > 0$ (because space is stretching in 45° direction), and blueshift ($\delta T > 0$) if $\dot{h}_\times < 0$ (because space is contracting in 45° direction).

5. If photons are propagating in 135° direction, then there would be blueshift ($\delta T > 0$) if $\dot{h}_\times > 0$ (because space is contracting in 135° direction), and redshift ($\delta T < 0$) if $\dot{h}_\times < 0$ (because space is stretching in 135° direction).

In general, as you derive in the homework problem, $\dot{h}_{ij}$ changes momentum of photons as

$$\frac{1}{p}\frac{dp}{dt} = -\frac{\dot{a}}{a} - \frac{1}{2}\sum_{ij} \dot{h}_{ij}\gamma^i\gamma^j, \tag{2.97}$$

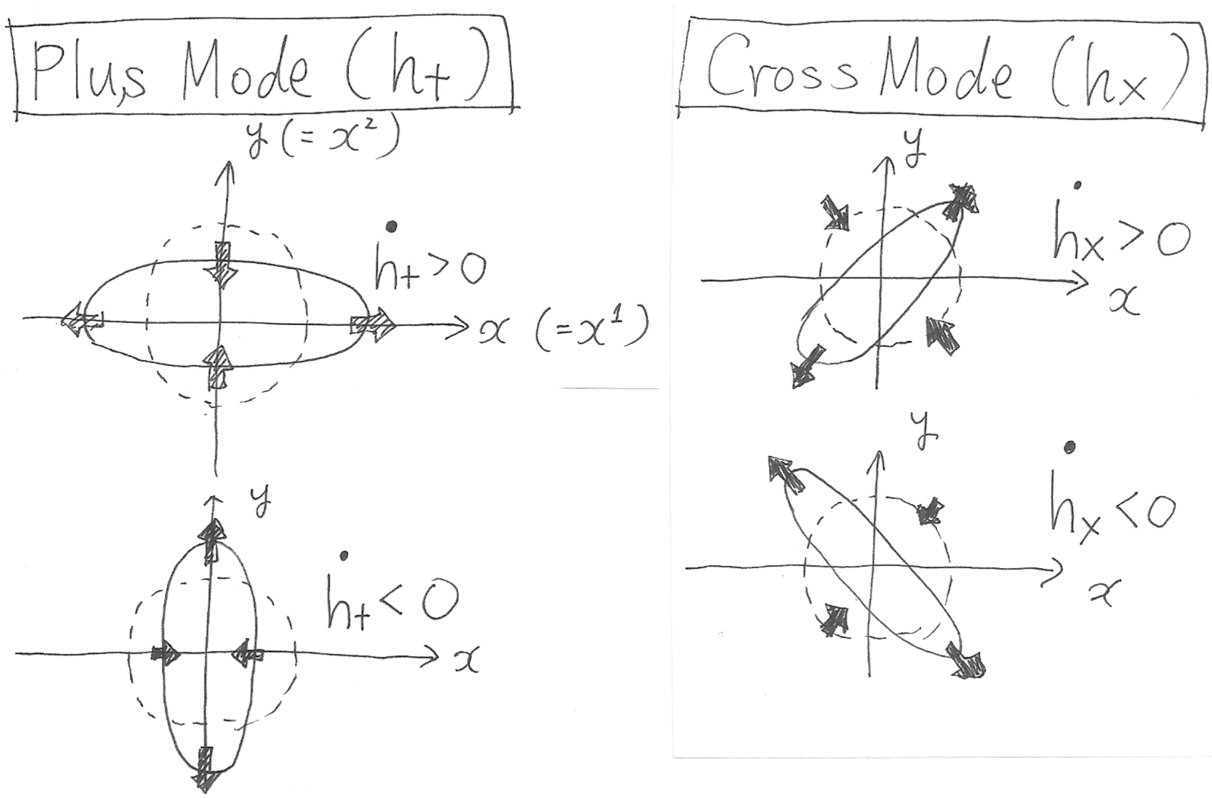
where as usual γ^i is the unit vector for a propagation direction of photons.



For a clock-wise rotation of coordinates by an angle ϕ, h_+ & $h_\times$ transform as

SPIN 2

$$\begin{pmatrix} h_+ \\ h_\times \end{pmatrix} \to \begin{pmatrix} h_+' \\ h_\times' \end{pmatrix} = \boxed{\begin{pmatrix} \cos 2\phi & -\sin 2\phi \\ \sin 2\phi & \cos 2\phi \end{pmatrix}} \begin{pmatrix} h_+ \\ h_\times \end{pmatrix}$$



Once again, by converting momentum to temperature anisotropy, and recalling $\hat{n}^i = -\gamma^i$ where $\hat{n}^i$ is the line-of-sight unit vector, we obtain

$$\left.\frac{\delta T}{\bar{T}}\right|_{\mathcal{O}}(\hat{n}^i) = \left.\frac{\delta T}{\bar{T}}\right|_{\mathcal{E}}(\hat{n}^i) - \frac{1}{2}\sum_{i'j'}\hat{n}^{i'}\hat{n}^{j'}\int_{t_{\mathcal{E}}}^{t_{\mathcal{O}}} dt\ \dot{h}_{i'j'}(r(t)\hat{n}^i, t), \tag{2.98}$$

where we have made explicit that h_{ij} depends on spatial coordinates and time, and that $x^i = \hat{n}^i r(t)$ where $r(t)$ is the comoving distance to the time t. Note that this formula is valid for h_{ij} propagating in any directions (not just z direction).

We now need to know how h_{ij} changes with time. For this purpose, we need to solve Einstein's equation for h_{ij}. This can be done in a straightforward way: Einstein's equation is (with $c = 1$)

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}. \tag{2.99}$$

We simply calculate the left hand side of this equation using the metric given by equation (2.90), with the transverse ($\sum_j \partial h_{ij}/\partial x^j = 0$) and traceless ($\sum_i h_{ii} = 0$) conditions. The result is remarkably simple:

$$-\frac{1}{2}\Box h_{ij} = 8\pi G\delta T_{ij}, \tag{2.100}$$

where δT_{ij} is the linear perturbation to T_{ij} that would affect gravitational waves, and $\Box h_{ij}$ is

$$\Box h_{ij} \equiv g^{\mu\nu} h_{ij;\mu\nu}. \tag{2.101}$$

For a perfect fluid, $\delta T_{ij} = 0$, and thus we have an equation describing waves propagating in vacuum:

$$\Box h_{ij} = 0. \tag{2.102}$$

For the Friedmann-Robertson-Walker metric, this equation becomes

$$\boxed{\ddot{h}_{ij} + 3\frac{\dot{a}}{a}\dot{h}_{ij} - \frac{1}{a^2}\nabla^2 h_{ij} = 0} \tag{2.103}$$

By Fourier-transforming h_{ij}:

$$h_{ij}(x^i, t) = \int \frac{d^3k}{(2\pi)^3}\tilde{h}_{ij}(k^i, t)e^{i\sum_j k^j x^j}, \tag{2.104}$$

the wave equation becomes

$$\ddot{\tilde{h}}_{ij} + 3\frac{\dot{a}}{a}\dot{\tilde{h}}_{ij} + \frac{k^2}{a^2}\tilde{h}_{ij} = 0. \tag{2.105}$$

For a matter-dominated universe, $a \propto t^{2/3}$, the solutions of this equation are

$$\tilde{h}_{ij}(k^i, t) = A_{ij}(k^i)\frac{3j_1(k\eta)}{k\eta} + B_{ij}(k^i)\frac{y_1(k\eta)}{k\eta}, \tag{2.106}$$

where A_{ij} and B_{ij} are constant matrices (which represent initial conditions) and η is defined as

$$\eta \equiv \int \frac{dt}{a(t)} = 3t_0^{2/3}t^{1/3}, \tag{2.107}$$

which is, in a flat universe, related to the comoving distance as $r(t) = \eta_0 - \eta(t)$ (with $c = 1$ and $\eta_0 = 3t_0$ is the present-day value). The functions j_1 and y_1 are the spherical Bessel functions of the first and second kind, respectively:

$$j_1(x) = \frac{\sin(x)}{x^2} - \frac{\cos(x)}{x}, \tag{2.108}$$

$$y_1(x) = -\frac{\cos(x)}{x^2} - \frac{\sin(x)}{x}. \tag{2.109}$$

In order to determine the initial conditions, let us take the limit of $t \to 0$. We find

$$\tilde{h}_{ij}(k^i, t \to 0) \to A_{ij}(k^i) - \frac{B_{ij}(k^i)}{(k\eta)^3}. \tag{2.110}$$

The second term blows up as $t \to 0$, which is unphysical. Therefore, we take $B_{ij} = 0$ as the initial condition. The final form of the solution during the matter era is then

$$\boxed{\tilde{h}_{ij}(k^i, t) = A_{ij}(k^i)\frac{3j_1(k\eta)}{k\eta}} \tag{2.111}$$

and its time derivative is

$$\boxed{\dot{\tilde{h}}_{ij}(k^i, t) = -\frac{kA_{ij}(k^i)}{a(t)}\frac{3j_2(k\eta)}{k\eta}} \tag{2.112}$$

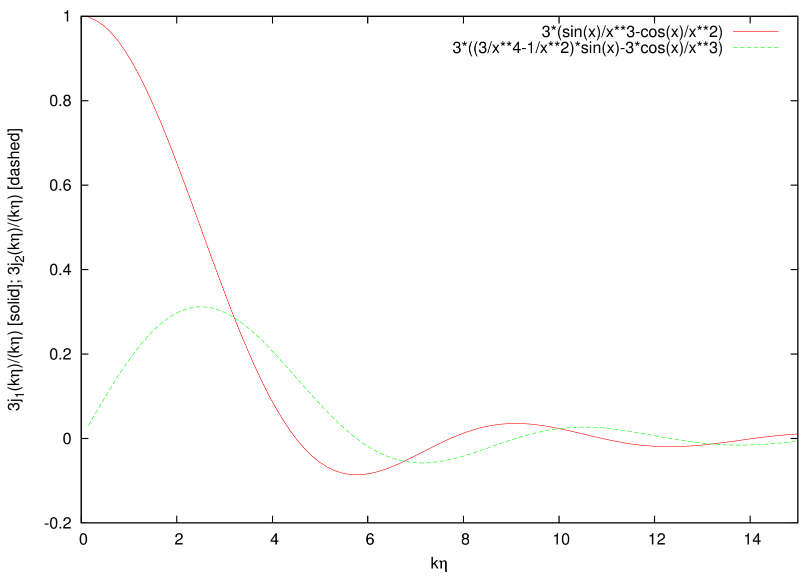
where

$$j_2(x) \equiv \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin(x) - \frac{3}{x^2}\cos(x). \tag{2.113}$$

With this result, we can finally write the temperature anisotropy as

$$\begin{aligned}\left.\frac{\delta T}{\bar{T}}\right|_{\mathcal{O}}(\hat{n}^i) &= \left.\frac{\delta T}{\bar{T}}\right|_{\mathcal{E}}(\hat{n}^i) + \frac{3}{2}\sum_{i'j'}\hat{n}^{i'}\hat{n}^{j'}\int\frac{d^3k}{(2\pi)^3}A_{i'j'}(\vec{k})\int_{t_\mathcal{E}}^{t_\mathcal{O}}\frac{kdt}{a(t)}\frac{j_2(k\eta)}{k\eta}e^{i\vec{k}\cdot\hat{n}r(t)} \\ &= \left.\frac{\delta T}{\bar{T}}\right|_{\mathcal{E}}(\hat{n}^i) + \frac{3}{2}\sum_{i'j'}\hat{n}^{i'}\hat{n}^{j'}\int\frac{d^3k}{(2\pi)^3}A_{i'j'}(\vec{k})\int_{x_\mathcal{E}}^{x_\mathcal{O}}dx\frac{j_2(x)}{x}e^{i\hat{k}\cdot\hat{n}(x_\mathcal{O}-x)}, \end{aligned} \tag{2.114}$$

where $x \equiv k\eta$. Since $j_2(x)/x$ peaks at $x \approx 2$, the integral over x is dominated by the modes with $k\eta \approx 2$, with higher k modes highly suppressed. This will be reflected on the shape of the angular power spectrum of temperature anisotropy from gravitational waves.



Calculation of C_l from equation (2.114) is a bit involved, so let us just give the result:

$$\boxed{C_l = \frac{(l+2)!}{(l-2)!}\int\frac{dk}{2\pi}P_h(k)\left[\int_{\eta_\mathcal{E}}^{\eta_\mathcal{O}}kd\eta\frac{3j_2(k\eta)}{k\eta}\frac{j_l[k(\eta_\mathcal{O}-\eta)]}{k^2(\eta_\mathcal{O}-\eta)^2}\right]^2} \tag{2.115}$$

where $P_h(k)$ is the power spectrum of each polarization state of the gravitational wave:

$$\langle h_+(\vec{k})h_+^*(\vec{k}')\rangle = \langle h_\times(\vec{k})h_\times^*(\vec{k}')\rangle = (2\pi)^3P_h(k)\delta_D^{(3)}(\vec{k}-\vec{k}'). \tag{2.116}$$

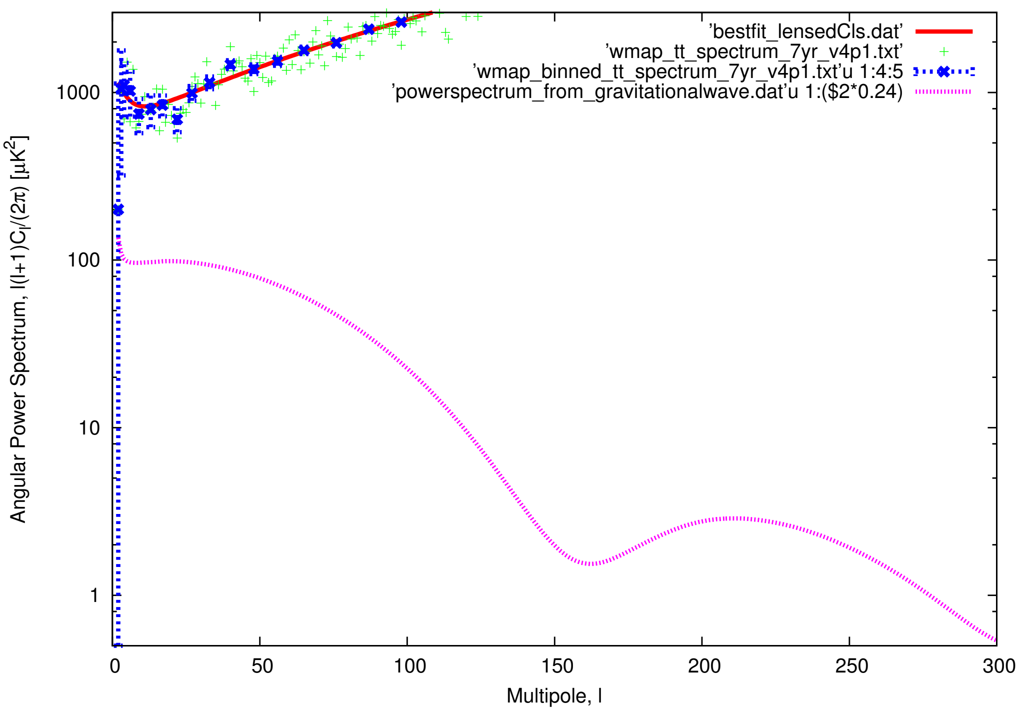
Similarly to what we have done for the scalar perturbation (gravitational potential contribution), we usually parametrize P_h as

$$P_h(k) = \frac{2\pi^2}{k^3}\Delta_h^2(k_0)\left(\frac{k}{k_0}\right)^{n_t}, \tag{2.117}$$

and n_t is called the **tensor spectral tilt**. We then define the so-called **tensor-to-scalar ratio**, r, defined by

$$r \equiv \frac{4\Delta_h^2(k_0)}{\Delta_\zeta^2(k_0)}, \tag{2.118}$$

where a factor of four is there for a historical reason.



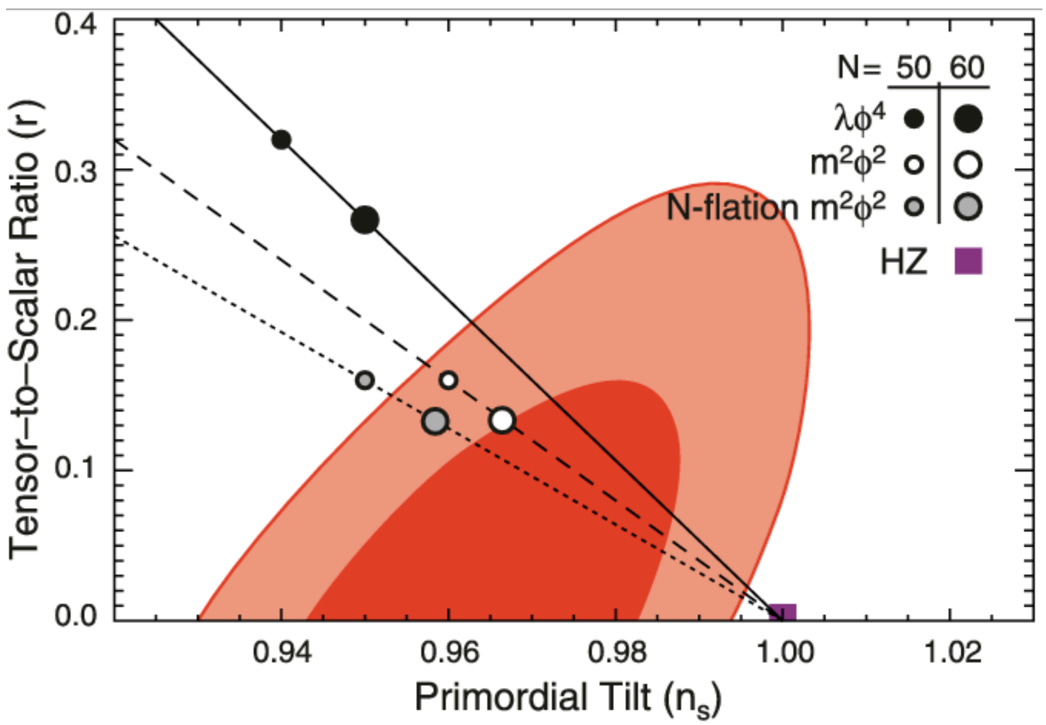
Simple single-field inflation models predict a relation between the tensor tilt and the tensor-to-scalar ratio as

$$\boxed{r = -8n_t} \tag{2.119}$$

Therefore, using this relation, we have only three parameters for characterizing the primordial perturbation spectra produced by inflation: $\Delta_\zeta(k_0)$, r, and n_s. Among these, r is particularly important because a detection of non-zero r means a detection of primordial gravitational wave created during inflation. Many experts think that the detection of r would be a proof of inflation. Currently, we have not detected r, and the latest limit on r is (Komatsu et al., Astrophysical Journal Supplement Series, 192, 18 (2011))

$$\boxed{r < 0.24 \text{ (95\% C.L.)}} \tag{2.120}$$

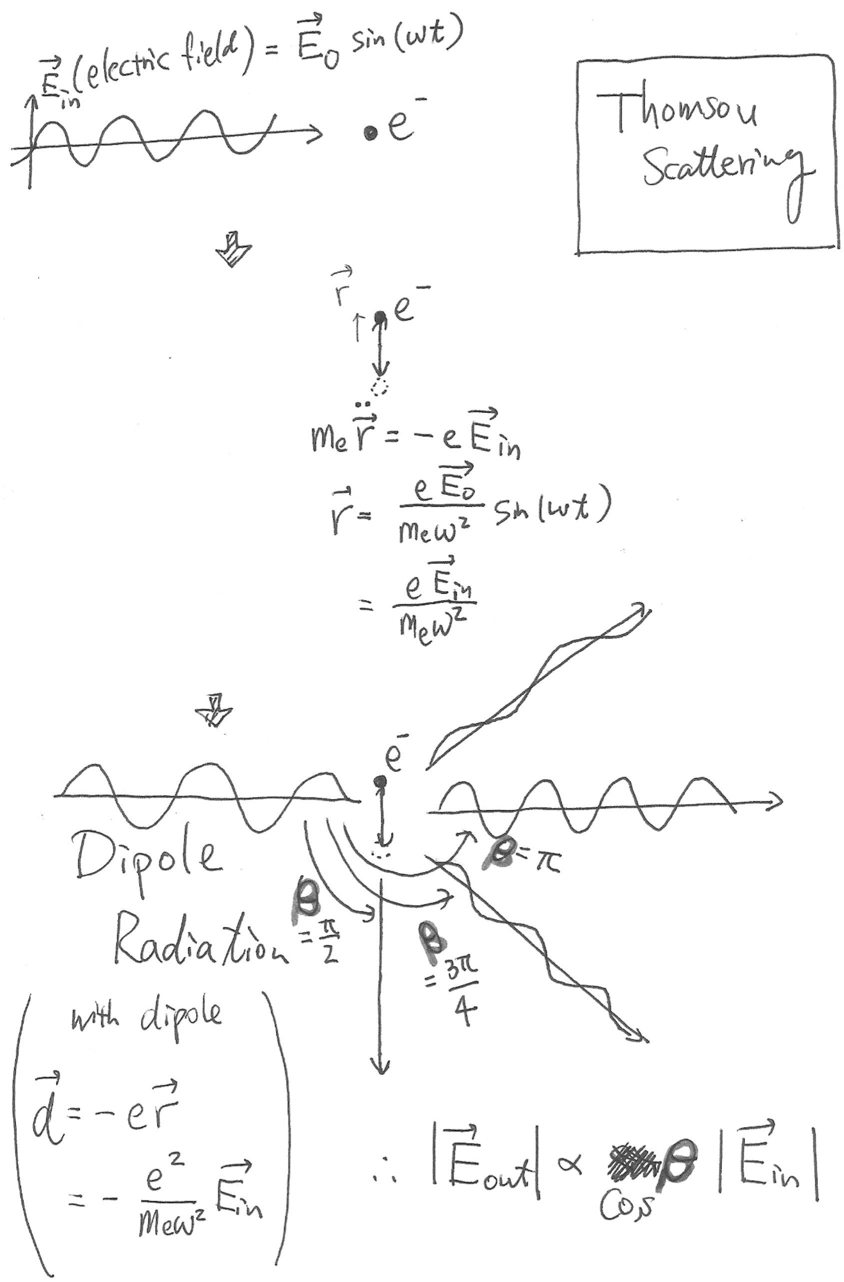
This constraint comes from the temperature anisotropy spectra that we have learned so far. Since the gravitational wave spectrum adds power to lower multipoles (mainly $l < 50$), it tilts the total power spectrum. This effect can be absorbed by increasing n_s (which will make the contribution from ζ at low multipoles smaller), and thus there is a positive correlation between n_s and r. This gives a fundamental limit on $r \approx 0.1$ we can reach by using the temperature power spectrum alone. In order to break this correlation, one must use not only the temperature power spectrum, but only the power spectrum of **polarization** of the cosmic microwave background.



