"Hunting for primordial non-Gaussianity in CMB"

~ Lecture on non-Gaussianity, given at IESC ~

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GRAND THEME

— Detection of any forms of PRIMORDIAL non-Gaussianity* is a breakthrough in cosmology.

* Non-Gaussianity = deviation from a Gaussian distribution.

Outline of the lecture

1. Basics of Gaussian and non-Gaussian statistics. (July 7)

2. Effects of non-Gaussianity on cosmological fluctuations (July 8)

3. Measuring non-Gaussianity from the cosmological data (July 9)

Reference

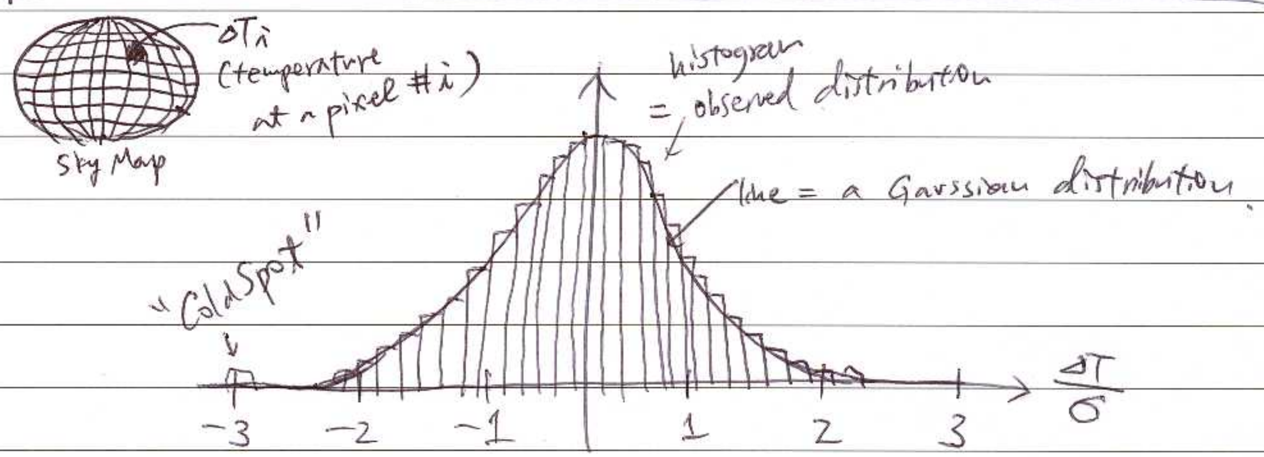
- E. Komatsu, Classical and Quantum Gravity, 27, 124010 (2010) (arXiv: 1003.6097)
- E. Komatsu, astro-ph/0206039 (also see arXiv: 0902.4759)

1. Basics of Gaussian and non-Gaussian statistics

1.1. Gaussian distribution

The observed one point distribution of temperature fluctuations of CMB on the sky is very close to a Gaussian distribution:

$$\boxed{\text{1-point distribution} = \frac{1}{(2\pi)^{N_{pix}/2}\prod_i \sigma_i}\exp\left[-\frac{1}{2}\sum_i \frac{(\Delta T)_i^2}{\sigma_i^2}\right]}$$



The basic properties of a one-point Gaussian distribution:

For $p(x) \equiv \frac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/2\sigma^2}$ where $x \equiv \Delta T$,

$$\left\{\begin{array}{ll}
\langle x \rangle \equiv \int_{-\infty}^{\infty} x\, p(x)\, dx = 0 & [\text{zero mean}] \\
\langle x^2 \rangle \equiv \int_{-\infty}^{\infty} x^2 p(x)\, dx = \sigma^2 & [\text{Variance}] \\
\langle x^3 \rangle \equiv \int_{-\infty}^{\infty} x^3 p(x)\, dx = 0 & [\text{zero } \underbrace{\textbf{skewness}}_{\equiv \langle x^3 \rangle}] \\
\langle x^4 \rangle \equiv \int_{-\infty}^{\infty} x^4 p(x)\, dx = 3\sigma^4 & [\text{zero } \underbrace{\textbf{Kurtosis}}_{\equiv \langle x^4 \rangle - 3\langle x^2 \rangle^2}] \\
\langle x^5 \rangle \equiv \int_{-\infty}^{\infty} x^5 p(x)\, dx = 0, \text{ etc.} &
\end{array}\right.$$

IMPORTANT
σ determines everything about a Gaussian!!

