"Hunting for primordial non-Gaussianity in CMB"

~ Lecture on non-Gaussianity, given at IESC ~

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GRAND THEME

— Detection of any forms of PRIMORDIAL non-Gaussianity* is a breakthrough in cosmology.

* Non-Gaussianity = deviation from a Gaussian distribution.

Outline of the lecture

1. Basics of Gaussian and non-Gaussian statistics. (July 7)
2. Effects of non-Gaussianity on cosmological fluctuations (July 8)
3. Measuring non-Gaussianity from the cosmological data. (July 9)

Reference

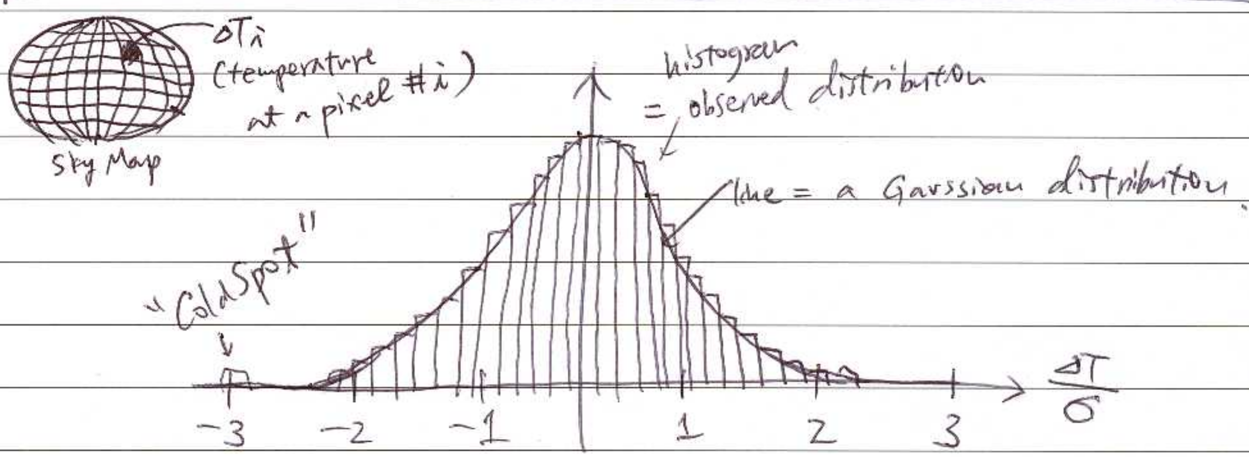
- E. Komatsu, Classical and Quantum Gravity, 27, 124010 (2010) (arXiv: 1003.6097)
- E. Komatsu, astro-ph/0206039 (also see. arXiv: 0902.4759)

1. Basics of Gaussian and non-Gaussian statistics

1.1. Gaussian distribution

The observed one point distribution of temperature fluctuations of CMB on the sky is very close to a Gaussian distribution:

$$\boxed{\text{1-point distribution} = \frac{1}{(2\pi)^{N_{pix}/2}\prod_i \sigma_i} \exp\left[-\frac{1}{2}\sum_i \frac{(\Delta T)_i^2}{\sigma_i^2}\right]}$$



The basic properties of a one-point Gaussian distribution:

For $\quad p(x) \equiv \frac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/2\sigma^2} \quad$ where $x \equiv \frac{\Delta T}{}$,

$$\left\{\begin{array}{ll}
\langle x\rangle \equiv \int_{-\infty}^{\infty} x\, p(x)\, dx = 0 & [\text{zero mean}] \\
\langle x^2\rangle \equiv \int_{-\infty}^{\infty} x^2 p(x)\, dx = \sigma^2 & [\text{Variance}] \\
\langle x^3\rangle \equiv \int_{-\infty}^{\infty} x^3 p(x)\, dx = 0 & [\text{zero } \underset{\equiv\langle x^3\rangle}{\textbf{skewness}}] \\
\langle x^4\rangle \equiv \int_{-\infty}^{\infty} x^4 p(x)\, dx = 3\sigma^4 & [\text{zero } \underset{\equiv\langle x^4\rangle - 3\langle x^2\rangle^2}{\textbf{Kurtosis}}] \\
\langle x^5\rangle \equiv \int_{-\infty}^{\infty} x^5 p(x)\, dx = 0, \text{ etc.} &
\end{array}\right.$$

IMPORTANT
σ determines everything about a Gaussian!!

But, the 1-point distribution cannot be a full description of CMB, as CMB temperatures at 2 points on the sky are <u>not</u> independent, but CORRELATED.