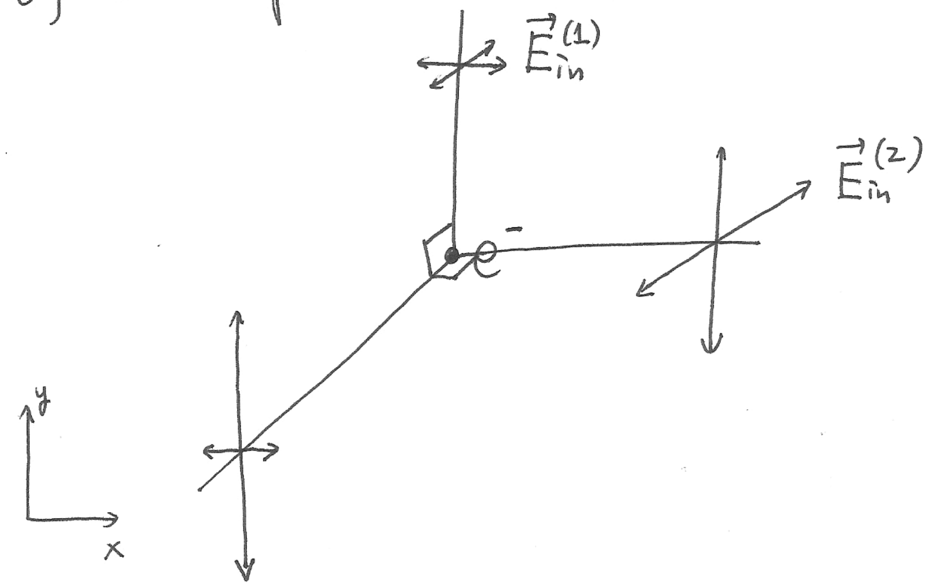
2.5 Polarization

The cosmic microwave background is weakly polarized. Polarization is generated by Thomson scattering, and thus it is generated at the last scattering surface ($z = 1090$) and during the epoch of reionization ($z < 15$).

The way Thomson scattering generates can be understood easily by recalling the dipole radiation.

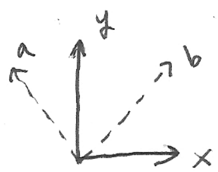


Generating Polarization by Thomson Scattering of Quadrupole Anisotropy

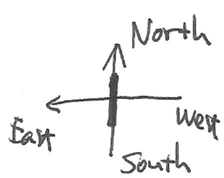


$$Q \equiv |E_y^{out}|^2 - |E_x^{out}|^2 = |\vec{E}_{in}^{(2)}|^2 - |\vec{E}_{in}^{(1)}|^2 > 0$$

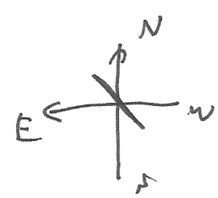
[For this example]



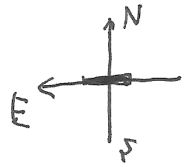
$$U \equiv |E_a^{out}|^2 - |E_b^{out}|^2 = 0$$



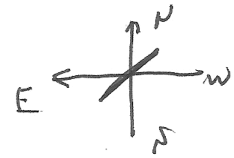
$Q>0$ (North-South)



$U>0$ (NE-SW)



$Q<0$ (East-West)



$U<0$ (NW-SE)

Now, let us place an electron at the origin of coordinates, have some temperature anisotropy around it, and calculate the polarization pattern produced by that electron. It is easy to imagine this problem by thinking about it from a point of view of the electron at the origin.

Consider the Sachs–Wolfe effect, $\delta T/T = \frac{1}{3}\Psi$, with Ψ being a plane wave going in z direction.

$$\frac{\delta T(\hat{n})}{\bar{T}} = \frac{1}{3}A\cos(kz\cos\theta), \tag{2.121}$$

where $A > 0$ is a constant representing the amplitude of Ψ, and θ is the usual polar angle measured from the z direction. The origin ($z = 0$) is hotter ($\delta T > 0$).

The polarization produced by a scalar perturbation (such as Ψ) scattered by an electron is given by

$$Q + iU = -\frac{\sqrt{6}}{10}\,{}_2Y_2^0(\theta,\phi)\int d\tilde{\Omega}\frac{\delta T(\tilde{\theta},\tilde{\phi})}{\bar{T}}(Y_2^0)^*(\tilde{\theta},\tilde{\phi}), \tag{2.122}$$

$$Q - iU = -\frac{\sqrt{6}}{10}\,{}_{-2}Y_2^0(\theta,\phi)\int d\tilde{\Omega}\frac{\delta T(\tilde{\theta},\tilde{\phi})}{\bar{T}}(Y_2^0)^*(\tilde{\theta},\tilde{\phi}). \tag{2.123}$$

Here, Y_l^m is the spherical harmonics, and ${}_2Y_l^m$ is the spin-2 harmonics. For $l = 2$ and $m = 0$, we have

$$Y_2^0 = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1), \tag{2.124}$$

$${}_2Y_2^0 = \sqrt{\frac{15}{32\pi}}\sin^2\theta, \tag{2.125}$$

$${}_{-2}Y_2^0 = {}_2Y_2^0. \tag{2.126}$$

Note that, in general, ${}_{-s}Y_l^m = (-1)^{m+s}({}_sY_l^{-m})^*$.

The reason why $Q \pm iU$ is described by spin-2 harmonics is that $Q \pm iU$ is the spin-2 quantity. For a clock-wise rotation of coordinates by an angle φ, Q and U transform as

$$\begin{pmatrix} Q \\ U \end{pmatrix} \to \begin{pmatrix} Q' \\ U' \end{pmatrix} = \begin{pmatrix} \cos 2\varphi & -\sin 2\varphi \\ \sin 2\varphi & \cos 2\varphi \end{pmatrix}\begin{pmatrix} Q \\ U \end{pmatrix}, \tag{2.127}$$

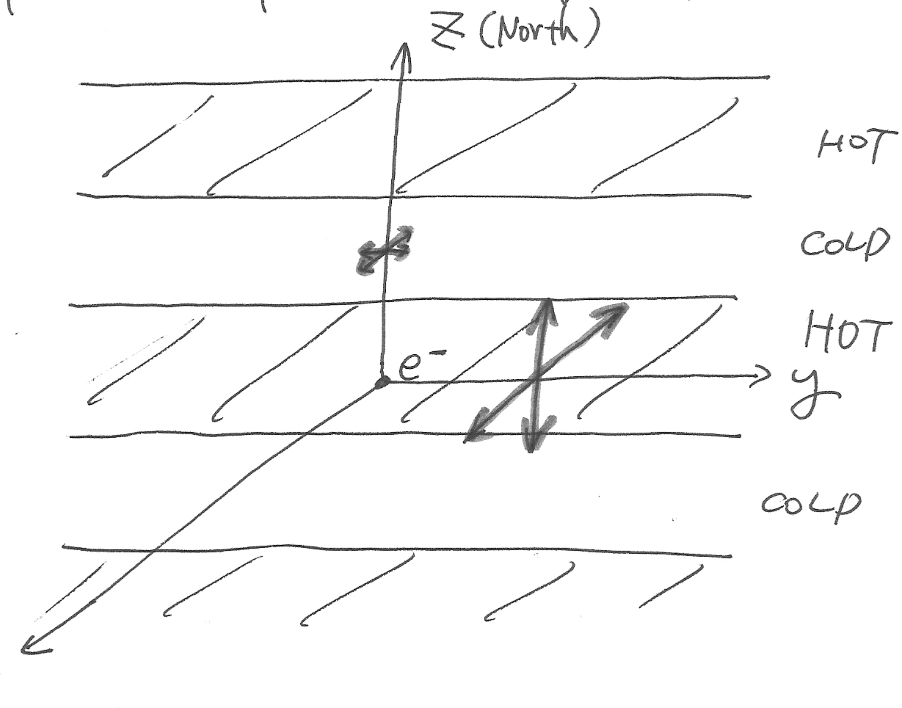
or equivalently $Q \pm iU \to Q' \pm iU' = e^{\pm 2i\varphi}(Q \pm iU)$. Therefore, gravitational waves are indeed a spin-2 field.

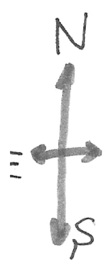
First of all, it follows from ${}_{-2}Y_2^0 = {}_2Y_2^0$ that $U = 0$. Then, the Q polarization is given by

$$\begin{aligned} Q &= -\frac{\sqrt{6}}{10}\sqrt{\frac{15}{32\pi}}\sin^2\theta\int_{-1}^{1} d\cos\theta\frac{1}{3}A\cos(kz\cos\theta)Y_2^0(\theta)\int_0^{2\pi} d\phi \\ &= \frac{1}{4}Aj_2(kz)\sin^2\theta. \end{aligned} \tag{2.128}$$

Therefore, an observer at $\theta = 0$ does not see any polarization, while an observer at $\theta = \pi/2$ sees the maximum polarization with $Q > 0$ (polarization in the north-south direction). All of this can be understood graphically (see the next page).

For a plane wave temperature anisotropy going in z direction:



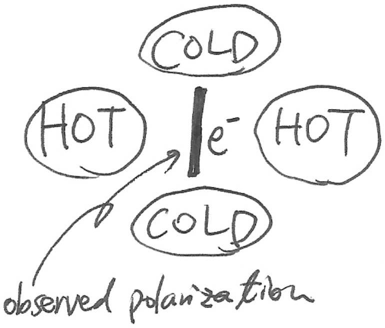


$Q>0$

$U=0$

Therefore the polarization direction is

Perpendicular to cold, and parallel to hot!



PROBLEM SET 2

1.1 Cosmic Microwave Background - I

While the speed of light is kept for completeness below, you may set $c = 1$ if you wish.

1.1.1 Propagation of photons in a clumpy universe

How does the momentum of photons change as photons propagate through space? First, every photon suffers from the mean cosmological redshift, and thus its magnitude, p, will decrease as $p \propto 1/a$. In addition, as photons pass through potential wells and troughs, they gain or lose momentum. Finally, not only the magnitude, p, but also the direction of momentum, γ^i, will change when photons are deflected gravitationally.

We can calculate the evolution of four-dimensional momentum, $p^\mu \equiv dx^\mu/d\lambda$, using the following **geodesic equation**:

$$\frac{dp^\mu}{d\lambda} + \sum_{\alpha\beta} \Gamma^\mu_{\alpha\beta} p^\alpha p^\beta = 0. \tag{1.129}$$

Here, λ is a parameter which gives the location along the path of photons. Using $p^0 = d(ct)/d\lambda$, one may rewrite the geodesic equation in terms of the total time derivative of p^μ:

$$\frac{dp^\mu}{dt} + c \sum_{\alpha\beta} \Gamma^\mu_{\alpha\beta} \frac{p^\alpha p^\beta}{p^0} = 0. \tag{1.130}$$

In order to calculate $\Gamma^\mu_{\alpha\beta}$, we need to specify the metric. To describe a clumpy universe, we perturb the Robertson-Walker metric in the following way:

$$ds^2 = -[1 + 2\Psi(t, x^i)]c^2 dt^2 + a^2(t)[1 + 2\Phi(t, x^i)] \sum_{ij} \delta_{ij} dx^i dx^j. \tag{1.131}$$

Here, Ψ is the usual Newtonian potential (divided by c^2 to make it dimensionless), and Φ is called the **curvature perturbation**. For this metric, all of the components of $\Gamma^\mu_{\alpha\beta}$ are non-zero.

From now on, we will assume that the magnitudes of these variables are small: $|\Psi| \ll 1$ and $|\Phi| \ll 1$, and calculate everything only up to the first order in these variables.

Question 1.1: Calculate Γ^0_{00}, Γ^0_{0i}, Γ^0_{ij}, Γ^i_{00}, Γ^i_{0j}, and Γ^i_{jk}, up to the first order in Φ and Ψ. You may use the short-hand notation such as

$$\dot{\Psi} \equiv \frac{\partial \Psi}{\partial t}, \quad \Psi_{,i} \equiv \frac{\partial \Psi}{\partial x^i}.$$

The components of the metric and its inverse are given by

$$g_{00} = -(1 + 2\Psi); \quad g^{00} = -(1 - 2\Psi); \quad g_{ij} = a^2(1 + 2\Phi)\delta_{ij}; \quad g^{ij} = \frac{1}{a^2}(1 - 2\Phi)\delta^{ij}. \tag{1.132}$$

Question 1.2: Write down the geodesic equations in the following form:

$$\frac{dp^0}{dt} = \dots,$$
$$\frac{dp^i}{dt} = \dots,$$

up to the first order in Φ and Ψ. The final answers should not contain $\sum_{ij}\delta_{ij}p^ip^j$. You can eliminate this by using the normalization condition for momentum of massless particles, $\sum_{\alpha\beta} g_{\alpha\beta}p^\alpha p^\beta = 0$, which gives, for the above perturbed metric,

$$a^2\sum_{ij}\delta_{ij}p^ip^j = (1-2\Phi+2\Psi)(p^0)^2. \tag{1.133}$$

Question 1.3: Now, we want to derive the evolution equations for the magnitude of momentum, p, and its direction, γ^i. First, we define the magnitude as

$$p^2 \equiv \sum_{ij} g_{ij}p^ip^j. \tag{1.134}$$

Also, we normalize the direction such that

$$\sum_{ij}\delta_{ij}\gamma^i\gamma^j = 1. \tag{1.135}$$

Using this information, write p in terms of p_0 and Ψ, and write γ^i in terms of p, p^i, a, and Φ, up to the first order in Φ and Ψ.

Question 1.4: Write down the geodesic equations in the following form:

$$\frac{dp}{dt} = \dots,$$
$$\frac{d\gamma^i}{dt} = \dots,$$

up to the first order in Φ and Ψ. The answers should not contain p^0 or p^i. Whenever you find them, replace them with p and γ^i, respectively. You can check the result for the deflection equation, $d\gamma^i/dt$, by making sure that the result satisfies $\sum_i \gamma^i d\gamma^i/dt = 0$. (You can derive this by differentiating the normalization condition, $\sum_{ij}\delta_{ij}\gamma^i\gamma^j = 1$, with respect to time.) Note that the total time derivative of a variable is related to the partial derivatives as, e.g.,

$$\frac{d\Phi}{dt} = \dot{\Phi} + \sum_i \frac{dx^i}{dt}\Phi_{,i} = \dot{\Phi} + \sum_i \frac{cp^i}{p^0}\Phi_{,i}. \tag{1.136}$$

1.1.2 Perturbed Conservation Equations For A Pressure-less Fluid

Consider the stress-energy tensor for a perfect fluid. We then take the limit that the pressure is much less than the energy density, which would be a good approximation for a non-relativistic fluid. The stress-energy tensor for such a pressure-less fluid is

$$T_{\mu\nu} = \rho \frac{(\sum_\alpha g_{\mu\alpha} u^\alpha)(\sum_\beta g_{\nu\beta} u^\beta)}{c^2}. \tag{1.137}$$

As usual, $u^\mu \equiv dx^\mu/d\tau$ is a four-dimensional velocity and τ is the proper time.

Suppose that the fluid is moving at a non-relativistic physical three-dimensional velocity of $V^i \ll c$. By "physical" velocity, we mean

$$V^i \equiv a u^i = a\frac{dx^i}{d\tau}. \tag{1.138}$$

We also expand the energy density into the mean, $\bar{\rho}$, and the fluctuation around the mean, δ:

$$\rho = \bar{\rho}(1+\delta). \tag{1.139}$$

These perturbation variables, δ and V^i/c, are small in the same sense that Φ and Ψ are small. Therefore, we shall expand everything only up to the first order in Φ, Ψ, δ, and V^i/c. For example, T_{ij} is of order $(V/c)^2$, and thus can be ignored. On the other hand, T_{0i} is of order (V/c), and thus cannot be ignored unless it is multiplied by other perturbation variables.