But, the 1-point distribution cannot be a full description of CMB, as CMB temperatures at 2 points on the sky are not independent, but CORRELATED.

So, the exact distribution that describes the Gaussian CMB temperature anisotropy is given by the so-called "multi-variate" Gaussian distribution :

$$\boxed{P(\delta T) = \frac{1}{(2\pi)^{N_{pix}/2} |\xi|^{1/2}} \exp\left[-\frac{1}{2}\sum_{ij} \delta T_i (\xi^{-1})_{ij} \delta T_j\right]}$$

where ξ_{ij} is the 2-point correlation, or the covariance matrix, defined by

$$\boxed{\xi_{ij} \equiv \langle \delta T_i \delta T_j \rangle}$$

Of course, it still preserves the basic property of a Gaussian distribution : ξ_{ij} determines everything, and e.g.,

$$\langle \delta T_i \delta T_j \delta T_k \rangle = 0.$$

Wick's theorem :

$$\langle x^4 \rangle = 3 \langle x^2 \rangle^2$$

more generally,

$$\langle x_1 x_2 x_3 x_4 \rangle = \langle x_1 x_2 \rangle \langle x_3 x_4 \rangle + \langle x_1 x_3 \rangle \langle x_2 x_4 \rangle + \langle x_1 x_4 \rangle \langle x_2 x_3 \rangle$$

when $x_1 = x_2 = x_3 = x_4 \equiv x$, then, of course, we recover $\langle x^4 \rangle = 3\langle x^2 \rangle^2$.

☆ For Gaussian fluctuations, all even moments are given by products of 2-point functions, and all odd moments vanish.

We often work with the harmonic coefficient of the temperature anisotropy: $\Delta T(\hat{n}_i) = \sum_{\ell m} a_{\ell m} Y_{\ell m}(\hat{n}_i)$, or

$$a_{\ell m} = \int d\hat{n}\, \Delta T(\hat{n})\, Y^*_{\ell m}(\hat{n}).$$

In terms of $a_{\ell m}$, we have

$$\boxed{P(a_{\ell m}) = \frac{1}{(2\pi)^{N_{\rm harm}/2}|C|} \exp\left[-\frac{1}{2}\sum_{\ell m, \ell' m'} a^*_{\ell m} (C^{-1})^{\ell m}_{\ell' m'} a_{\ell' m'}\right]}$$

where

$$\boxed{C_{\ell m \atop \ell' m'} \equiv \langle a^*_{\ell m} a_{\ell' m'} \rangle}$$

This distribution provides the full description of a Gaussian CMB: it contains all information on it. Again, it is fully determined by the covariance matrix, $C_{\ell m, \ell' m'}$, and we have, e.g.,

$$\langle a_{\ell m} a_{\ell' m'} a_{\ell'' m''} \rangle = 0.$$

Under the special circumstance that CMB is statistically homogeneous & isotropic, or equivalently, CMB is invariant under translation & rotation on the sky, then

$$\boxed{C_{\ell m, \ell' m'} = C_\ell\, \delta_{\ell\ell'}\, \delta_{mm'}.}$$ translation & rotation invariance.

(Q: derive this from $\xi_{ij} = \langle \Delta T_i \Delta T_j \rangle = \xi(|\hat{n}_i - \hat{n}_j|)$, i.e., ξ depends only on the separation between i & j.)

1.2. Non-Gaussian Distribution

Now, what about a non-Gaussian distribution?

In order to study it, we must know what its probability distribution is. However — once we deviate from a Gaussian distribution, we face infinite number of possibilities!!

Are we totally clueless?

Fortunately not --- because we know, from observations, that CMB is very close to a Gaussian distribution!!

In other words, it makes sense to "Taylor-expand" the probability distribution around a Gaussian distribution.

How do we do that in practice??

Taylor expansion of a function $f(x)$ around $x=0$:

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \frac{d^n f}{dx^n}.$$

☆ Can we do the same thing for a Gaussian distribution?

→ Yes, and it is called the "Gram-Charlier expansion."