So, the exact distribution that describes the Gaussian CMB temperature anisotropy is given by the so-called "multi-variate" Gaussian distribution:

$$\boxed{P(\delta T) = \frac{1}{(2\pi)^{N_{pix}/2}\,|\xi|^{1/2}} \exp\left[-\frac{1}{2}\sum_{ij} \delta T_i \left(\xi^{-1}\right)_{ij} \delta T_j\right]}$$

where ξ_{ij} is the <u>2-point correlation</u>, or <u>the covariance matrix</u>, defined by

$$\boxed{\xi_{ij} \equiv \langle \delta T_i\, \delta T_j \rangle.}$$

Of course, it still preserves the basic property of a Gaussian distribution: ξ_{ij} determines everything, and e.g.,

$$\langle \delta T_i\, \delta T_j\, \delta T_k \rangle = 0.$$

Wick's theorem:

$$\langle x^4 \rangle = 3\langle x^2 \rangle^2$$

more generally,

$$\langle x_1 x_2 x_3 x_4 \rangle = \langle x_1 x_2\rangle\langle x_3 x_4\rangle + \langle x_1 x_3\rangle\langle x_2 x_4\rangle + \langle x_1 x_4\rangle\langle x_2 x_3\rangle$$

when $x_1 = x_2 = x_3 = x_4 \equiv x$, then, of course, we recover $\langle x^4\rangle = 3\langle x^2\rangle^2$.

☆ For Gaussian fluctuations, <u>all</u> even moments are given by products of 2-point functions, and <u>all</u> odd moments vanish.

We often work with the harmonic coefficient of the temperature anisotropy: $\Delta T(\hat{n}_i) = \sum_{\ell m} a_{\ell m} Y_{\ell m}(\hat{n}_i)$, or

$$a_{\ell m} = \int d\hat{n}\, \Delta T(\hat{n})\, Y^*_{\ell m}(\hat{n}).$$

In terms of $a_{\ell m}$, we have

$$\boxed{p(a_{\ell m}) = \frac{1}{(2\pi)^{N_{\rm harm}/2}|C|} \exp\left[-\frac{1}{2}\sum_{\ell m, \ell' m'} a^*_{\ell m} (C^{-1})^{\ell m}_{\ell' m'} a_{\ell' m'}\right]}$$

where

$$\boxed{C_{\ell m, \ell' m'} \equiv \langle a^*_{\ell m} a_{\ell' m'} \rangle}$$

This distribution provides the full description of a Gaussian CMB: it contains all information on it. Again, it is fully determined by the covariance matrix, $C_{\ell m, \ell' m'}$, and we have, e.g,

$$\langle a_{\ell m} a_{\ell' m'} a_{\ell'' m''} \rangle = 0.$$

Under the special circumstance that CMB is statistically homogeneous & isotropic, or equivalently, CMB is invariant under translation & rotation on the sky, then

$$\boxed{C_{\ell m, \ell' m'} = C_\ell\, \delta_{\ell\ell'} \delta_{mm'}}$$ translation & rotation invariance.

(Q: derive this from $\xi_{ij} = \langle \Delta T_i \Delta T_j \rangle = \xi(|\hat{n}_i - \hat{n}_j|)$, i.e., ξ depends only on the separation between i & j.)

1.2. Non-Gaussian Distribution

Now, what about a non-Gaussian distribution?

In order to study it, we must know what its probability distribution is. However — once we deviate from a Gaussian distribution, we face infinite number of possibilities!!

Are we totally clueless?

Fortunately not --- because we know, from observations, that CMB is very close to a Gaussian distribution!!

In other words, it makes sense to "Taylor-expand" the probability distribution around a Gaussian distribution.

How do we do that in practice??

Taylor expansion of a function $f(x)$ around $x=0$:

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \frac{d^n f}{dx^n}.$$

☆ Can we do the same thing for a Gaussian distribution?

→ Yes, and it is called the "Gram-Charlier expansion."

Gram-Charlier expansion

[See Blinnikov & Moessner, A&A Suppl., 130, 193 (1998) for a review.]

Suppose that $G(x)$ is a Gaussian distribution with a unit variance:

$$G(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

(One can think of x as $x = \delta T/\sigma$.)

Then, give the probability distribution, $p(x)$, as a Taylor series:

$$\boxed{p(x) = \sum_{n=0}^{\infty} C_n \frac{d^n G}{dx^n}.}$$

"Gram-Charlier expansion".