Question 1.5: Expand the following conservation equations up to the first order in Φ, Ψ, δ, and V^i/c:

1. Energy conservation equation, $\sum_{\alpha\beta} g^{\alpha\beta} T_{0\alpha;\beta} = 0$

2. Momentum conservation equation, $\sum_{\alpha\beta} g^{\alpha\beta} T_{i\alpha;\beta} = 0$

Use the conservation equation for the mean density, $\dot{\bar{\rho}} + 3\frac{\dot{a}}{a}\bar{\rho} = 0$, to eliminate the mean contributions from the above equations, and then rewrite these equations in the following form:

$$\begin{aligned} \dot{\delta} &= \ldots, \\ \frac{\dot{V}^i}{c} &= \ldots. \end{aligned}$$

1.1.3 Large-scale Solutions of Einstein Equations During Matter Era

The energy and momentum conservation equations contain four unknown perturbation variables, δ, V^i, Ψ, and Φ. Therefore, we cannot find solutions unless we have (at least) two more equations. Such equations are provided by perturbed Einstein equations. Don't worry - you are not asked to derive them (though I would not stop you from deriving them). Here are the two equations that can be derived by combining perturbed Einstein equations:[‡‡]

$$\frac{k^2}{a^2}\tilde{\Phi} = \frac{4\pi G}{c^4}\bar{\rho}\left(\tilde{\delta}+\frac{3\dot{a}\tilde{V}}{kc^2}\right), \tag{1.140}$$

$$\tilde{\Psi} = -\tilde{\Phi}. \tag{1.141}$$

Here, $\tilde{\Phi}$, $\tilde{\Psi}$, $\tilde{\delta}$, and $\tilde{V}$ are all in Fourier space, i.e., $\tilde{\Phi}=\tilde{\Phi}(\vec{k},t)$, $\tilde{\Psi}=\tilde{\Psi}(\vec{k},t)$, $\tilde{\delta}=\tilde{\delta}(\vec{k},t)$, and $\tilde{V}=\tilde{V}(\vec{k},t)$, and $\vec{k}$ is the **comoving wavenumber vector**. They are related to the original variables in position space by, e.g.,

$$\tilde{\Psi}(\vec{k},t) = \int d^3x\Psi(\vec{x},t)e^{-i\vec{k}\cdot\vec{x}}, \tag{1.142}$$

$$\Psi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3}\tilde{\Psi}(\vec{k},t)e^{i\vec{k}\cdot\vec{x}}. \tag{1.143}$$

Here, $\vec{k}\cdot\vec{x}\equiv\sum_{ij}\delta_{ij}k^ix^j$. For example, the left hand side of the first perturbed Einstein equation, $(k^2/a^2)\Phi$, came from the Laplacian of Φ:

$$\begin{aligned}\frac{1}{a^2}\nabla^2\Phi(\vec{x},t) &= \frac{1}{a^2}\int\frac{d^3k}{(2\pi)^3}\tilde{\Phi}(\vec{k},t)\left(\nabla^2e^{i\vec{k}\cdot\vec{x}}\right)\\ &= \frac{1}{a^2}\int\frac{d^3k}{(2\pi)^3}\tilde{\Phi}(\vec{k},t)\left(-k^2e^{i\vec{k}\cdot\vec{x}}\right),\end{aligned} \tag{1.144}$$

where $\nabla^2\equiv\sum_{ij}\delta^{ij}\frac{\partial^2}{\partial x^i\partial x^j}$, and $k^2\equiv\sum_{ij}\delta_{ij}k^ik^j$. Also, $\tilde{V}$ in the right hand side of the first perturbed Einstein equation is defined as $\tilde{V}(\vec{k},t)\equiv i\hat{k}\cdot\vec{\tilde{V}}(\vec{k},t)$ (where $\hat{k}\equiv\vec{k}/k$ is a unit vector), i.e.,

$$\begin{aligned}\vec{\nabla}\cdot\vec{V}(\vec{x},t) &= \int\frac{d^3k}{(2\pi)^3}\vec{\tilde{V}}(\vec{k},t)\cdot\left(\vec{\nabla}e^{i\vec{k}\cdot\vec{x}}\right)\\ &= \int\frac{d^3k}{(2\pi)^3}\vec{\tilde{V}}(\vec{k},t)\cdot\left(i\vec{k}e^{i\vec{k}\cdot\vec{x}}\right)\end{aligned} \tag{1.145}$$

$$\equiv \int\frac{d^3k}{(2\pi)^3}k\tilde{V}(\vec{k},t)e^{i\vec{k}\cdot\vec{x}}. \tag{1.146}$$

Here, $\vec{\nabla}\cdot\vec{V}\equiv\sum_k V^k_{,k}$.

Now, let us use the above four equations to find solutions for Ψ, Φ, V, and δ. From now on, we shall drop the tildes on variables in Fourier space for simplicity. It is convenient to change the independent variable from t to the scale factor, a. Finally, let us define the following variable:

$$\epsilon(a)\equiv\frac{ck}{\dot{a}}, \tag{1.147}$$

[‡‡]To those who wish to derive these results: the first equation can be obtained by combining perturbed $G_{00}=(8\pi G/c^4)T_{00}$ and $G_{0i}=(8\pi G/c^4)T_{0i}$, while the second equation can be obtained from the traceless part of $G_{ij}=(8\pi G/c^4)T_{ij}$.

which goes as $\epsilon \propto \sqrt{a}$ during the matter-dominated era. This quantity is useful, as it is much less than unity for fluctuations whose wavelength is longer than the Hubble length (≈horizon size):

$$\epsilon \ll 1 \quad \text{for super-horizon fluctuations, } k \ll aH/c,$$

where $H = \dot{a}/a$ is the Hubble expansion rate. Therefore, we can find large-scale (long-wavelength; super-horizon) solutions by consistently ignoring higher-order terms of ϵ.

Question 1.6: Using the Fourier-space variables and ϵ, show that the energy- and momentum-conservation equations can be re-written as follows:

$$\delta' = -\frac{\epsilon}{a}\frac{V}{c} - 3\Phi', \tag{1.148}$$

$$\frac{V'}{c} = -\frac{1}{a}\frac{V}{c} + \frac{\epsilon}{a}\Psi, \tag{1.149}$$

where the primes denote derivatives with respect to a.

Question 1.7: Using $\Phi = -\Psi$, we now have the following three equations for three unknown variables:

$$\delta' = -\frac{\epsilon}{a}\frac{V}{c} - 3\Phi', \tag{1.150}$$

$$\frac{V'}{c} = -\frac{1}{a}\frac{V}{c} - \frac{\epsilon}{a}\Phi, \tag{1.151}$$

$$\epsilon^2\Phi = \frac{3}{2}\left(\delta + \frac{3V}{\epsilon c}\right). \tag{1.152}$$

Once again, during the matter era, $\epsilon \propto \sqrt{a}$. Solve these equations on super-horizon scales, $\epsilon \ll 1$, and show that non-decaying solutions are given by

$$\delta = 2\Phi, \tag{1.153}$$

$$\frac{V}{c} = -\frac{2}{3}\epsilon\Phi. \tag{1.154}$$

By "non-decaying solutions" we mean the solutions that go as $\propto a^n$ where $n \geq 0$. Finally, show that Φ (and hence Ψ) is a constant and does not depend on a in the super-horizon limit.

Hint: you cannot ignore ϵ when two different variables are involved, e.g., $A + \epsilon B \neq A$, because you do not know a priori how A compares with B. You can ignore the terms of order ϵ only when you are sure that ϵ is compared to order unity, e.g., $A' + \frac{A}{a} + \epsilon\frac{A}{a} \approx A' + \frac{A}{a}$.

Do not use Mathematica to solve these coupled differential equations! Use your brain, please.

PROBLEM SET 3

1.1 Cosmic Microwave Background - II

While the speed of light is kept for completeness below, you may set $c = 1$ if you wish.

1.1.1 Temperature Anisotropy From Gravitational Waves

Gravitational waves stretch space as they propagate through space. This deformation of space is characterized by the following metric:

$$ds^2 = -c^2 dt^2 + a^2(t) \sum_{ij} (\delta_{ij} + h_{ij}) dx^i dx^j,$$

where h_{ij} is the so-called **tensor metric perturbation**. (On the other hand, Φ and Ψ that we have dealt with before are called "scalar metric perturbations".) The tensor metric perturbation is symmetric ($h_{ij} = h_{ji}$), traceless ($\sum_{i=1}^{3} h_{ii} = 0$), and transverse ($\sum_{j=1}^{3} \frac{\partial h_{ij}}{\partial x^j} = 0$).

At the first-order of perturbations, scalar and tensor perturbations are decoupled, and thus we can ignore the scalar perturbations when analyzing the tensor perturbations.

Question 1.1: Write down the geodesic equation for $p \equiv (\sum_{ij} g_{ij} p^i p^j)^{1/2}$ with the metric given above, up to the first order in h_{ij}. Then, by integrating the geodesic equation over time, derive the formula for the observed temperature anisotropy from gravitational waves as

$$\left.\frac{\delta T}{\bar{T}}\right|_{\mathcal{O}} = \left.\frac{\delta T}{\bar{T}}\right|_{\mathcal{E}} + \int_{t_{\mathcal{E}}}^{t_{\mathcal{O}}} dt \ (\dots)$$

where $(\dots)$ should contain only $\dot{h}_{ij}$ and γ^i (where γ^i is the unit vector of the direction of photons, satisfying $\sum_{ij} \delta_{ij} \gamma^i \gamma^j = 1$). *Hint*: you should check the result by making sure that you can recover a part of the scalar integrated Sachs–Wolfe effect, $-\dot{\Phi}$, by using the scalar metric perturbation, $h_{ij} = 2\Phi\delta_{ij}$. (You cannot recover the terms containing Ψ because $g_{00} = -1$ for the above metric.)

From now on, set $\left.\frac{\delta T}{T}\right|_{\mathcal{E}} = 0$.

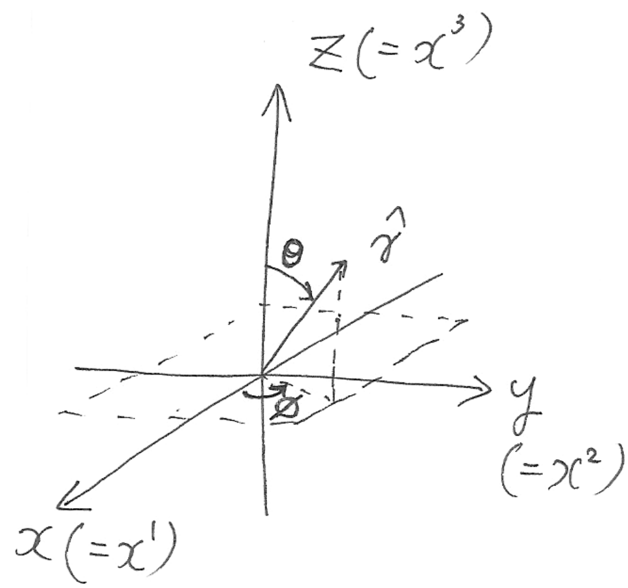
Question 1.2: Consider a gravitational wave propagating in the z $(= x^3)$ direction. For this special case, the components of the tensor metric perturbation are given by

$$h_{ij} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

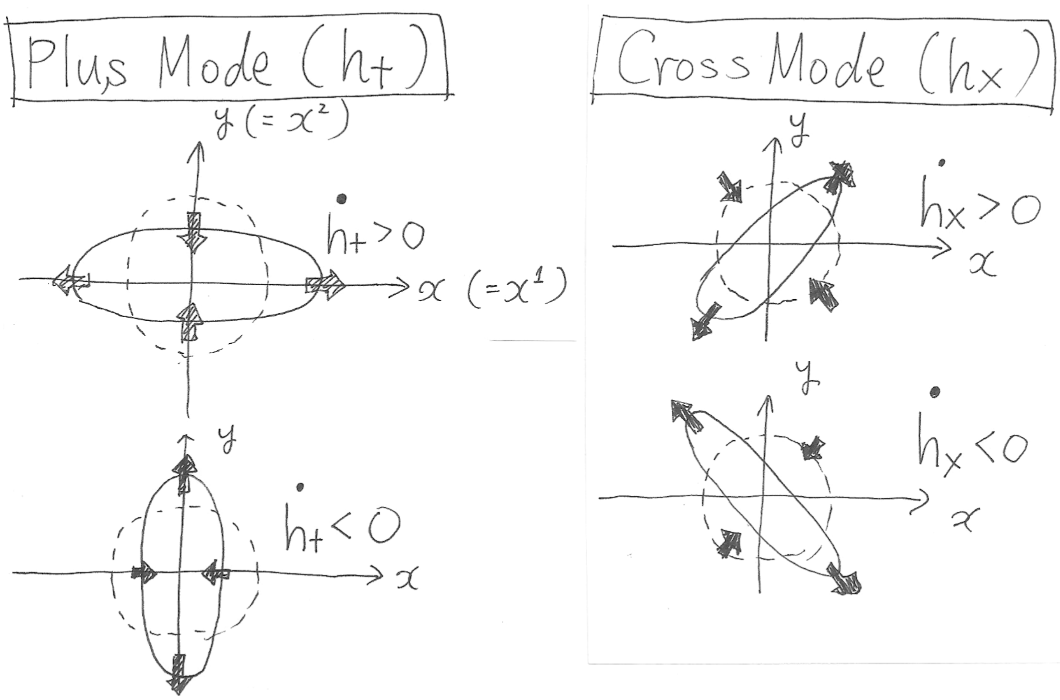
where h_+ and $h_\times$ denote two *linear* polarization states of a gravitational wave. Using polar coordinates for the propagation direction of photons with respect to the gravitational wave:

$$\gamma^i = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta),$$

rewrite the equation for $\frac{\delta T}{T}\big|_{\mathcal{O}}$ in terms of $\int \dot{h}_+ dt$, $\int \dot{h}_\times dt$, and trigonometric functions.



Question 1.3: A gravitational wave with $\dot{h}_+ > 0$ stretches space in x direction, while that with $\dot{h}_\times > 0$ stretches space in 45° direction (see the figure below). This stretching of space causes gravitational redshifts and blueshifts in the corresponding directions. Using this picture, give **physical** explanations for the result obtained in Question 1.2. (In other words, now that you have an equation, how much physical interpretation can you get out of this equation?) *For example*: in which cases do you find hot ($\Delta T > 0$) or cold ($\Delta T < 0$), and why?; compare the results for $\theta = 0$ and $\theta = \pi/2$, and give a physical explanation for the difference; compare the results for $\phi = 0$, $\pi/4$, $\pi/2$, and $3\pi/4$, and give a physical explanation for the difference. Use graphics as needed. It is easier to think about this from a point of view of photons: if you were a photon, how would you experience redshift or blueshift, depending on the angle between your propagation direction and the direction of the gravitational wave, or depending on the azimuthal angle?



Question 1.4: As it is evident from the above figure, a gravitational wave produces a quadrupolar ($l = 2$) temperature anisotropy. To see this more clearly, it is convenient to define the following *circular* polarization amplitudes, h_R (right-handed) and h_L (left-handed), as

$$h_+ = \frac{1}{\sqrt{2}}(h_R + h_L), \tag{1.155}$$

$$h_\times = \frac{i}{\sqrt{2}}(h_R - h_L). \tag{1.156}$$

Using h_R and h_L, and the definitions for spherical harmonics, Y_l^m, with $l = 2$:

$$Y_2^{\pm 2}(\theta, \phi) = \sqrt{\frac{15}{32\pi}} \sin^2 \theta e^{\pm 2i\phi}, \tag{1.157}$$

$$Y_2^{\pm 1}(\theta, \phi) = (\pm 1)\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta e^{\pm i\phi}, \tag{1.158}$$

$$Y_2^{0}(\theta, \phi) = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1), \tag{1.159}$$

rewrite the equation for $\frac{\delta T}{T}\big|_{\mathcal{O}}$ in terms of $\int \dot{h}_R dt$, $\int \dot{h}_L dt$, and Y_2^m.

1.1.2 Polarization From Gravitational Waves

Thomson scattering of a quadrupolar temperature anisotropy by an electron can produce linear polarization. In terms of the Stokes parameters produced by a scattering, $Q(\theta, \phi)$ and $U(\theta, \phi)$, there is a formula relating the temperature quadrupole to polarization by a single scattering:

$$Q + iU = -\frac{\sqrt{6}}{10} \sum_{m=\pm 2} {}_2Y_2^m(\theta,\phi) \int d\tilde{\Omega} \left.\frac{\delta T}{\bar{T}}\right|_{\mathcal{O}} (\tilde{\theta},\tilde{\phi}) Y_2^{m*}(\tilde{\theta},\tilde{\phi}), \tag{1.160}$$

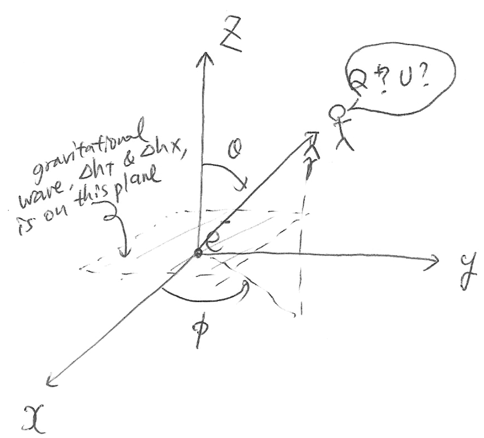
$$Q - iU = -\frac{\sqrt{6}}{10} \sum_{m=\pm 2} {}_{-2}Y_2^m(\theta,\phi) \int d\tilde{\Omega} \left.\frac{\delta T}{\bar{T}}\right|_{\mathcal{O}} (\tilde{\theta},\tilde{\phi}) Y_2^{m*}(\tilde{\theta},\tilde{\phi}), \tag{1.161}$$

where $d\tilde{\Omega} = d\cos\tilde{\theta} d\tilde{\phi}$, and ${}_2Y_l^m$ is a spin-2 harmonics given by

$$ {}_2Y_2^{\pm 2} = \sqrt{\frac{5}{64\pi}} (1 \mp \cos\theta)^2 e^{\pm 2i\phi}, \tag{1.162}$$

$$ {}_{-2}Y_2^{\pm 2} = \sqrt{\frac{5}{64\pi}} (1 \pm \cos\theta)^2 e^{\pm 2i\phi}. \tag{1.163}$$

Note that an electron is at the origin, and photons are scattered by this electron at the origin into various directions, (θ,ϕ). In other words, these are the Stokes parameters of polarization that would be observed by observers at various directions from this electron.



Now, to simplify the analysis, let us assume that we have $\Delta h_R \equiv \int \dot{h}_R dt$ and $\Delta h_L \equiv \int \dot{h}_L dt$ at the origin, and similarly define the linear polarization amplitudes of gravitational waves:

$$\Delta h_+ \equiv \frac{1}{\sqrt{2}} (\Delta h_R + \Delta h_L), \tag{1.164}$$

$$\Delta h_\times \equiv \frac{i}{\sqrt{2}} (\Delta h_R - \Delta h_L). \tag{1.165}$$

Question 1.5: Calculate $Q(\theta,\phi)$ and $U(\theta,\phi)$ in terms of $\Delta h_{+,\times}$ and trigonometric functions.

Question 1.6: Give physical explanations for the results obtained in Question 1.5. *For example*: compare Q and U at $\theta = \pi/2$ and $\phi = 0$, and explain the origin of the difference; compare the results at different ϕ, and give a physical explanation for the behavior. Use graphics as needed.

For this problem, it is easier to think about this from a point of view of an electron at the origin: if you were an electron scattering photons into various directions, what polarization would you produce depending on the scattering direction and the direction of the gravitational wave, or depending on the azimuthal angle?

Chapter 3

Large-scale Structure of the Universe

In this chapter, we shall learn how matter density fluctuations grow and how structures are formed in an expanding universe.

3.1 Evolution of Density Fluctuations

3.1.1 Matter Era

Super-horizon solutions

As you saw in the homework problem, the evolution of matter density fluctuations during the matter era is given by the following 3 equations (with $c = 1$):

- Energy (Mass) Conservation:
$$\delta' = -\frac{\epsilon}{a}V - 3\Phi', \tag{3.1}$$
where $\epsilon \equiv k/\dot{a} = k/(aH) \propto \sqrt{a}$ during the matter era, and the primes denote partial derivatives with respect to a, i.e., $\delta' \equiv \partial\delta/\partial a$.

- Momentum Conservation:
$$V' = -\frac{1}{a}V - \frac{\epsilon}{a}\Phi. \tag{3.2}$$

- Einstein Equation:
$$\epsilon^2\Phi = \frac{3}{2}\left(\delta + \frac{3V}{\epsilon}\right). \tag{3.3}$$

Then, we have seen that, on **super-horizon scales** ($\epsilon \ll 1$), the solutions are given by

$$\delta = 2\Phi = \text{constant}, \tag{3.4}$$
$$V = -\frac{2}{3}\epsilon\Phi \propto \sqrt{a}, \tag{3.5}$$
$$\Phi = \text{constant}. \tag{3.6}$$

These super-horizon solutions were important when computing temperature anisotropy, as the large-scale temperature anisotropy (Sachs–Wolfe effect and Doppler effect on angular scales larger than about 3 degrees) is given by these solutions.

Sub-horizon solutions

On the other hand, when studying the large-scale structure of the universe (such as the clustering of galaxies), we are always dealing with the scales much smaller than the horizon size. Therefore, let us take the opposite limit, $\epsilon \gg 1$, and find solutions. In this **sub-horizon limit**, the equations become

$$\delta' = -\frac{\epsilon}{a}V, \tag{3.7}$$

$$V' = -\frac{1}{a}V - \frac{\epsilon}{a}\Phi, \tag{3.8}$$

$$\epsilon^2\Phi = \frac{3}{2}\delta. \tag{3.9}$$

These equations can be solved easily, and the growing-mode solutions are

$$\delta = \frac{2}{3}\epsilon^2\Phi \propto a, \tag{3.10}$$

$$V = -\frac{2}{3}\epsilon\Phi \propto \sqrt{a}, \tag{3.11}$$

$$\Phi = \text{constant}. \tag{3.12}$$

There are 3 important observations one can make:

- The matter density perturbation, δ, is constant outside the horizon, but it grows linearly with a inside the horizon.

- The solution for the matter velocity perturbation, V, is the same in the super-horizon and sub-horizon limits, and grows as $\sqrt{a}$ at all scales.

- Φ $(= -\Psi)$ is constant both in the super-horizon and sub-horizon limits, and thus it is constant at all scales.

3.1.2 Radiation Era

Super-horizon solutions

How the matter density perturbations grow during the radiation era? As the energy density in radiation is much greater than that in matter during the radiation era, we need to take the radiation energy density perturbation in Einstein equation:

$$\delta' = -\frac{\epsilon}{a}V - 3\Phi' \tag{3.13}$$

$$V' = -\frac{1}{a}V - \frac{\epsilon}{a}\Phi, \tag{3.14}$$

$$\epsilon^2\Phi = \frac{3}{2}\left(\delta_R + \frac{\bar{\rho}_M}{\bar{\rho}_R}\delta + \frac{3V}{\epsilon}\right), \tag{3.15}$$

where $\epsilon \equiv k/\dot{a} = k/(aH) \propto a$ during the radiation era. Here, δ_R is the fractional perturbation in the radiation energy density, $\bar{\rho}_R$ the mean radiation energy density, and $\bar{\rho}_M$ the mean matter energy density. As usual, δ is the fractional perturbation in the matter density.

First, let us look at the super-horizon ($\epsilon \ll 1$) solutions. On super-horizon scales, the radiation perturbation δ_R and the matter perturbation δ are related by the adiabatic initial condition:

$$\delta = \frac{3}{4}\delta_R \qquad \text{(super horizon)}. \tag{3.16}$$

Since $\bar{\rho}_M/\bar{\rho}_R \ll 1$ during the radiation era, this simply means that the term involving $\bar{\rho}_M/\bar{\rho}_R$ in Einstein's equation can be ignored. Therefore, on super-horizon scales, we have

$$\delta' = -\frac{\epsilon}{a}V - 3\Phi' \tag{3.17}$$

$$V' = -\frac{1}{a}V - \frac{\epsilon}{a}\Phi \tag{3.18}$$

$$\epsilon^2\Phi = \frac{3}{2}\left(\frac{4}{3}\delta + \frac{3V}{\epsilon}\right), \tag{3.19}$$

where we have used $\delta_R = \frac{4}{3}\delta$.

Using the same technique we used for the matter era, we can solve these equations to find the **super-horizon solutions during the radiation era**:

$$\delta = \frac{9}{8}\Phi = \text{constant}, \tag{3.20}$$

$$V = -\frac{1}{2}\epsilon\Phi \propto a, \tag{3.21}$$

$$\Phi = \text{constant}. \tag{3.22}$$

Therefore, similarly to the matter era, Φ and δ remain constant outside the horizon during the radiation era. However, the values of Φ and δ are not the same as those during the matter era. In other words, the values of Φ and δ change when the universe becomes matter dominated.

To see this, recall that, on super horizon scales, there is a conserved quantity ζ given by

$$\zeta \equiv \Phi - \frac{V}{\epsilon}. \tag{3.23}$$

This quantity is given by

$$\zeta = \Phi + \frac{1}{2}\Phi = \frac{3}{2}\Phi, \tag{3.24}$$

during the radiation era. In other words, $\Phi = \frac{2}{3}\zeta$ during the radiation era. By comparing this to the matter-era relation, $\Phi = \frac{3}{5}\zeta$, we find

$$\Phi(\text{matter era}) = \frac{9}{10}\Phi(\text{radiation era}), \tag{3.25}$$

and

$$\delta(\text{matter era}) = \frac{8}{5}\delta(\text{radiation era}), \tag{3.26}$$

on super horizon scales.

Sub-horizon solutions

Next, let us consider the sub-horizon solutions. The relevant equations are

$$\delta' = -\frac{\epsilon}{a}V, \tag{3.27}$$

$$V' = -\frac{1}{a}V - \frac{\epsilon}{a}\Phi, \tag{3.28}$$

$$\epsilon^2\Phi = \frac{3}{2}\left(\delta_R + \frac{\bar{\rho}_M}{\bar{\rho}_R}\delta\right). \tag{3.29}$$

Now, on sub-horizon scales, the adiabatic condition does not have to be held because it is the initial condition, and the density perturbations can evolve away from the initial condition inside the horizon.

Interestingly, on sub-horizon scales during the radiation, we can ignore δ_R compared to $\frac{\bar{\rho}_M}{\bar{\rho}_R}\delta$, despite that $\frac{\bar{\rho}_M}{\bar{\rho}_R} \gg 1$. This is because radiation cannot cluster (cannot form clumps) inside the horizon due to a large amount of pressure it has (recall that the radiation pressure is given by $P_R = \rho_R/3$, which is comparable to the energy density). In other words, the distribution of radiation is quite smooth inside the horizon, and it does not contribute very much to Φ compared to the matter density perturbation. As a result, we can write

$$\delta' = -\frac{\epsilon}{a}V, \tag{3.30}$$

$$V' = -\frac{1}{a}V - \frac{\epsilon}{a}\Phi, \tag{3.31}$$

$$\epsilon^2\Phi = \frac{3}{2}\frac{\bar{\rho}_M}{\bar{\rho}_R}\delta = \frac{3}{2}\frac{a}{a_{\rm EQ}}\delta, \tag{3.32}$$

where $a_{\rm EQ}$ is the scale factor at which $\bar{\rho}_M = \bar{\rho}_R$. (Recall $\bar{\rho}_M \propto 1/a^3$ and $\bar{\rho}_R \propto 1/a^4$.)

By combining these equations, one finds

$$\delta'' + \frac{1}{a}\delta' - \frac{3}{2aa_{\rm EQ}}\delta = 0. \tag{3.33}$$

This is not quite straightforwardly solvable, so we use a trick: defining $y \equiv \delta/a$, we rewrite this equation in terms of y:

$$y'' + \frac{3}{a}y' + \frac{1}{a^2}\left(1 - \frac{3}{2}\frac{a}{a_{\rm EQ}}\right)y = 0. \tag{3.34}$$

Since we are considering the radiation era, we can ignore $a/a_{\rm EQ} \ll 1$.

$$y'' + \frac{3}{a}y' + \frac{1}{a^2}y = 0. \tag{3.35}$$

The solution is $y = \frac{A}{a} + \frac{B}{a}\ln a$. Therefore,

$$\delta = A + B\ln(a/a_{\rm EQ}) \tag{3.36}$$

where A and B are integration constants. While the matter density perturbation grows linearly with a during the matter era, **it grows only logarithmically during the radiation era**. This has a very important implication, which we shall learn in a moment.

The other solutions are given by

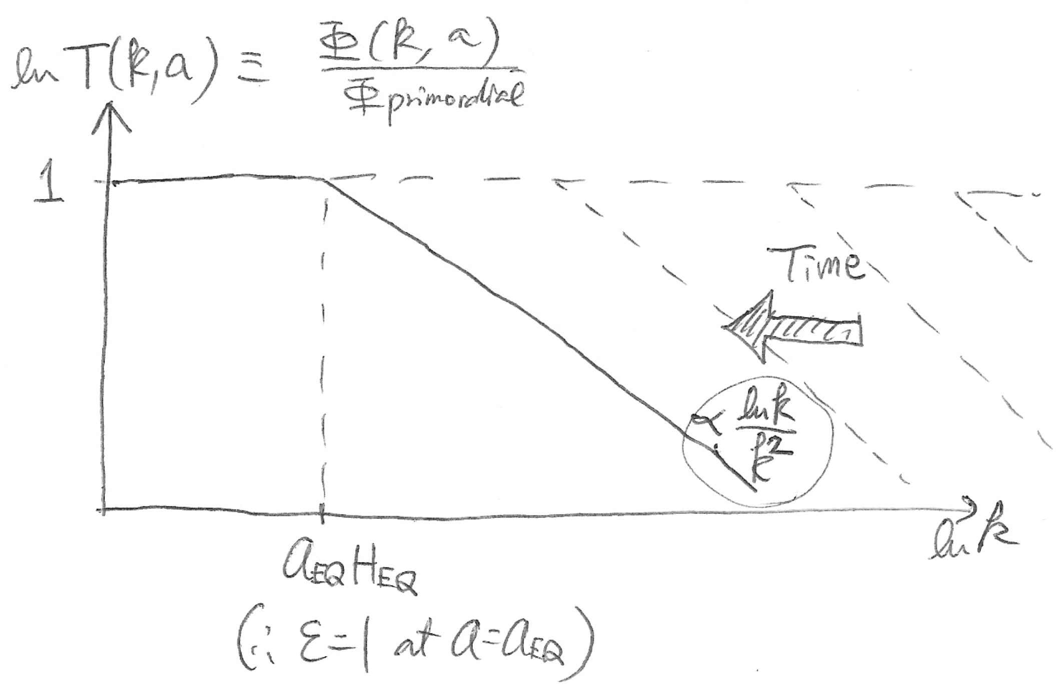
$$V = -\frac{B}{\epsilon} \propto \frac{1}{a}, \tag{3.37}$$

$$\Phi = \frac{3}{2}\frac{a}{a_{\rm EQ}\epsilon^2}\left[A + B\ln(a/a_{\rm EQ})\right] \propto \frac{\ln(a/a_{\rm EQ})}{a}. \tag{3.38}$$

As Φ inside the horizon **decays** during the radiation era, **the small-scale perturbation (which entered the horizon earlier) is suppressed relative to the primordial one.** It is conventional to characterize this effect by using the so-called **transfer function**:

$$T(k,a) \equiv \frac{\Phi(k,a)}{\Phi_{\rm primordial}}. \tag{3.39}$$

Since Φ becomes constant at all scales after the matter-radiation equality, the shape of $T(k,a)$ gets frozen after the matter-radiation equality. Therefore, from now on, we shall simply write it as $T(k)$ without time dependence.



3.2 Matter Density Power Spectrum

3.2.1 Shape

In most cases, the large-scale structure is characterized by the power spectrum of matter density fluctuations, $P(k,t)$:

$$\langle\delta(\vec{k},t)\delta^*(\vec{k},t)\rangle = (2\pi)^3\delta_D^{(3)}(\vec{k}-\vec{k}')P(k,t). \tag{3.40}$$

Using Einstein's equation during the matter era,

$$\epsilon^2(k,t)\Phi(\vec{k},t) = \frac{3}{2}\delta(\vec{k},t), \tag{3.41}$$

we can relate $P(k,t)$ to the power spectrum of Φ:

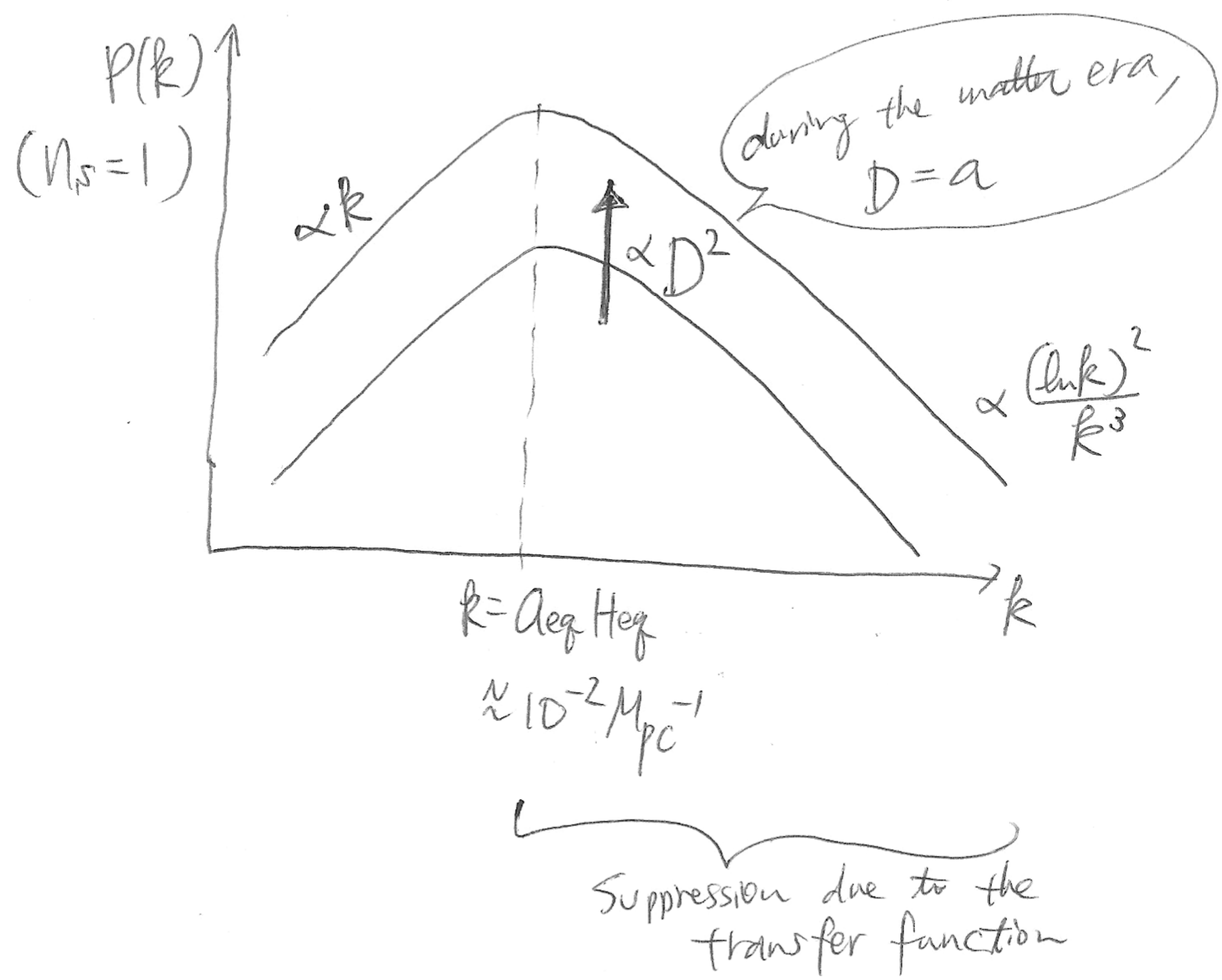
$$P(k,t) = \frac{4}{9}\left(\frac{k}{aH}\right)^4 P_\Phi(k,t). \tag{3.42}$$

As we have seen before, on super horizon scales, it is convenient to relate Φ to the conserved quantity ζ. During the matter era, we have $\Phi = \frac{3}{5}\zeta$. Finally, we need to take into account the matter density evolution during the radiation era using the transfer function. The final result is

$$\begin{aligned} P(k,t) &= \frac{4}{9}\left(\frac{k}{aH}\right)^4 \times \frac{9}{25}\frac{2\pi^2}{k^3}\Delta_\zeta^2(k_0)\left(\frac{k}{k_0}\right)^{n_s-1} T^2(k)D^2(t) \\ &= \frac{8\pi^2 k}{25(aH)^4}\Delta_\zeta^2(k_0)\left(\frac{k}{k_0}\right)^{n_s-1} T^2(k)D^2(t). \end{aligned} \tag{3.43}$$

Here, $D(t)$ is a time-dependent function giving the amount of growth of δ. During the matter era, $D \propto a$. Therefore, on very large scales where the transfer function is approximately unity, the scale-invariant power spectrum ($n_s = 1$) yields the matter density power spectrum of $P(k) \propto k$. Then, $P(k)$ peaks at $k = a_{\rm EQ}H_{\rm EQ} \approx 0.01~{\rm Mpc}^{-1}$, and then decreases toward large values of k. The small-scale limit is given by $P(k) \propto [\ln(k)]^2/k^3$ (for $n_s = 1$).

Therefore, in principle, if one can measure the matter power spectrum accurately, one can determine the parameters such as Δ_ζ^2 and n_s. As we learn later, the growth function $D(t)$ also encodes important cosmological information. However, the accuracy of the measurement of $P(k)$ is not yet good enough compared to the cosmic microwave background, and thus the information on Δ_ζ^2, n_s, etc., is dominated by the microwave background data.



3.2.2 Baryon Acoustic Oscillation

Up until now, we have ignored interactions between matter and radiation. Since photons and electrons interact efficiently via Thomson scattering, it is conceivable that this interaction leaves some signatures in the microwave background as well as in the matter power spectrum.

Once we include interactions between matter and radiation, we can no longer treat these components separately. As a result, the equation system becomes a bit more involved.

Since electrons and baryons (protons and helium nuclei) are also interacting efficiently via Coulomb interaction, we can treat photons and baryons as a coupled fluid. We should not forget also dark matter, which provides most of the gravitational potential during the matter era.

Then the relevant equations are

- Energy Conservation:

$$\dot{\delta}_D = -\frac{k}{a}V_D - 3\dot{\Phi}, \tag{3.44}$$
$$\dot{\delta}_B = -\frac{k}{a}V_B - 3\dot{\Phi}, \tag{3.45}$$
$$\dot{\delta}_\gamma = -\frac{4}{3}\frac{k}{a}V_\gamma - 4\dot{\Phi}, \tag{3.46}$$

where δ_D, δ_B, and δ_γ are the dark matter, baryon, and photon densities, respectively. Since we consider the matter-dominated era, we shall ignore $\dot{\Phi}$.

- Momentum Conservation:

$$\dot{V}_D = -\frac{\dot{a}}{a}V_D - \frac{k}{a}\Phi, \tag{3.47}$$
$$\dot{V}_B = -\frac{\dot{a}}{a}V_B - \frac{k}{a}\Phi + \frac{\sigma_T n_e}{R}(V_\gamma - V_B), \tag{3.48}$$
$$\dot{V}_\gamma = \frac{1}{4}\frac{k}{a}\delta_\gamma - \frac{k}{a}\Phi + \sigma_T n_e(V_B - V_\gamma), \tag{3.49}$$

where R is the baryon-to-photon energy density ratio defined as

$$R \equiv \frac{3\bar{\rho}_B}{4\bar{\rho}_\gamma}. \tag{3.50}$$

- Einstein's Equation:

$$\frac{k^2}{a^2}\Phi = 4\pi G\left(\bar{\rho}_D\delta_D + \bar{\rho}_B\delta_B\right), \tag{3.51}$$

where we have ignored the radiation contribution in the right hand side because we are considering the matter-dominated era.

Now, while there are many equations, one can simplify the equation system considerably when the coupling between photons and baryons is very efficient. In such a case, baryons and photons basically move together, i.e., $V_B \approx V_\gamma$.