☆ One can rewrite this in an elegant form. To see this, define the "Chebyshev-Hermite polynomials" as

$$He_n(x) \equiv (-1)^n e^{+x^2/2} \frac{d^n}{dx^n} e^{-x^2/2},$$

which gives

$$\begin{cases} He_0(x) = 1 \\ He_1(x) = x \\ He_2(x) = x^2-1 \\ He_3(x) = x^3-3x \\ He_4(x) = x^4-6x^2+3 \\ \quad \vdots \end{cases}$$

This is different from the usual, more familiar Hermite polynomials:

☆ $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$

$\therefore He_n(x) = 2^{-\frac{n}{2}} H_n(x/\sqrt{2})$

▶ The above expansion gives

$$p(x) = G(x)\left[\underbrace{C_0}_{C_0 He_0} + \underbrace{C_1(-1)x}_{(-1)C_1 He_1} + \underbrace{C_2(x^2-1)}_{C_2 He_2} + \underbrace{C_3(-1)(x^3-3x)}_{(-1)C_3 He_3} + \cdots\right]$$

Therefore, we obtain

$$\boxed{p(x) = G(x)\left[\sum_{n=0}^{\infty} C_n (-1)^n He_n(x)\right]}$$

In other words, the Gram-Charlier expansion expands the ratio, $p(x)/G(x)$, in terms the Chebishev-Hermite polynomials times $(-1)^n$.

$$\frac{p(x)}{G(x)} = \sum_{n=0}^{\infty} C_n \left[\underbrace{(-1)^n He_n(x)}_{\text{Basis function for } p/G}\right]$$

Now, determine C_n. Note the inverse transform:

$$\int_{-\infty}^{\infty} p(x) He_n(x)\,dx = \sum_{m=0}^{\infty} C_m (-1)^m \underbrace{\int_{-\infty}^{\infty} G(x) He_m(x) He_n(x)\,dx}$$

$$= \sum_{m=0}^{\infty} C_m (-1)^m \; m!\,\delta_{nm}$$

$$= (-1)^n n!\, C_n$$

$$\boxed{\therefore\ C_n = \frac{(-1)^n}{n!}\int_{-\infty}^{\infty} p(x) He_n(x)\,dx}$$

What are the components of C_n?

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Recall: $He_0(x) = 1$, $He_1(x) = x$, $He_2(x) = x^2 - 1$, $He_3(x) = x^3 - 3x$
$He_4(x) = x^4 - 6x^2 + 3$, ...

Then, $C_n = \frac{(-1)^n}{n!}\int_{-\infty}^{\infty} p(x)\, He_n(x)\, dx$ gives:

$$\left\{\begin{aligned}
C_0 &= \int_{-\infty}^{\infty} p(x)\, dx = 1 \\
C_1 &= -\int_{-\infty}^{\infty} p(x)\, x\, dx = 0 \quad \text{[zero mean]} \\
C_2 &= \frac{1}{2!}\int_{-\infty}^{\infty} p(x)\,(x^2-1)\, dx = 1 - 1 = 0 \quad \text{[unit variance]} \\
C_3 &= \frac{(-1)}{3!}\int_{-\infty}^{\infty} p(x)\,(x^3 - 3x)\, dx = -\frac{1}{6}\kappa_3 \quad [\kappa_3 : \text{"skewness"},\ \kappa_3 \equiv \langle x^3 \rangle] \\
C_4 &= \frac{1}{4!}\int_{-\infty}^{\infty} p(x)\,(x^4 - 6x^2 + 3)\, dx \\
&= \frac{1}{4!}\Big[\underbrace{\Big(\int_{-\infty}^{\infty} p(x)\, x^4\, dx\Big) - 3}_{\equiv \kappa_4}\Big] \quad [\kappa_4 : \text{"kurtosis"}] \\
&= \frac{1}{24}\kappa_4 .
\end{aligned}\right.$$

⋮

Collecting all terms, we obtain the Gram-Charlier expansion of a probability distribution around a Gaussian with the zero mean and the unit variance:

$$\boxed{p(x) = G(x)\left[1 + \frac{1}{6}\kappa_3 He_3(x) + \frac{1}{24}\kappa_4 He_4(x) + \cdots\right]}$$

Now, extend this formula to the case where the variance is not unity, i.e., $\sigma \neq 1$.

For this case, it turns out that the following quantity:

$$S_n \equiv \frac{K_n}{\sigma^{2n-2}}$$

is a good expansion parameter for better convergence of the series. This is called the "**Edgeworth expansion**" (see, e.g., Bernardeau & Kofman, ApJ, 443, 479 (1995); Blinnikov & Moessner, A&A Suppl., 130, 193 (1998))

$$P(y) = G\left(\frac{y}{\sigma}\right)\left[1 + \sigma\frac{S_3}{6}He_3(y/\sigma) + \sigma^2\frac{S_4}{24}He_4(y/\sigma) + \cdots\right]$$

where y can be the CMB anisotropy, galaxy density fluctuation, etc...