We rewrite Eq. (5.48) as

$$\begin{aligned} V_B &= V_\gamma - \frac{R}{\sigma_T n_e}\left(\dot{V}_B + \frac{\dot{a}}{a}V_B + \frac{k}{a}\Phi\right) \\ &\approx V_\gamma - \frac{R}{\sigma_T n_e}\left(\dot{V}_\gamma + \frac{\dot{a}}{a}V_\gamma + \frac{k}{a}\Phi\right). \end{aligned} \tag{3.52}$$

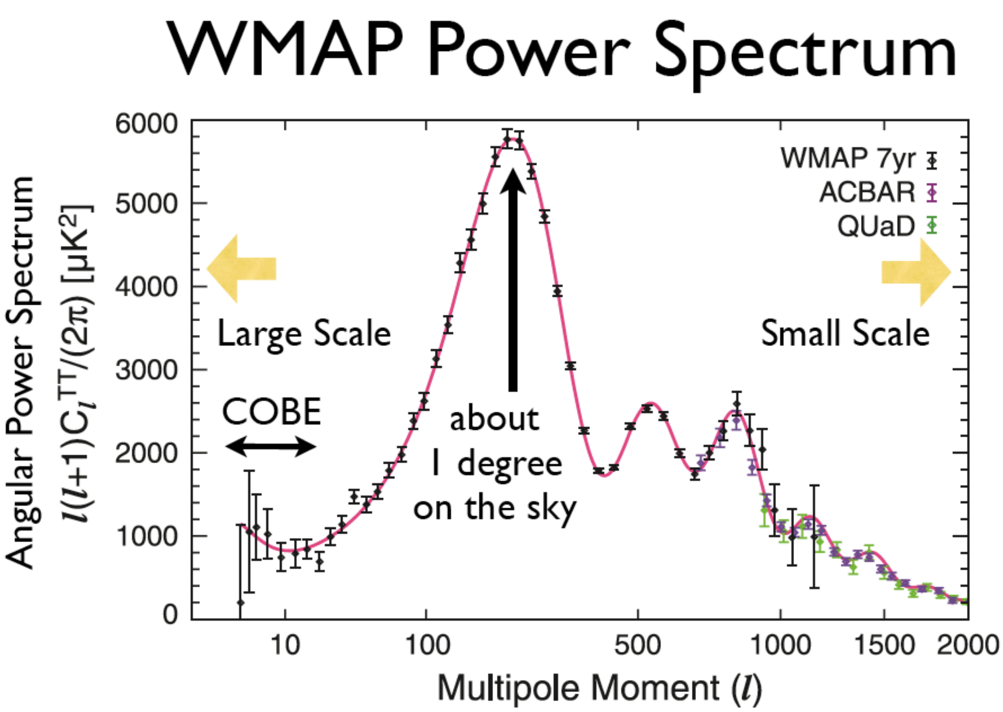
Here, since we assume that the difference between V_B and V_γ is small, we have replaced V_B with V_γ in the right hand side. We then use the photon momentum conservation equation, Eq. (5.49), in the right hand side:

$$
\begin{aligned}
V_B &= V_\gamma - \frac{R}{\sigma_T n_e}\left[\frac{1}{4}\frac{k}{a}\delta_\gamma + \frac{\dot{a}}{a}V_\gamma + \sigma_T n_e(V_B - V_\gamma)\right] \\
(1+R)V_B &= (1+R)V_\gamma - \frac{R}{\sigma_T n_e}\left[\frac{1}{4}\frac{k}{a}\delta_\gamma + \frac{\dot{a}}{a}V_\gamma\right] \\
V_B &= V_\gamma - \frac{R}{1+R}\frac{1}{\sigma_T n_e}\left[\frac{1}{4}\frac{k}{a}\delta_\gamma + \frac{\dot{a}}{a}V_\gamma\right] \\
\sigma_T n_e(V_B - V_\gamma) &= -\frac{R}{1+R}\left[\frac{1}{4}\frac{k}{a}\delta_\gamma + \frac{\dot{a}}{a}V_\gamma\right]. \qquad (3.53)
\end{aligned}
$$

Now, using this in the photon momentum conservation equation, and using the photon energy conservation equation $V_\gamma = -\frac{3}{4}\frac{a}{k}\dot{\delta}_\gamma$, we arrive at the following differential equation for the photon energy density:

$$
\ddot{\delta}_\gamma + \frac{1+2R}{1+R}\frac{\dot{a}}{a}\dot{\delta}_\gamma + \frac{1}{3(1+R)}\frac{k^2}{a^2}\delta_\gamma = \frac{4}{3}\frac{k^2}{a^2}\Phi. \qquad (3.54)
$$

This is a wave equation for δ_γ; thus, a coupling between baryons and photons results in the acoustic oscillations in the photon density perturbations. Since baryons and photons are coupled, the same oscillations must also be present in the baryon density perturbations as well. Indeed, the acoustic oscillations have been observed both in photons (microwave background) and the distribution of matter (galaxies).



Acoustic Oscillations in CMB

To have a deeper understanding of the structures of the acoustic oscillation, let us focus on the regime where the oscillation frequency is much greater than the expansion rate of the universe. In this case, the wave equation simplifies to

$$\ddot{\delta}_\gamma + \frac{1}{3(1+R)}\frac{k^2}{a^2}\delta_\gamma = \frac{4}{3}\frac{k^2}{a^2}\Phi. \tag{3.55}$$

Since $\dot{\Phi} = 0$ during the matter era, one may rewrite this equation in a suggestive way:

$$\frac{\partial^2}{\partial t^2}\left[\frac{1}{4}\delta_\gamma - (1+R)\Phi\right] + \frac{k^2 c_s^2}{a^2}\left[\frac{1}{4}\delta_\gamma - (1+R)\Phi\right] = 0, \tag{3.56}$$

where c_s is the **speed of sound**:

$$c_s^2 \equiv \frac{1}{3(1+R)} = \frac{1}{3(1+\frac{3\bar{\rho}_B}{4\bar{\rho}_\gamma})}. \tag{3.57}$$

Note that this speed of sound is *less than* that for the relativistic fluid, $c_s^2 = 1/3$. This is due to the coupling to baryons: the inertia of baryons reduces the speed of sound of photon-baryon fluid relative to that of the relativistic fluid. The solution to the above wave equation is

$$\frac{1}{4}\delta_\gamma = (1+R)\Phi + A\cos(kr_s) + B\sin(kr_s), \tag{3.58}$$

where r_s is the **sound horizon** defined by

$$r_s \equiv \int_0^{t_*} \frac{dt}{a} c_s(a) = 147 \text{ Mpc}, \tag{3.59}$$

for the cosmological parameters best-fit to the WMAP data and t_* is the decoupling time.

How do we determine the integration constants, A and B? We determine these coefficients by noting that, on super horizon scales, these solutions should match the adiabatic initial condition:

$$\frac{1}{4}\delta_\gamma = \frac{1}{3}\delta_m = \frac{2}{3}\Phi \qquad \text{on super-horizon scales } (kc_s \ll 1) \tag{3.60}$$

Therefore,

$$A = \frac{2}{3}\Phi - (1+R)\Phi = -\left(\frac{1}{3}+R\right)\Phi, \tag{3.61}$$

$$B = 0, \tag{3.62}$$

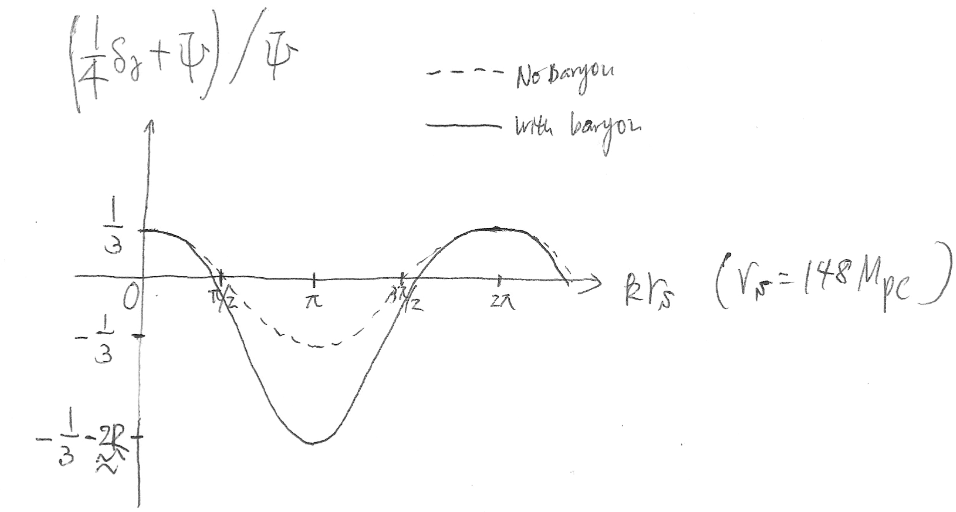
and

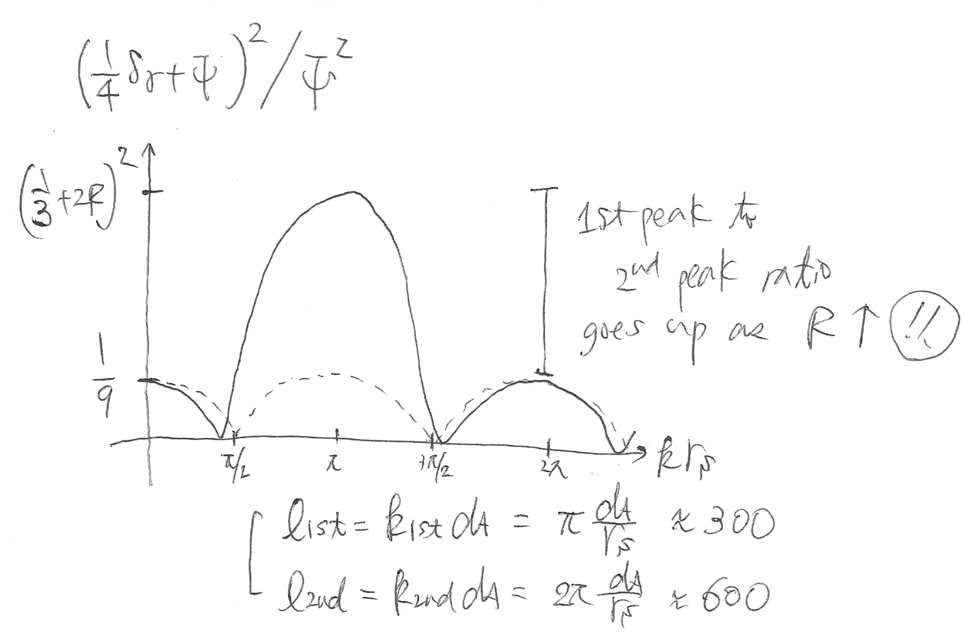
$$\frac{1}{4}\delta_\gamma = (1+R)\Phi - \left(\frac{1}{3}+R\right)\Phi\cos(kr_s). \tag{3.63}$$

Since $\rho_\gamma \propto T^4$, we can relate $\frac{1}{4}\delta_\gamma$ to $\delta T/T$ as $\frac{1}{4}\delta_\gamma = \delta T/T$. Moreover, since the observed temperature anisotropy is the sum of $\delta T/T$ at the bottom of the potential well and the potential Ψ, we write, using $\Psi = -\Phi$

$$\frac{1}{4}\delta_\gamma + \Psi = -R\Psi + \left(\frac{1}{3} + R\right)\Psi\cos(kr_s). \tag{3.64}$$

Since what we observe is the power spectrum, which is the temperature squared, we may plot this result squared as a function of kr_s. We then notice that the 1st peak to the 2nd peak ratio goes up as one increases R; thus, the 1st peak to the 2nd peak ratio can be used to determine the baryon density!





Acoustic Oscillations in Baryons

We have seen that a coupling between photons and baryons induces acoustic oscillations in the distribution of photons. How about baryons?

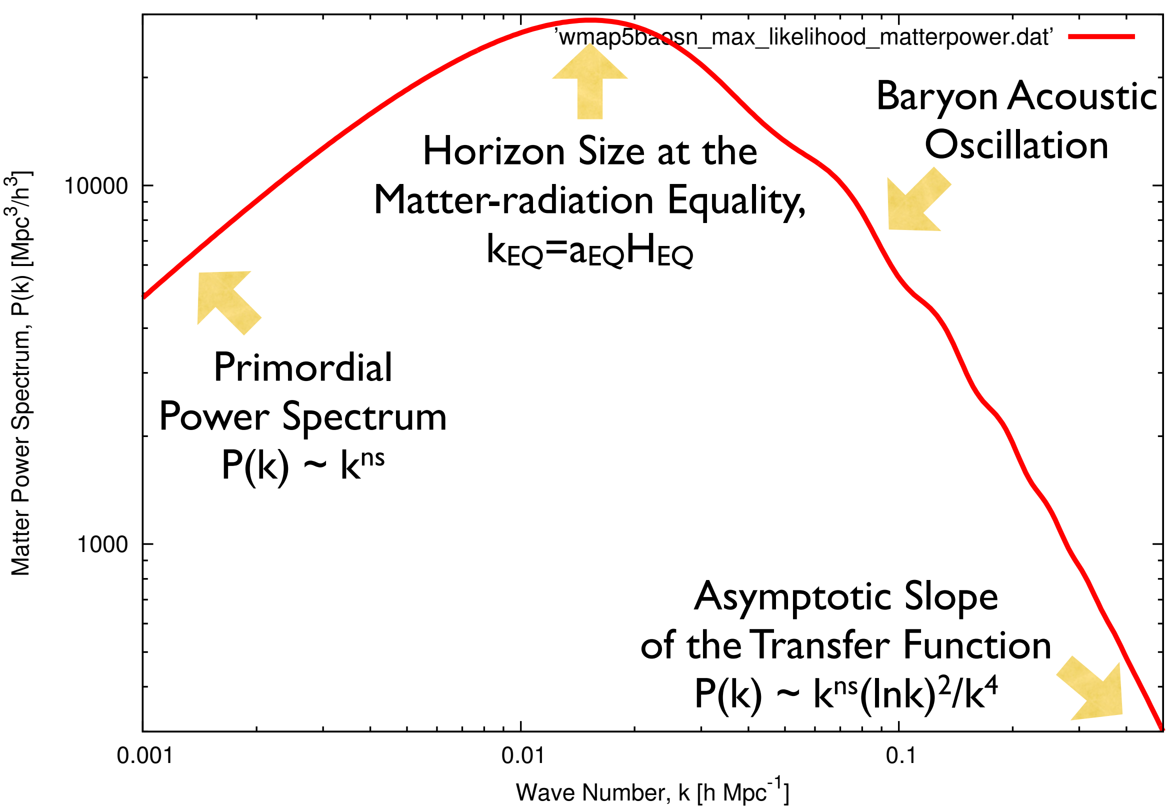
By repeating a similar analysis, one can obtain the wave-like equation for baryon density fluctuations:

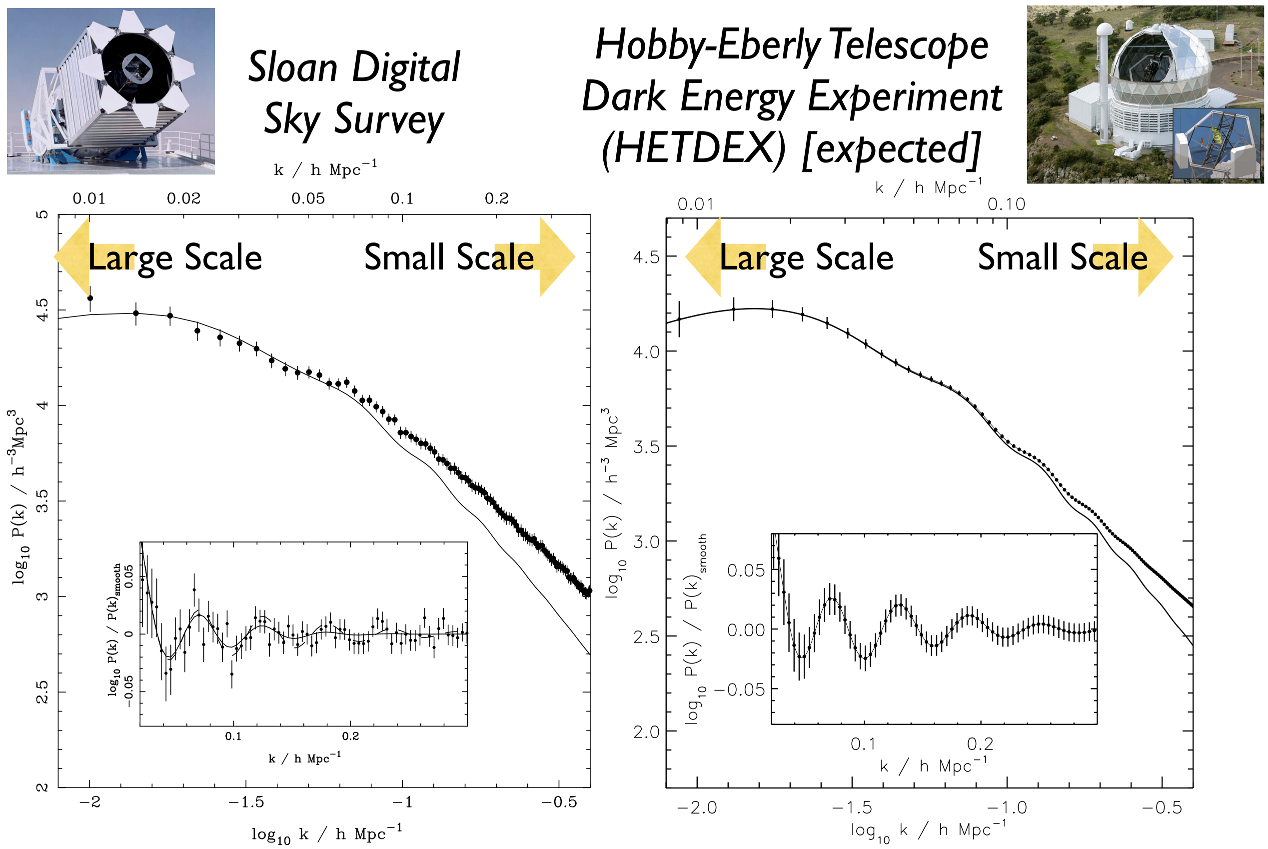
$$\ddot{\delta}_B + \frac{1+2R}{1+R}\frac{\dot{a}}{a}\dot{\delta}_B + \frac{1}{3(1+R)}\frac{k^2}{a^2}\frac{3}{4}\delta_\gamma = \frac{k^2}{a^2}\Phi. \tag{3.65}$$

Note that we have δ_γ instead of δ_B in the third term on the left hand side, and thus it is not quite the wave equation for δ_B. However, we know that, on super horizon scales, the adiabatic initial condition gives $\frac{3}{4}\delta_\gamma = \delta_B$. Therefore, with this initial condition, the baryons have the same acoustic oscillations as photons.

The most remarkable thing about this equation is that baryons acquire a significant speed of sound, $c_s = 1/\sqrt{3(1+R)}$, via a coupling with photons. If baryons were not coupled to photons, their speed of sound would be simply given by $c_s = \sqrt{T/m_p}$. For the decoupling temperature, $T \approx 3000$ K ≈ 0.26 eV, this is tiny: $c_s \approx 2 \times 10^{-5}$. So, the coupling between baryons and photons changes the behaviour of baryons completely.

This oscillation is imprinted on the power spectrum of galaxies today, and is often called the **baryon acoustic oscillations** (BAO).





Since we know the value of r_s (from measurements of the CMB anisotropy), we can use BAO of the matter power spectrum at a given redshift to determine the angular diameter distance and the expansion rate at that redshift. Namely:

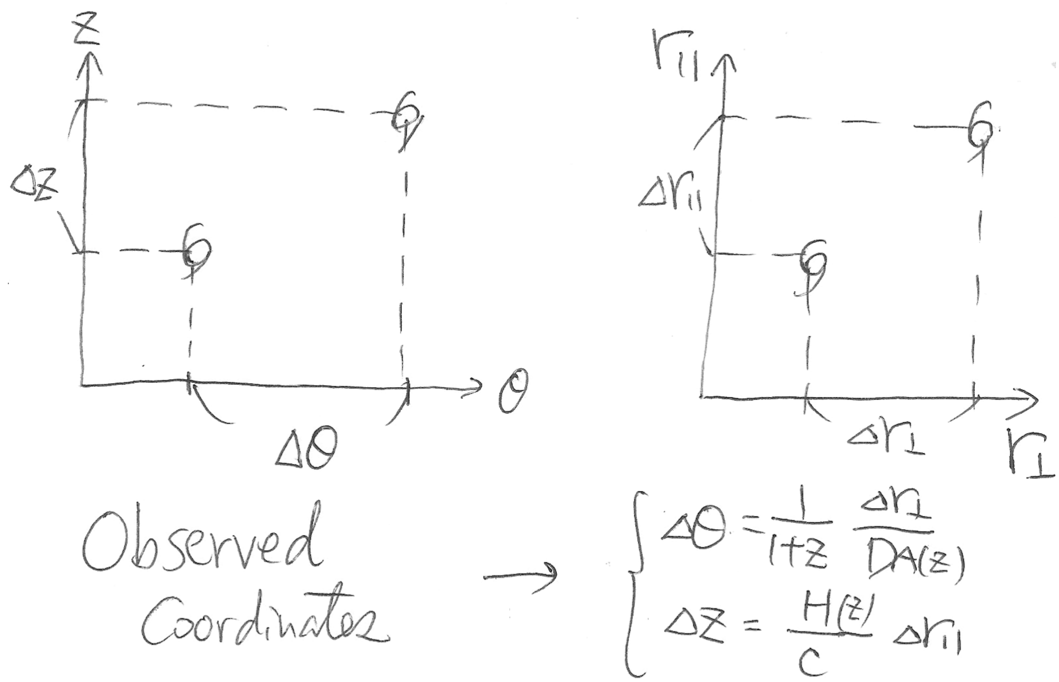
$$\delta\theta_{\rm BAO} = \frac{1}{1+z}\frac{r_s}{D_A(z)}, \tag{3.66}$$

$$\delta z_{\rm BAO} = \frac{r_s H(z)}{c}, \tag{3.67}$$

where $\delta\theta_{\rm BAO}$ and $\delta z_{\rm BAO}$ are the observed angular separations and redshift separations corresponding to r_s. From these, it is clear that we can measure $D_A(z)$ and $H(z)$ separately - BAO is the standard ruler that we discussed in Section 1.5! Therefore, in order to fully utilize the power of BAO, we must consider the power spectrum in 2-dimensional space: angular directions and redshift direction.