However, this is <u>not</u> exactly what we wanted, as we are interested in the FULL distribution, rather than the 1-point distribution.

This can be found in a similar way:
(see Taylor & Watts, MNRAS, 328, 1027 (2001))
(Amendola, MNRAS, 283, 983 (1996))

$$p(a) = \left[1 - \frac{1}{6}\sum_{\substack{l_1m_1\\l_2m_2\\l_3m_3}} \langle a_{l_1m_1} a_{l_2m_2} a_{l_3m_3}\rangle \frac{\partial}{\partial a_{l_1m_1}}\frac{\partial}{\partial a_{l_2m_2}}\frac{\partial}{\partial a_{l_3m_3}} + \dots\right]$$

$$\times \frac{1}{(2\pi)^{N_{harm}/2}|C|^{1/2}} \exp\left[-\frac{1}{2}\sum_{\substack{lm\\l'm'}} a^*_{lm} (C^{-1})^{lm}_{l'm'} a_{l'm'}\right]$$

Doing the derivatives ...

$$p(a) = \frac{1}{(2\pi)^{N_{harm}/2}|C|^{1/2}} \exp\left[-\frac{1}{2}\sum_{\substack{lm\\l'm'}} a^*_{lm} (C^{-1})^{lm}_{l'm'} a_{l'm'}\right]$$

$$\times \left\{1 + \frac{1}{6}\sum_{\substack{l_1m_1\\l_2m_2\\l_3m_3}} \langle a_{l_1m_1} a_{l_2m_2} a_{l_3m_3}\rangle \left[(C^{-1}a)_{l_1m_1}(C^{-1}a)_{l_2m_2}(C^{-1}a)_{l_3m_3}\right]\right.$$

$$\left. - \frac{1}{6}\sum_{\substack{l_1m_1\\l_2m_2\\l_3m_3}} \langle a_{l_1m_1} a_{l_2m_2} a_{l_3m_3}\rangle \left[3(C^{-1})^{l_1m_1}_{l_2m_2}(C^{-1}a)_{l_3m_3}\right] + \dots\right\}$$

※ Compare this with the Edgeworth expansion:

$$p(a) = G(a) \times \left\{1 + \frac{1}{6}K_3\left(\frac{a^3}{\sigma^6} - 3\frac{a}{\sigma^4}\right)\right\}$$

So, the correspondence is obvious.

2. Effects of non-Gaussianity on cosmological fluctuations

There are four types of non-Gaussianity that have been considered/studied in the literature:

COSMOLOGICAL

1. Primordial non-Gaussianity
   - The non-Gaussianity that was generated during inflation (or some alternatives)

   MOST IMPORTANT

2. Second-order non-Gaussianity
   - The non-Gaussianity that was generated by the second-order effect at the decoupling epoch (or earlier)
3. Secondary non-Gaussianity
   - The non-Gaussianity that was generated by the late-time ($z \ll 1000$) effect.

NON-COSMOLOGICAL

4. Foreground non-Gaussianity
   - The non-Gaussianity that was generated by the Galactic & extra-Galactic foreground sources

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1. Generation of primordial fluctuation, ζ.
2. Conversion to ΔT (non-linear)
3. secondary effect
4. foreground

$$\zeta \longrightarrow \Delta T \longrightarrow \Delta T + \Delta T_{2nd} \longrightarrow \Delta T + \Delta T_{2nd} + \Delta T_{FG}$$

$\downarrow$

observed.

2.1 Primordial non-Gaussianity

Inflation generates the "primordial fluctuation" in curvature of space, ζ. [See P. Creminelli's lecture!]

(In terms of the metric,

$$ds^2 = -dt^2 + a^2(t)\, e^{2\zeta}\, d\vec{x}^2$$
)

(At the linear order, ζ would generate δT on large scales via the Sachs-Wolfe effect:
[See J.P. Uzan's lecture!]

$$\frac{\Delta T}{T} = -\frac{1}{5}\zeta .$$
)

Inflation predicts that its power spectrum is nearly scale-invariant:

$$k^3 P_\zeta(k) \sim \text{const.}$$

or

$$P_\zeta(k) = \frac{A}{k^{n_s-4}} \quad \text{where} \quad n_s \sim 1.$$

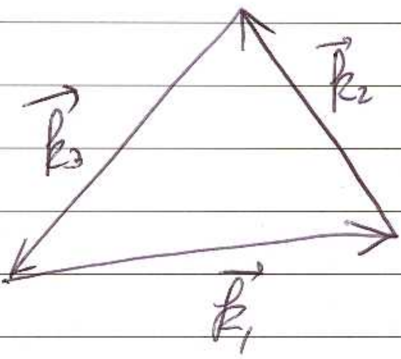
The WMAP 7-year results show that

$$n_s = 0.968 \pm 0.012 \quad (68\%\ \text{CL})$$

[E. Komatsu et al, arXiv:1001.4538]

How about the 3-point function (bispectrum)?