3.2.3 2-dimensional Power Spectrum: Alcock-Paczyński test

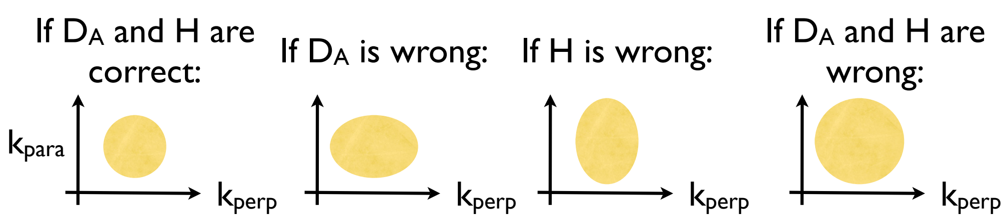
When discussing the power spectrum of matter density fluctuations (traced by, e.g., galaxies), $P(k)$, it is important to realize that we cannot directly measure the wave numbers, k. In order to go to Fourier space, we first need to know 3-dimensional positions of galaxies; however, in order to know those, we must know the angular diameter distances and the expansion rates, as our observables are the angular coordinates and redshift coordinates, rather than the actual 3-dimensional positions.



As a result, the observed power spectrum would not be a function of k, but would always be a function of two wave numbers: $k_\parallel$ and $k_\perp$. (Of course, $k = \sqrt{k_\perp^2 + k_\parallel^2}$.) However, the *underlying* matter power spectrum, $P(k)$, must be isotropic and depend only on the magnitude of k, and thus we can use this property to determine $D_A H$. This is precisely the Alcock-Paczyński test that we studied in Section 1.5. Combining BAO and AP is a powerful method for measuring D_A and H.

The AP Test: How That Works

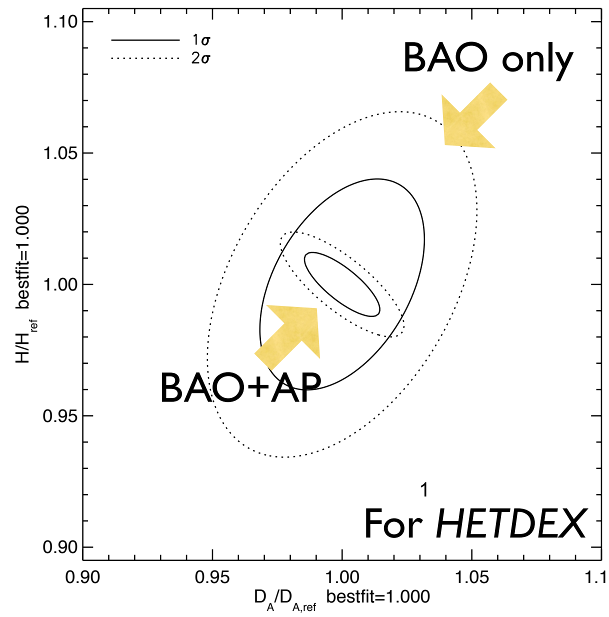
- **D_A**: (RA,Dec) to the transverse separation, r_{perp}, to the transverse wavenumber
  - $k_{perp} = (2\pi)/r_{perp} = (2\pi)$[Angle on the sky]/**$D_A$**
- **H**: redshifts to the parallel separation, r_{para}, to the parallel wavenumber
  - $k_{para} = (2\pi)/r_{para} = (2\pi)$**H**/(c$\Delta$z)



Shoji, Jeong & Komatsu, Astrophysical Journal, 693, 1404 (2009)

BAO-only vs BAO+AP Test

- BAO+AP improves upon the determinations of D_A & H by more than a factor of two.
- On the D_A-H plane, the size of the ellipse shrinks by more than a factor of four.



3.2.4 2-dimensional Power Spectrum: Redshift Space Distortion

However, things are not so simple. Motion of galaxies adds a complication. While we rely on the measured redshifts for inferring the locations of galaxies along the line of sight, the measured redshifts are in fact the sum of the cosmological redshifts and peculiar velocities. Namely, when galaxies moving toward us, they appear to be closer to us than they actually are, and when galaxies are moving away from us, they appear to be farther away than they actually are. This has an effect of **increasing** the clustering of galaxies (hence the power spectrum) along the line of sight on large scales, and **decreasing** the power spectrum on small scales. (See the diagram below.) The large-scale effect is called the **Kaiser effect**, while the small-scale effect is called the **fingers-of-God** effect. The latter is still too complicated to model reliably, so we shall focus only on the Kaiser effect.

As you derive in the homework, the observed power spectrum in redshift space is related to the underlying power spectrum in real space as

$$\boxed{P_{\rm obs}(k, k_\parallel) = \left(1 + f\frac{k_\parallel^2}{k^2}\right)^2 P(k)} \tag{3.68}$$

on large scales (Kaiser effect). Here, f is the logarithmic derivative of the growth of density fluctuations:

$$f \equiv \frac{d \ln \delta}{d \ln a}. \tag{3.69}$$

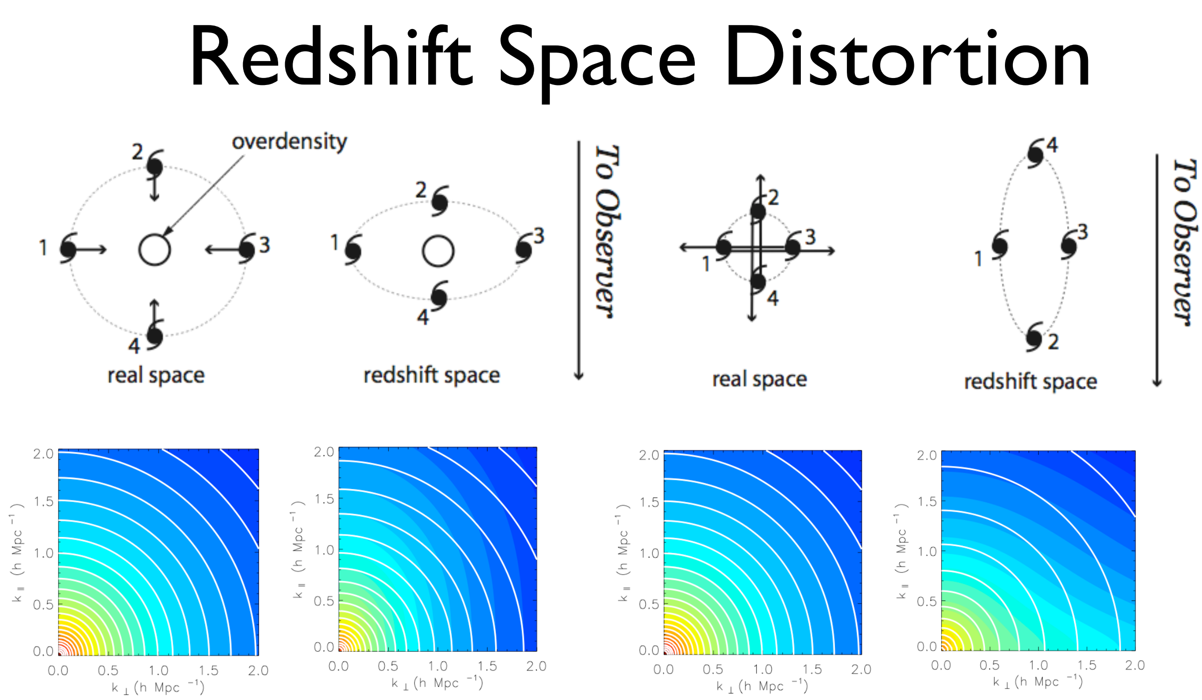
For the directions perpendicular to the line of sight, $k_{\|} = 0$, the observed power spectrum is equal to the underlying spectrum:

$$P_{\rm obs}(k, k_{\|} = 0) = P(k). \tag{3.70}$$

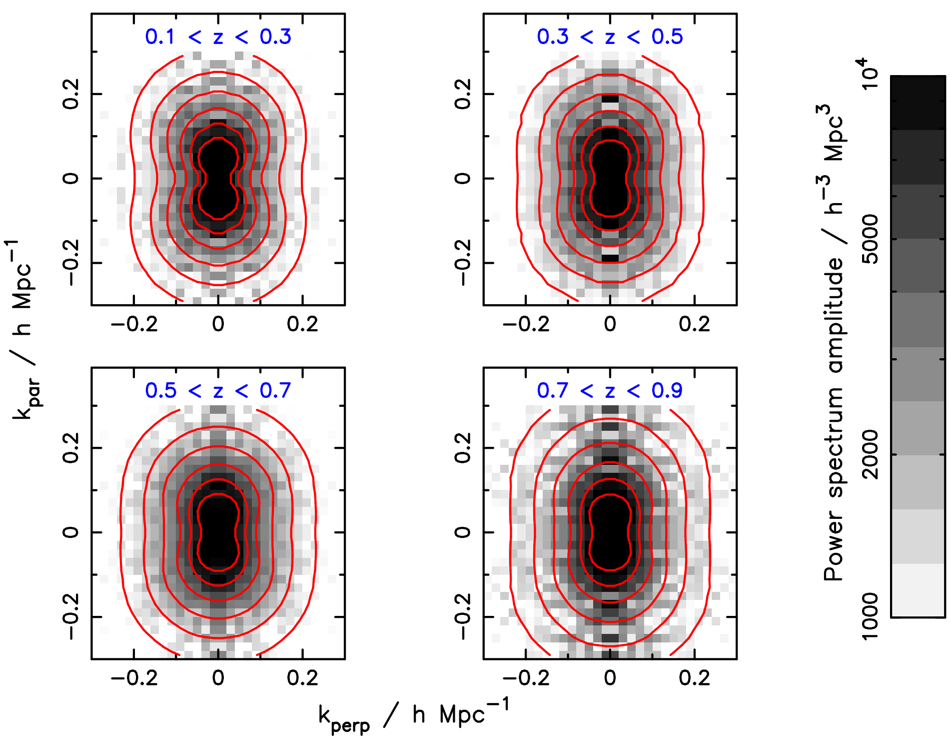
For the directions parallel to the line of sight, $k_{\|} = k$, the observed power spectrum is enhanced relative to the underlying spectrum:

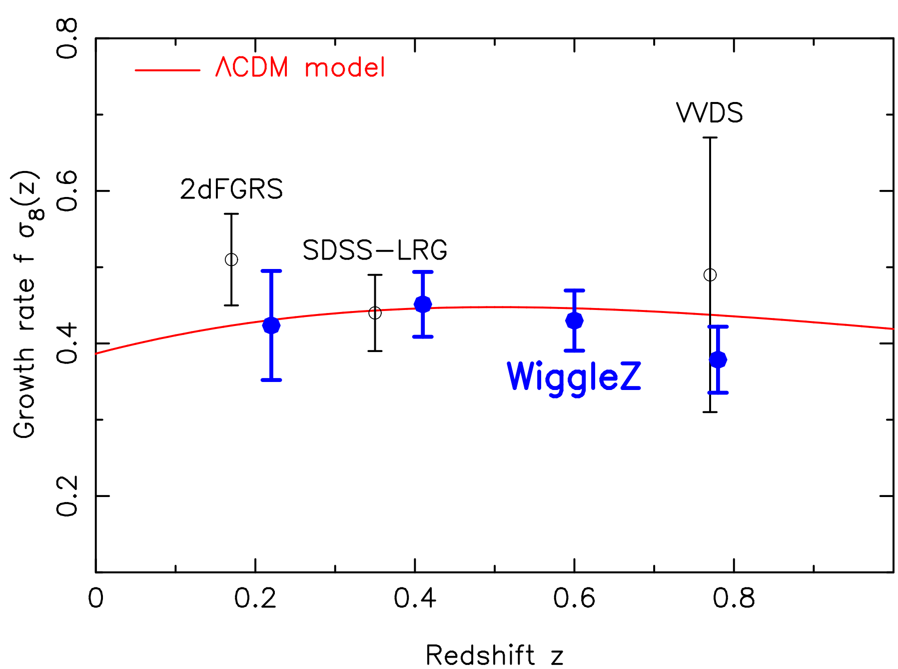
$$P_{\rm obs}(k, k_{\|} = k) = (1 + f)^2 P(k). \tag{3.71}$$

One can use this property to extract the information on the growth of structures. For a universe dominated by matter, $f = 1$; however, for a universe containing matter and dark energy, such as the universe that we live in, f decreases toward low redshifts, providing an important information on the effect of dark energy on the growth of structures.



This effect has been measured routinely from large-scale structure surveys. The measured values of f are consistent with the predictions from the standard ΛCDM model. The plots shown below are taken from the latest paper on "Wiggle Z" survey (Blake et al., arXiv:1104.2948).

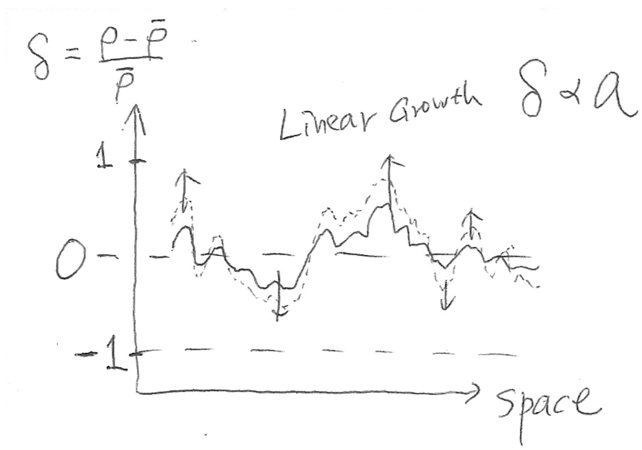

0.1 < z < 0.3
0.3 < z < 0.5
0.5 < z < 0.7
0.7 < z < 0.9
k_par / h Mpc^-1
k_perp / h Mpc^-1
Power spectrum amplitude / h^-3 Mpc^3
10^4
5000
2000
1000

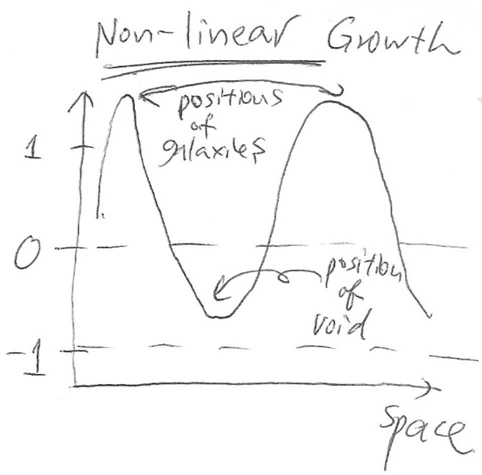

ΛCDM model
2dFGRS
SDSS−LRG
VVDS
WiggleZ
Growth rate f σ8(z)
Redshift z

3.3 Non-linear Evolution of Matter Density Fluctuations

During the matter era, the linear matter density fluctuation, δ, grows as $\delta \propto a$. However, this analysis is valid only for $\delta \ll 1$, and thus cannot be used to follow the evolution of non-linear density fluctuations that would eventually form objects such as galaxies.

Specifically, when $\delta \gg 1$, density fluctuations collapse gravitationally and form "halos" (i.e., gravitationally bound objects). Galaxies are hosted by these halos. Since the total matter must be conserved, the fact that some regions have $\delta \gg 1$ implies that other regions have $\delta < 1$. From the definition, $\delta \equiv \rho/\bar{\rho} - 1$, the minimum value of δ is $\delta = -1$. These empty regions (or nearly empty regions) correspond to "voids."





The exact treatment of non-linear processes is difficult, and we usually use computer simulations (such as N-body simulations) to study the formation and evolution of halos. Before we go into some of the results obtained from simulations, it is useful to work out a simplified case known as the **spherical collapse**.

Consider a spherical region with mass M and radius r. Due to the expansion of the universe, initially $\dot{r} > 0$. As the mass enclosed within r must be conserved, we have $\dot{M} = 0$. During the matter era, the equation of motion is given as the usual Newtonian formula*

$$\ddot{r} = -\frac{GM}{r}. \tag{3.72}$$

Multiplying both sides by $\dot{r}$ and integrating, we get

$$\frac{1}{2}\dot{r}^2 = \frac{GM}{r} + E, \tag{3.73}$$

where E is an integration constant. This should be quite familiar to you: (kinetic energy) + (potential energy) = E, where E is the total energy. Now, since we wish to analyze the case where

*Once again, the same result is obtained from General Relativity. There is a correction to this equation when we have components with a large pressure, such as radiation and dark energy.

the expansion of this region eventually stops, turns around, and collapses, we shall consider the case where $E < 0$. The solution to this equation is known as the **cycloid** and is given as

$$r = A(1 - \cos\theta), \tag{3.74}$$
$$t = B(\theta - \sin\theta), \tag{3.75}$$
$$A^3 = GMB^2, \tag{3.76}$$

where A and B are constants, and we have chosen the zero point of time such that $t \to 0$ as $\theta \to 0$.

The evolution of matter density within this region is given as a function of a new parameter θ:

$$\rho = \frac{M}{\frac{4\pi}{3}r^3} = \frac{3}{4\pi GB^2(1-\cos\theta)^3}. \tag{3.77}$$

Now, in order to calculate $\delta = \rho/\bar{\rho} - 1$, we need to know how the mean density $\bar{\rho}$ depends on θ. We do this by recalling that, from the Friedmann equation,

$$H^2 = \frac{8\pi G}{3}\bar{\rho} = \frac{4}{9}\frac{1}{t^2}, \tag{3.78}$$

during the matter era. Therefore,

$$\bar{\rho} = \frac{1}{6\pi Gt^2} = \frac{1}{6\pi GB^2(\theta - \sin\theta)^2}. \tag{3.79}$$

By taking the ratio,

$$\boxed{\delta = \frac{9}{2}\frac{(\theta - \sin\theta)^2}{(1-\cos\theta)^3} - 1} \tag{3.80}$$

This is the result. The collapse time corresponds to $\theta = 2\pi$, at which δ goes to infinity. Does δ really go to infinity in practice? No. This is an artifact of spherical symmetry: in reality, a finite angular momentum makes it impossible for particles to go straight down to the center $r = 0$, and thus an object with a finite size would be formed.

It is instructive to take an early-time limit, $\theta \ll 1$. We find

$$\delta \approx \frac{3}{20}\theta^2 \qquad (\theta \ll 1). \tag{3.81}$$

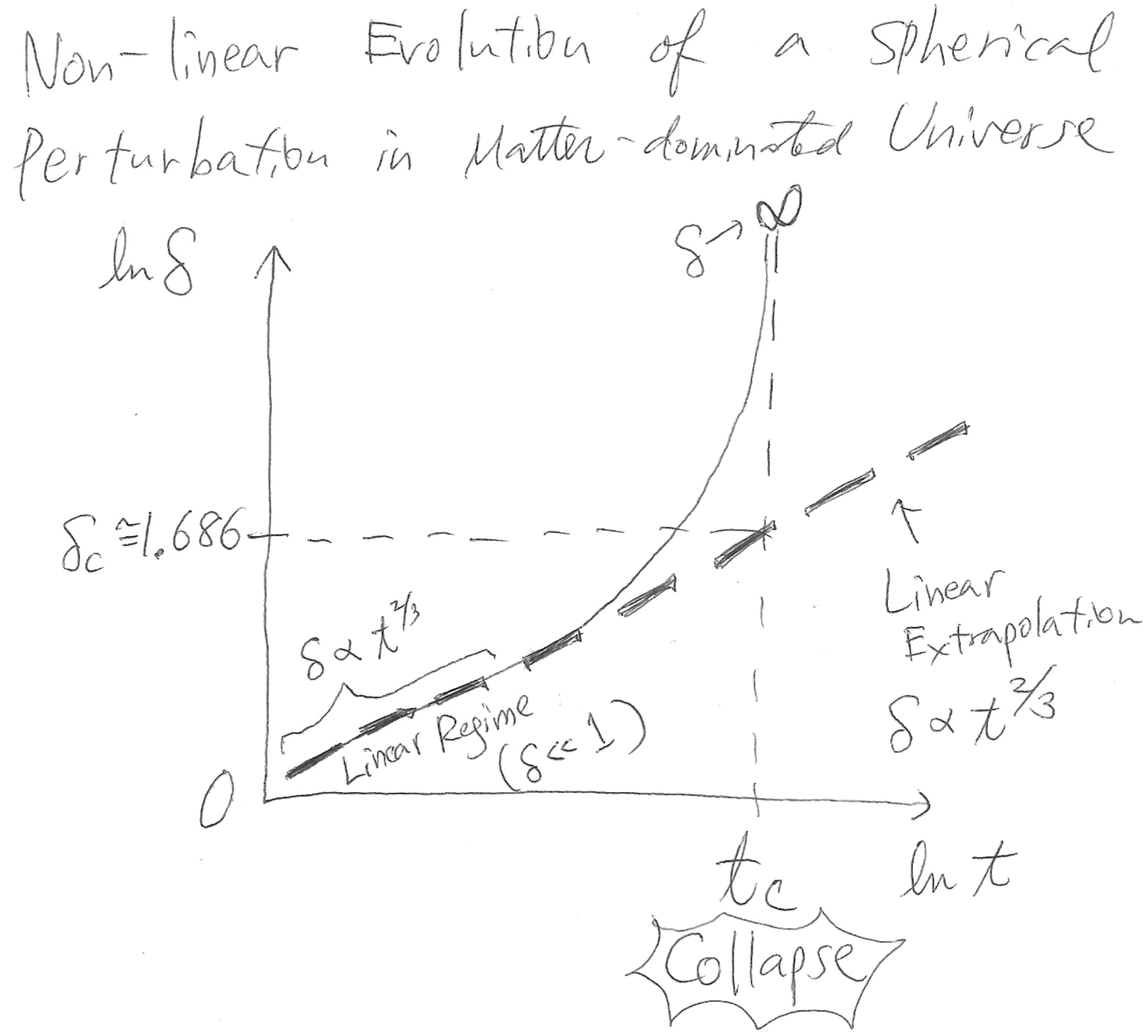
As $\delta \ll 1$ for this case, we should be able to recover the linear evolution, $\delta \propto a \propto t^{2/3}$. Looking at equation (5.75), $t \propto \theta^3$ for $\theta \ll 1$, and thus we indeed recover $\delta \propto t^{2/3} \propto \theta^2$.

The time at which a density fluctuation collapses ($\theta = 2\pi$; $\delta \to \infty$) is given by

$$t_c = 2\pi B. \tag{3.82}$$

While δ goes to infinity at $t = t_c$, what would be the value of δ if we assume the linear evolution? Using $\delta \approx (3/20)\theta^2$ and $t \approx (B/6)\theta^3$ for $\theta \ll 1$, the linear evolution is given by

$$\delta_L = \frac{3}{20}\left(\frac{6}{B}\right)^{2/3} t^{2/3}. \tag{3.83}$$



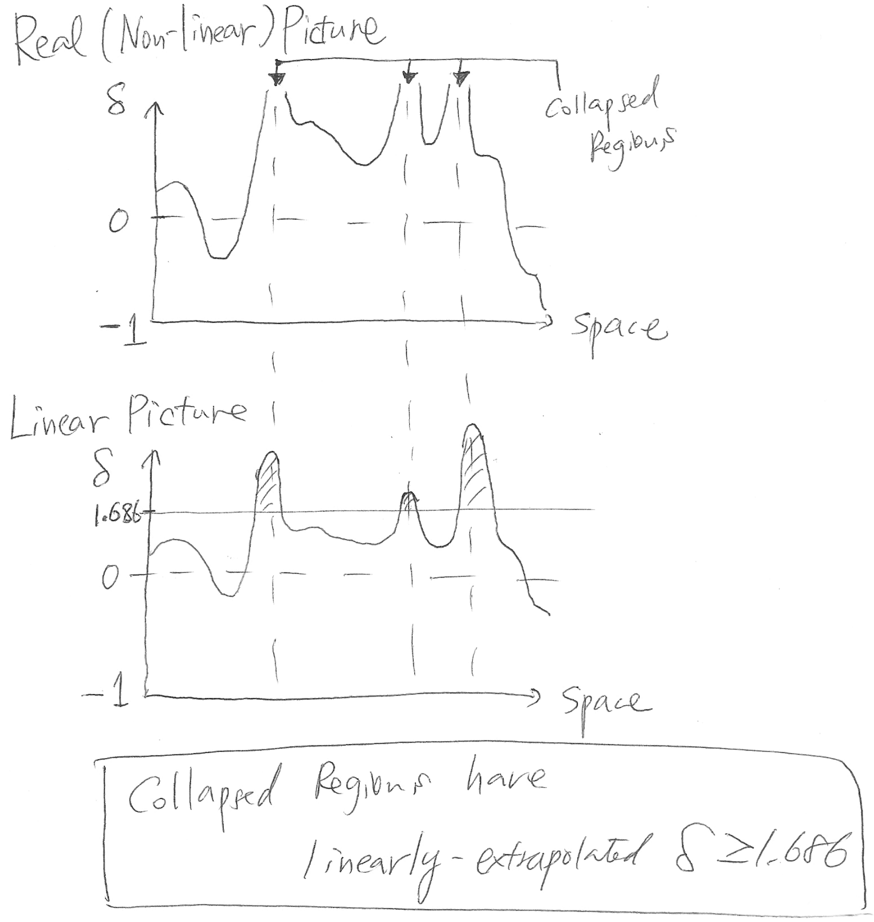
Then, inserting the collapse time, $t_c = 2\pi B$, into this result, we obtain

$$\boxed{\delta_c \equiv \delta_L(t_c) = \frac{3(12\pi)^{2/3}}{20} \simeq 1.686} \tag{3.84}$$

Why is this result interesting? While non-linear evolution of density fluctuations is generally quite complicated, the linear evolution is known. Now, suppose that we have some initial density fluctuations that are small, as a function of spatial coordinates ($\delta_{\rm ini}(\vec{x}) \ll 1$). These fluctuations evolve in time. Some of them collapse, and some of them do not. More specifically, some density *peaks* collapse. In real-world picture, these collapsed regions have very high density. On the other hand, in the corresponding linear world, these collapsed regions have $\delta_L(\vec{x}) \geq \delta_c \simeq 1.686$. This is a nice property, allowing us to calculate the number of collapsed objects at a given time.

3.4 Mass Function of Collapsed Halos

How do we calculate the number of collapsed objects at a given time, as a function of masses? The idea is the following:



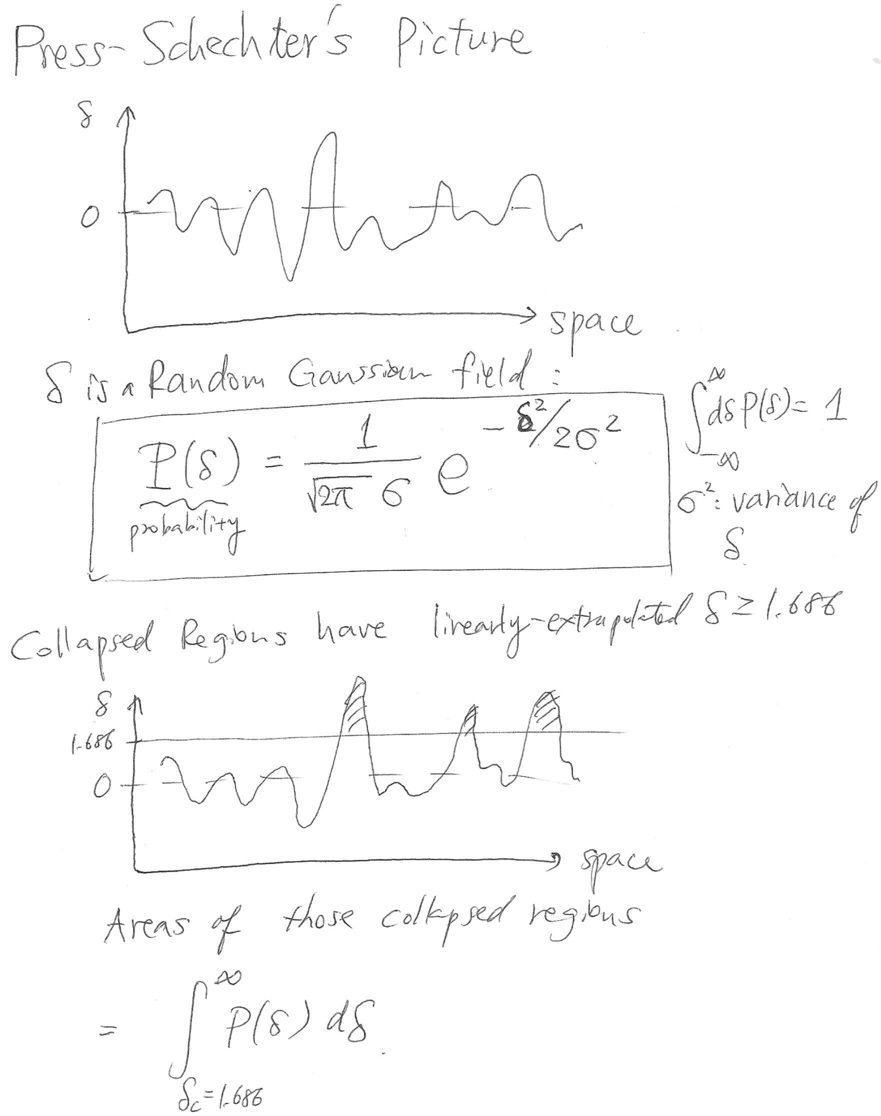
Suppose that the distribution of initial density fluctuations, $\delta_{\rm ini}(\vec{x})$, is given by $P(\delta_{\rm ini})$. Then, the distribution of *linearly-evolved density fluctuations*, δ_L, should also obey the same probability distribution function, $P(\delta)$. Then, a fraction of the volume occupied by the collapsed regions is given simply by

$$P(>\delta_c) = \int_{\delta_c}^{\infty} d\delta_L^{\infty}\ P(\delta_L). \tag{3.85}$$

This should be related to the number of collapsed objects at a given time. But how? To make progress, we must specify the form of $P(\delta_L)$. The current data (especially the cosmic microwave background) strongly suggest that the initial fluctuations obey a Gaussian distribution to high precision, which is consistent with the standard prediction of inflation. While it is possible that some level of non-Gaussianity (departure from a Gaussian distribution) were present, for this lecture we shall ignore non-Gaussianity and assume that the initial fluctuations obeyed a Gaussian distribution. Then, the linearly-evolved density fluctuations also obey a Gaussian distribution:

$$P(\delta_L) = \frac{e^{-\delta_L^2/(2\sigma_L^2)}}{\sqrt{2\pi}\sigma_L}, \tag{3.86}$$

with $\int_{-\infty}^{\infty} d\delta_L P(\delta_L) = 1$. Here, σ_L^2 is the variance of density fluctuations.



We have to pause here. How are peaks related to objects? In order to answer this question, we must recall that the above discussion on δ_c relied upon the spherical collapse model - we started by discussing the evolution of a spherical overdensity region with mass M. When the fluctuation was linear, this region had the mass density that is close to the mean mass density of the universe.

Therefore, the **initial comoving radius** of this region was given by

$$R = \left(\frac{3M}{4\pi\bar{\rho}_M}\right) = 4.0 \text{ Mpc} \left(\frac{0.135}{\Omega_M h^2}\frac{M}{10^{13}\ M_\odot}\right)^{1/3}, \tag{3.87}$$

where we have used $\bar{\rho}_M = 2.775 \times 10^{11}\Omega_M h^2\ M_\odot$ Mpc^{-3} (which is the present-day mass density of the universe). Therefore, for galactic scales ($M = 10^{12}\ M_\odot$), $R = 1.9$ Mpc; for clusters scales ($M = 10^{15}\ M_\odot$), $R = 19$ Mpc. Again, note that this is not the real radius of objects observed today with a given mass. Rather, this is the radius that objects *would* have, if they had the *mean* mass density of the universe today. This is the most relevant radius when we talk about the linear density fluctuations. These regions then expand, turn around, and then contract to form objects with physical radii much smaller than R given above.

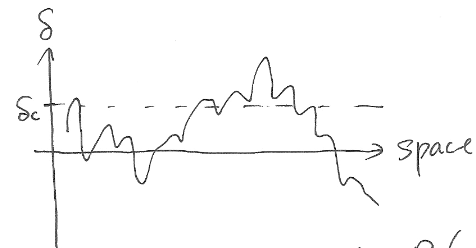
The next step is to find overdense regions that have a certain mass M. In order to do this, we need to "bin" the density fields with the corresponding radii R. Namely, we first average the density field as

$$\delta_R(\vec{x}) \equiv \frac{1}{\frac{4\pi R^3}{3}}\int_{|\vec{r}|\le R} d^3r\delta_L(\vec{x}+\vec{r}), \tag{3.88}$$

where $R = \left(\frac{3M}{4\pi\bar{\rho}_M}\right)$, and see if the averaged δ_R exceeds the critical overdensity δ_c. The Fourier transform of δ_R is related to that of the original linear density field δ_L as

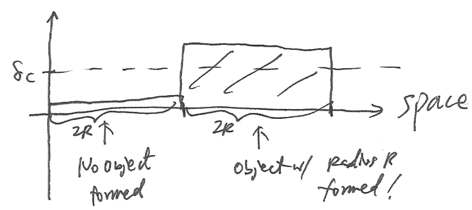
$$\begin{aligned}
\tilde{\delta}_R(\vec{k}) &= \int d^3x\ \delta_R(\vec{x})e^{-i\vec{k}\cdot\vec{x}} \\
&= \int d^3x \frac{1}{\frac{4\pi R^3}{3}}\int_{|\vec{r}|\le R} d^3r \int \frac{d^3k'}{(2\pi)^3}\tilde{\delta}_L(\vec{k}')e^{i\vec{k}'\cdot(\vec{x}+\vec{r})}e^{-i\vec{k}\cdot\vec{x}} \\
&= \frac{1}{\frac{4\pi R^3}{3}}\int_{|\vec{r}|\le R} d^3r \int d^3k'\ \delta_D^{(3)}(\vec{k}-\vec{k}')\tilde{\delta}_L(\vec{k}')e^{i\vec{k}\cdot\vec{r}} \\
&= \tilde{\delta}_L(\vec{k})\frac{1}{\frac{4\pi R^3}{3}}\int_{|\vec{r}|\le R} d^3r\ e^{i\vec{k}\cdot\vec{r}} \\
&= \tilde{\delta}_L(\vec{k})\frac{1}{\frac{2R^3}{3}}\int_0^R r^2 dr\int_{-1}^{1} d\mu\ e^{ikr\mu} \\
&= \tilde{\delta}_L(\vec{k})\left[\frac{3j_1(kR)}{kR}\right],
\end{aligned} \tag{3.89}$$

where $j_1(x) = \sin(x)/x^2 - \cos(x)/x$ is a spherical Bessel function of order 1. Therefore, the Fourier transform of δ_R is the Fourier transform of δ_L times the "window function" given by $\frac{3j_1(kR)}{kR}$.

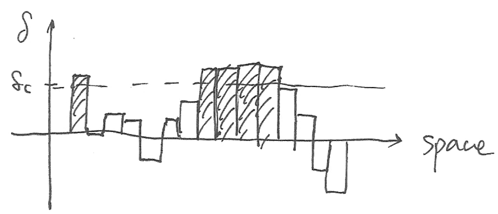


How many objects with a radius $R \left(= \left(\frac{3M}{4\pi\bar{\rho}}\right)^{1/3} \right)$ are formed ?

⇓ "bin" the density field with a width $2R$.



For much smaller radius R:



Much more objects are formed.

Therefore, for a given mass of interest, M, there is a corresponding radius R, and thus:

$$\boxed{\delta_R \equiv \frac{1}{\frac{4\pi R^3}{3}} \int_{|\vec{r}|\le R} d^3r\ \delta(\vec{r})}$$

is the relevant density fluctuation.

The variance of the averaged density field δ_R is then given by

$$
\begin{aligned}
\sigma_R^2 &\equiv \langle \delta_R^2(\vec{x}) \rangle \\
&= \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \langle \tilde{\delta}_R(\vec{k}) \tilde{\delta}_R^*(\vec{k}') \rangle e^{i(\vec{k}-\vec{k}')\cdot\vec{x}} \\
&= \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \langle \tilde{\delta}_L(\vec{k}) \tilde{\delta}_L^*(\vec{k}') \rangle \left[\frac{3j_1(kR)}{kR}\right] \left[\frac{3j_1(k'R)}{k'R}\right] e^{i(\vec{k}-\vec{k}')\cdot\vec{x}} \\
&= \int \frac{k^2 dk}{2\pi^2} P(k) \left[\frac{3j_1(kR)}{kR}\right]^2 .
\end{aligned}
$$

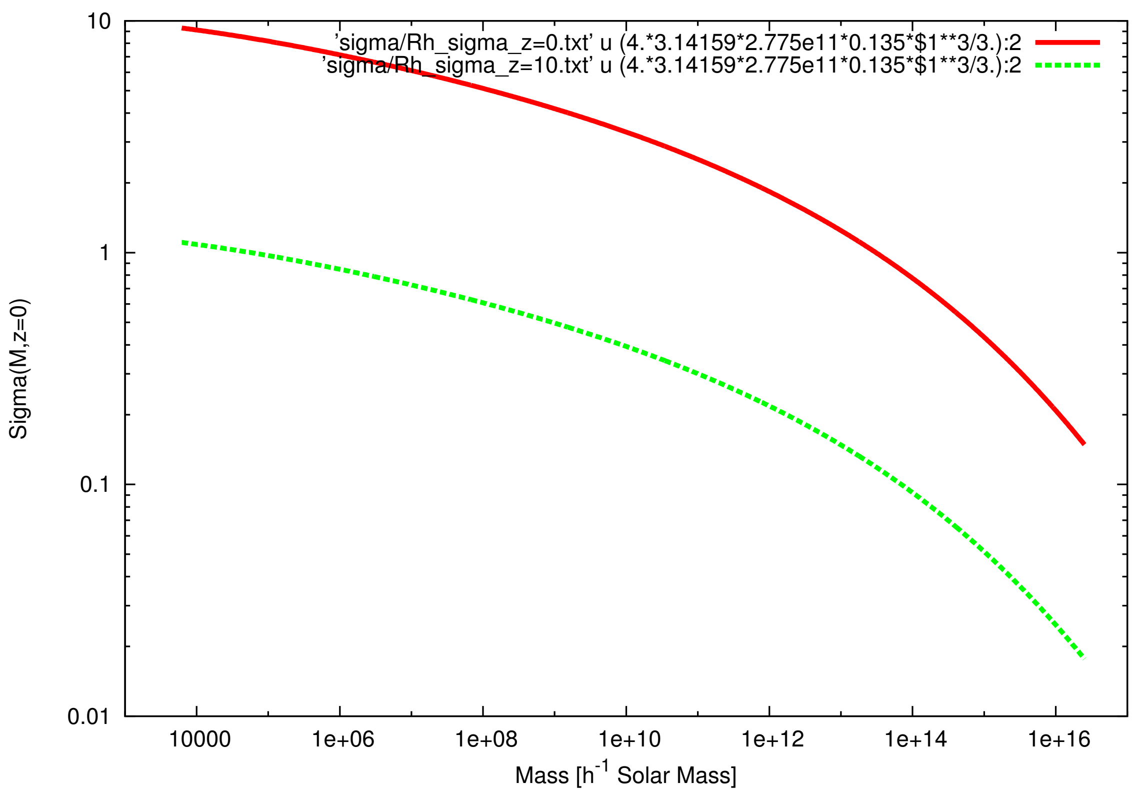
This integral can be estimated roughly as

$$\sigma_R^2 \approx \left.\frac{k^3 P(k)}{2\pi^2}\right|_{k=1/R}. \tag{3.90}$$

This means that, for a power-law power spectrum of $P(k) \propto k^m$, the variance scales as

$$\sigma_R^2 \propto R^{-(m+3)} \propto M^{-(m+3)/3}. \tag{3.91}$$

For example, the large-scale limit of the power spectrum is $P(k) \propto k^{n_s}$, and thus $\sigma_R^2 \propto M^{-(n_s+3)/3} \approx M^{-4/3}$. The small-scale limit of the power spectrum is $P(k) \propto k^{n_s-4}(\ln k)^2$, and thus $\sigma_R^2 \propto M^{-(n_s-1)/3} \approx M^0$ (except for a logarithmic factor). Finally, as σ^2 is proportional to the power spectrum, its growth is given by the growth of mass density fluctuation squared, i.e., $\sigma_R^2 \propto D^2$, where $D \propto a$ during the matter era.



The above figure shows σ_R as a function of $M = 4\pi\bar{\rho}_M R^3/3$ in units of h^{-1} $M_\odot$. The upper and lower curves correspond to $z = 0$ and $z = 10$, respectively. From this figure, one finds that, at $z = 10$, a 1-σ fluctuation corresponding to $M = 10^4$ h^{-1} $M_\odot$ has not yet reached the critical overdensity, $\delta_c \simeq 1.686$. This does not mean that these masses have not collapsed yet - according to a Gaussian distribution, there are fluctuations exceeding 1-σ fluctuations. They are just not very common. At $z = 0$, 1-σ fluctuations corresponding to $M \simeq 10^{13}\ h^{-1}\ M_\odot$ exceed δ_c, and thus typically collapsing halos at $z = 0$ have $M \simeq 10^{13}\ h^{-1}\ M_\odot$, and halos more massive than that (such as clusters of galaxies) can be collapsing but are still rare.

With these, one can now calculate the **mass function of halos**, dn/dM, which is the "comoving number density of collapsed halos per unit mass interval at a given time." This can be calculated as (Press & Schechter, Astrophysical Journal, 187, 425 (1974))

$$\frac{dn}{dM} = -\frac{\bar{\rho}_M}{M}\frac{d}{dM}P(>\delta_c). \tag{3.92}$$

Plugging in a Gaussian form of the probability distribution function, we can calculate the derivative:

$$\begin{aligned}\frac{dn}{dM} &= -\frac{\bar{\rho}_M}{M}\frac{d}{dM}\int_{\delta_c}^{\infty} d\delta_R \frac{1}{\sqrt{2\pi}\sigma_R}e^{-\delta_R^2/(2\sigma_R^2)} \\ &= -\frac{\bar{\rho}_M}{M}\frac{d}{dM}\int_{\delta_c/\sigma_R}^{\infty} dx \frac{1}{\sqrt{2\pi}}e^{-x^2/2} \\ &= \frac{\bar{\rho}_M}{M}\delta_c\frac{d\sigma_R^{-1}}{dM}\frac{1}{\sqrt{2\pi}}e^{-\delta_c^2/(2\sigma_R^2)}.\end{aligned} \tag{3.93}$$

This is the mass function.

Now, let us check this mass function. Under the assumption that all the mass in the universe are enclosed in halos, the mass function times mass integrated over masses should be equal to the mean mass density of the universe, i.e.,

$$\int_0^\infty dM\ M\frac{dn}{dM} = \bar{\rho}_M. \tag{3.94}$$

Is this satisfied by the above mass function? A straightforward calculation shows that

$$\int_0^\infty dM\ M\frac{dn}{dM} = \frac{1}{2}\bar{\rho}_M, \tag{3.95}$$

and thus the above formula fails to account for a half of the mass in the universe! Press and Schechter, who came up with the above formula, then arbitrarily multiplied the above formula by a factor of two, and came up with the formula now known as the "Press-Schechter mass function":

$$\boxed{\frac{dn}{dM} = \frac{\bar{\rho}_M}{M}\delta_c\frac{d\sigma_R^{-1}}{dM}\sqrt{\frac{2}{\pi}}e^{-\delta_c^2/(2\sigma_R^2)}} \tag{3.96}$$

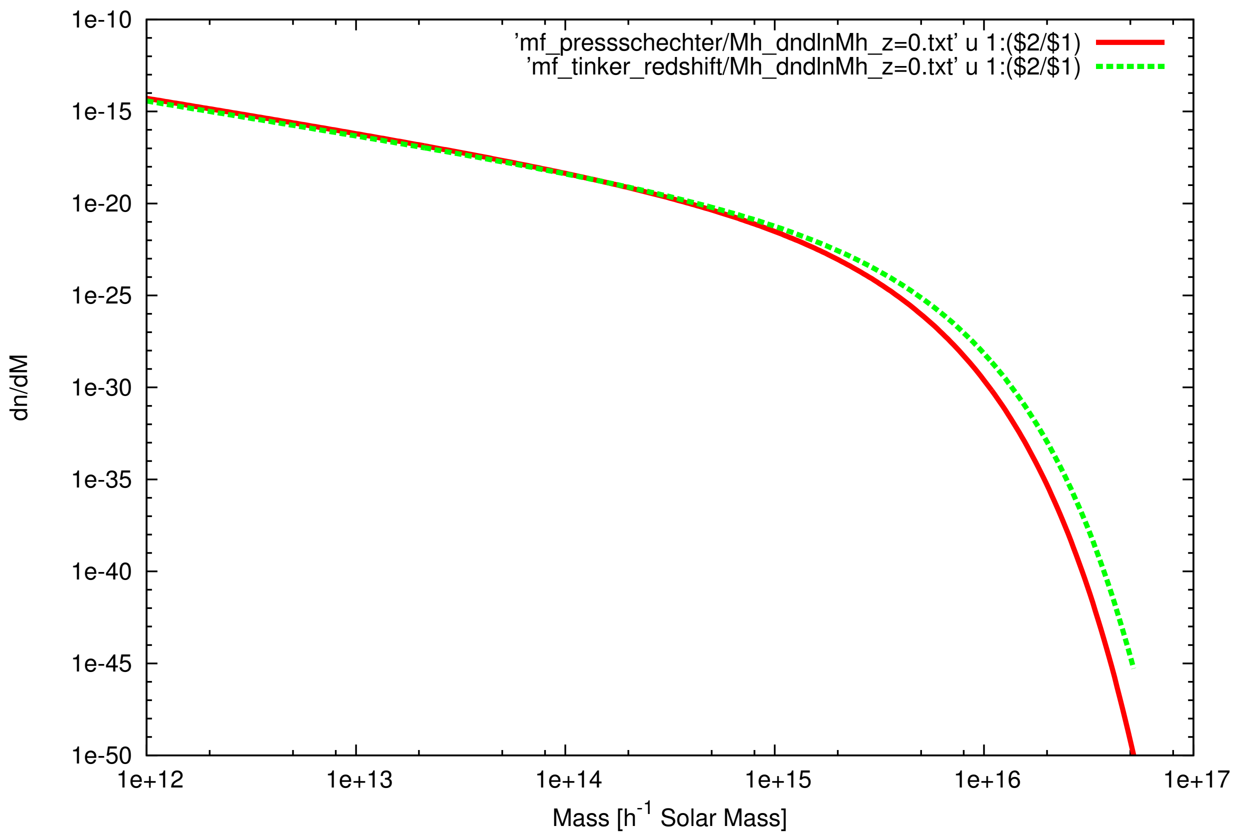
The arguments which have led to this formula are arguably simplistic - a spherical collapse - and even requires a fudge factor of two. However, a remarkable thing about this formula is that it gives more-or-less correct form of the mass function derived from N-body simulations.

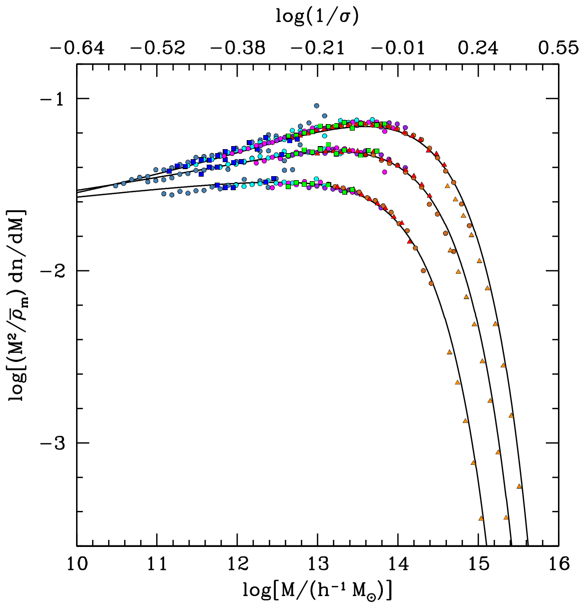
Many research groups have been trying to find a better formula for the mass function. A big motivation for getting a correct mass function is that the mass function is an observable quantity, and thus it can be used to infer the values of σ_R^2. Since it is proportional to the growth rate, $\sigma_R^2 \propto D^2$, the mass function can be used to infer D as a function of redshifts which, in turn, can be used to infer the nature of dark energy.

The latest fitting formula for the mass function is given by Tinker et al., Astrophysical Journal, 688, 709 (2008):

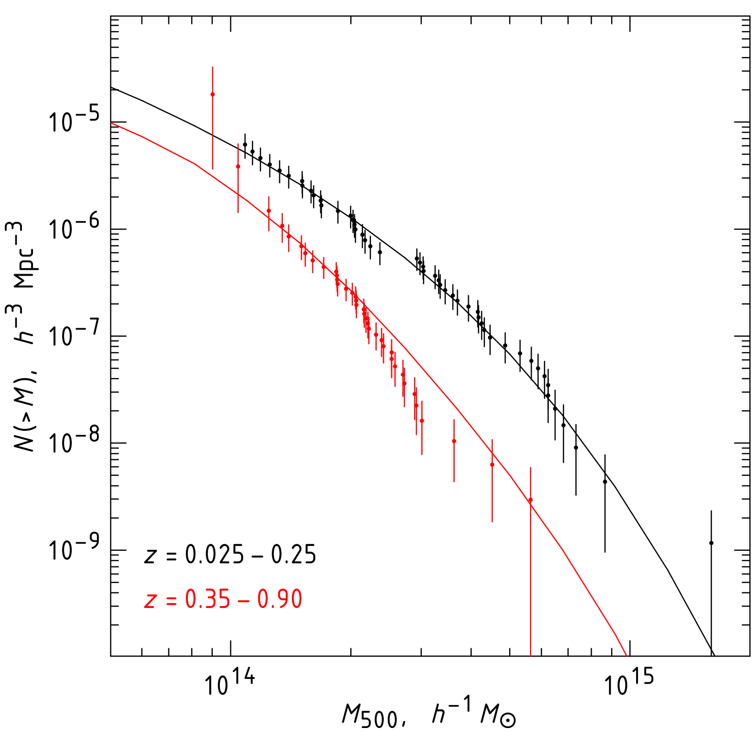
$$\frac{dn}{dM} = \frac{\bar{\rho}_M}{M}\frac{d\sigma_R^{-1}}{dM} A \left[\left(\frac{\sigma_R}{b}\right)^{-a} + 1 \right] e^{-c/\sigma_R^2}, \tag{3.97}$$

where $A = 0.186$, $a = 1.47$, $b = 2.57$, and $c = 1.19$. The comparison between Tinker et al.'s mass function and Press-Schechter mass function at $z = 0$ is given below.





The mass function has been derived from observations of the number of massive clusters of galaxies. *Chandra Cosmology Project* led by Alexey Vikhlinin (Vikhlinin et al., 692, 1060 (2009)) has yielded an impressive agreement between the cluster number counts and the prediction from the standard ΛCDM model, as shown below.



PROBLEM SET 4

1.1 Large-scale Structure of the Universe

1.1.1 Growth of Linear Density Fluctuations

Let us consider a universe containing matter and dark energy. Assuming that dark energy is a cosmological constant, the Friedmann equation gives

$$H^2 = \frac{8\pi G}{3}\left(\frac{\rho_{M0}}{a^3} + \rho_\Lambda\right), \tag{1.98}$$

where ρ_{M0} is the present-day value of the matter density, and ρ_Λ is the energy density associated with a cosmological constant (which is, of course, constant). How would the matter density fluctuations evolve in such a universe?

On the sub-horizon scales,

$$\epsilon \equiv \frac{k}{aH} \gg 1, \tag{1.99}$$

the evolution of the matter density fluctuations obeys the following equations:

$$\delta' = -\frac{\epsilon}{a}V, \tag{1.100}$$

$$V' = -\frac{1}{a}V - \frac{\epsilon}{a}\Phi, \tag{1.101}$$

$$\epsilon^2\Phi = \frac{4\pi G\rho_{M0}}{a^3H^2}\delta. \tag{1.102}$$

The primes denote derivatives with respect to a. Here, the right hand side of Poisson's equation (the third equation) contains only the matter density fluctuation, as a cosmological constant is spatially uniform and does not contribute to Φ.

Question 1.1: Combining equations (3), (4), and (5), obtain a single differential equation for δ. The answer should contain only δ, H, a, and their derivatives with respect to a. Once you obtain the desired equation, you should check that the solutions to that equation in the matter-dominated limit are given by $\delta = C_1 a + C_2/a^{3/2}$, where C_1 and C_2 are integration constants. *Hint*: You can relate ρ_{M0} to H'.

Question 1.2: Show that one of the solutions to the equation obtained above is $\delta \propto H$. This is a decaying solution.

Question 1.3: Show that another solution is given by

$$\delta \propto H \int \frac{da}{(aH)^3}. \tag{1.103}$$

This is a growing solution.

Question 1.4: Take the above growing solution, and define a new quantity, $g \equiv \frac{\delta}{a}$. This quantity must approach a constant during the matter era. Adjust the integration constant such that $g \to 1$ during the matter era. Writing the expansion rate as

$$H = H_0 \sqrt{\Omega_M (1+z)^3 + \Omega_\Lambda}, \tag{1.104}$$

where $1+z = \frac{1}{a}$, make a diagram showing the evolution of g as a function z from $z = 0$ to 3 for $\Omega_M = 0.27$ and $\Omega_\Lambda = 0.73$. The difference between $g = 1$ and g computed here is due to the effect of dark energy.

1.1.2 Redshift Space Distortion: Kaiser Effect

While the underlying power spectrum of density fluctuations, $P(k)$, should only depend on the magnitude of k owing to isotropy of the universe, the *observed* power spectrum can depend on directions of $\vec{k}$. This is due to the effect of peculiar velocity of matter (say, galaxies), and is called the Kaiser effect (N. Kaiser, Monthly Notices of Royal Astronomical Society, 277, 1 (1987)).

The Kaiser effect arises because we make observations in **redshift space**, rather than in real space. Specifically, we infer the location of galaxies along the line of sight from observed redshifts. However, redshifts receive contributions from both the cosmological expansion and peculiar velocity along the line of sight:

$$z_{\rm obs} = z_{\rm real} + \frac{1}{a}\frac{v_\parallel}{c}, \tag{1.105}$$

where $v_\parallel \equiv \hat{n} \cdot \vec{v}$. As a result, galaxies moving toward us appear to have smaller redshifts and to be closer than they actually are, while galaxies moving away from us appear to have larger redshifts and to be farther than they actually are.

As we learned in Section 1.5, the difference in redshifts is related to the comoving separation between two galaxies along the line of sight as

$$\delta r_\parallel = \frac{c\delta z}{H}. \tag{1.106}$$

Therefore, observationally inferred comoving separation along the line of sight is different from the real comoving separation by

$$\delta r_{\parallel,\rm obs} = \delta r_{\parallel,\rm real} + \frac{v_\parallel}{aH}. \tag{1.107}$$

On the other hand, nothing would happen to the directions perpendicular to the line of sight.

This can be summarized as coordinate transformation. The coordinates in redshift space, s^i, and those in real space, x^i, are related by

$$s^1 = x^1, \tag{1.108}$$
$$s^2 = x^2, \tag{1.109}$$
$$s^3 = x^3 + \frac{v_\parallel}{aH}, \tag{1.110}$$

where we have chosen the line of sight direction as the 3-direction. How does this affect the observed power spectrum?

Question 1.5: Since this is merely coordinate transformation, the mass within a unit volume must be conserved regardless of the choice of the coordinate system. We have

$$\rho_s d^3 s = \rho_x d^3 x, \tag{1.111}$$

where ρ_s and ρ_x are the mass densities in redshift space and real space, respectively. Expanding these into perturbations,

$$\bar{\rho}(1+\delta_s)d^3 s = \bar{\rho}(1+\delta_x)d^3 x. \tag{1.112}$$

Note that the mean density, $\bar{\rho}$, is the same in both real space and redshift space. From this, we obtain

$$\delta_s = \frac{1}{|J|}(1+\delta_x) - 1, \tag{1.113}$$

where $|J|$ is the determinant of the Jacobian matrix:

$$J \equiv \begin{pmatrix} \frac{\partial s^1}{\partial x^1} & \frac{\partial s^1}{\partial x^2} & \frac{\partial s^1}{\partial x^3} \\ \frac{\partial s^2}{\partial x^1} & \frac{\partial s^2}{\partial x^2} & \frac{\partial s^2}{\partial x^3} \\ \frac{\partial s^3}{\partial x^1} & \frac{\partial s^3}{\partial x^2} & \frac{\partial s^3}{\partial x^3} \end{pmatrix}. \tag{1.114}$$

By expanding equation (16) up to the first order in perturbations (including density and velocity), find the relation between δ_s, δ_x, and $v_\parallel$. Note that $v_\parallel$ depends on spatial coordinates x^i, whereas H does not but it depends only on time. *Hint*: does the result you obtained make sense? What are the conditions for $\delta_s < \delta_x$ or $\delta_s > \delta_x$? Can you explain why they are so?

Question 1.6: Now is the time to go to Fourier space. Use

$$\delta_s = \int \frac{d^3k}{(2\pi)^3}\, \tilde{\delta}_{s,\vec{k}}\, e^{i\vec{k}\cdot\vec{x}}, \tag{1.115}$$

$$\delta_x = \int \frac{d^3k}{(2\pi)^3}\, \tilde{\delta}_{x,\vec{k}}\, e^{i\vec{k}\cdot\vec{x}}, \tag{1.116}$$

$$\vec{v} = \int \frac{d^3k}{(2\pi)^3}\, \tilde{\vec{v}}_{\vec{k}}\, e^{i\vec{k}\cdot\vec{x}}, \tag{1.117}$$

$$\tag{1.118}$$

and write the relation between δ_s and δ_x in Fourier space.† Here, $\vec{v} = (v^1, v^2, v^3)$ and $v_\parallel = v^3$.

We need to relate $v_\parallel$ to δ_x. For this, we can use the mass conservation equation:

$$\dot{\delta}_x + \frac{1}{a}\vec{\nabla}\cdot\vec{v} = 0. \tag{1.119}$$

Fourier-transforming this, one finds

$$\dot{\tilde{\delta}}_{x,\vec{k}} + \frac{i\vec{k}}{a}\cdot\tilde{\vec{v}}_{\vec{k}} = 0. \tag{1.120}$$

†If you are careful, you might wonder why we can expand δ_s using x coordinates, rather than s coordinates. This is OK up to the first order - since δ_s is already a perturbation, the difference between x coordinates and s coordinates would appear only at the second order.

This equation is satisfied if

$$\tilde{\vec{v}}_{\vec{k}} = ia\frac{\vec{k}}{k^2}\dot{\tilde{\delta}}_{x,\vec{k}}. \tag{1.121}$$

As we have seen from the previous section, $\tilde{\delta}_{x,\vec{k}}$ evolves by the same factor at all scales (all k), so we may write $\tilde{\delta}_{x,\vec{k}} \propto D$. Then,

$$\dot{\tilde{\delta}}_{x,\vec{k}} = \frac{\dot{D}}{D}\tilde{\delta}_{x,\vec{k}} = H\frac{d\ln D}{d\ln a}\tilde{\delta}_{x,\vec{k}}. \tag{1.122}$$

From now on, let us write

$$f \equiv \frac{d\ln D}{d\ln a}, \tag{1.123}$$

so that

$$\tilde{\vec{v}}_{\vec{k}} = iaHf\frac{\vec{k}}{k^2}\tilde{\delta}_{x,\vec{k}}. \tag{1.124}$$

By putting these altogether, show

$$\tilde{\delta}_{s,\vec{k}} = \left(1 + f\frac{k_{\parallel}^2}{k^2}\right)\tilde{\delta}_{x,\vec{k}}, \tag{1.125}$$

where $k_{\parallel} = k^3$ for $\vec{k} = (k^1, k^2, k^3)$. The modification of density fluctuations in redshift space due to the peculiar velocity effect is known as the **redshift space distortion**, and is often called the **Kaiser effect**. Because of this, the observed power spectrum depends on $k_{\parallel}$:

$$P_s(k, k_{\parallel}) = \left(1 + f\frac{k_{\parallel}^2}{k^2}\right)^2 P_x(k). \tag{1.126}$$

This is a nice result, as one can use the dependence of the observed power spectrum on $k_{\parallel}$ to extract the information on the growth of structures via f. As we have seen in the previous section, D (hence $f = d\ln D/d\ln a$) changes if there is dark energy, and thus this information can be used to study the nature of dark energy.