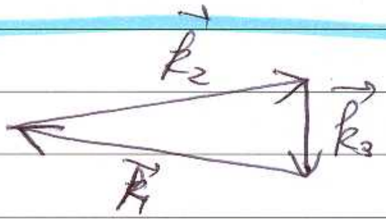
$$\langle \zeta(\vec{k}_1)\zeta(\vec{k}_2)\zeta(\vec{k}_3)\rangle = (2\pi)^3\delta^D(\vec{k}_1+\vec{k}_2+\vec{k}_3)\underbrace{B_\zeta(k_1,k_2,k_3)}_{\text{BISPECTRUM}}$$



It is important to realize that we have the freedom to choose for the <u>amplitude</u> and the <u>shape</u>.

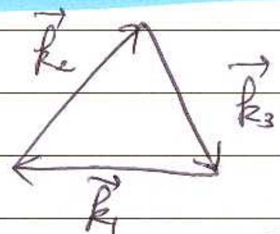
For example:

(i) Squeezed triangle
$|\vec{k}_3| \ll |\vec{k}_1| \approx |\vec{k}_2|$

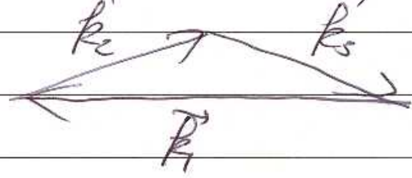


MOST IMPORTANT

(ii) Equilateral triangle
$|\vec{k}_1| \approx |\vec{k}_2| \approx |\vec{k}_3|$



(iii) Flat/Folded triangle
$|\vec{k}_3| \approx |\vec{k}_2| \approx |\vec{k}_1|/2$



(iv) etc, etc ...

The squeezed triangle has special significance:

"Detection of primordial non-Gaussianity in the squeezed limit rules out ALL single-field inflation models, regardless of details of models."

VERY IMPORTANT!

[See Creminelli & Zaldarriaga, JCAP, 10, 006 (2004)]

The theorem says: in the squeezed limit ($k_3 \ll k_1 \sim k_2$), ALL of single-field inflation models give:

$$\boxed{B_\zeta(k_1, k_1, k_3 \to 0) = (1-n_s) P_\zeta(k_1) P_\zeta(k_3)}$$

For $n_s = 0.968$, $1-n_s = 0.032$.

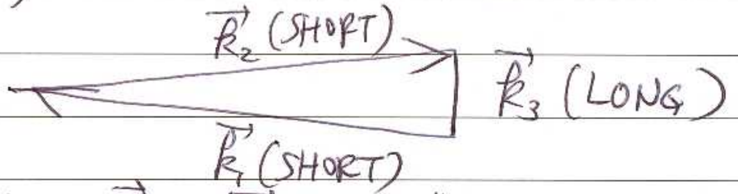
On the other hand, any foreseeable experiments cannot detect (this form of) non-Gaussianity unless $\frac{B_\zeta(k_1,k_1,0)}{P_\zeta(k_1)P_\zeta(k_3)} \gg 1$.

☆ How can we understand this theorem?

See Ganc & Komatsu, arXiv: 1006.5457, on a technical discussion of the proof.

The proof of the theorem as given by Creminelli & Zaldarriaga (2004).

Recall that the "squeezed triangle" correlates the very long-wavelength modes to short wave-length modes.



So, let's call them $\vec{k}_3 = \vec{k}_L$ (for "LONG") and $\vec{k}_1 = \vec{k}_2 = \vec{k}_S$ (for "SHORT"), respectively.

Then, we have:

$$\langle \zeta(\vec{k}_1)\zeta(\vec{k}_2)\zeta(\vec{k}_3)\rangle \xrightarrow[\text{squeezed}]{} \langle \zeta_S^2 \zeta_L \rangle$$

The theorem says:

$$\langle \zeta_S^2 \zeta_L \rangle = (1-n_S)\, P(k_S) P(k_L)$$

☆ The question is, "WHY ζ_S^2 ever cares about ζ_L?"

→ The theorem says that it doesn't, unless the fluctuations are exactly scale invariant, $n_S = 1$.

What does this tell us??

To analyze this problem, it is useful to decompose the calculation of $\langle \zeta_s^{o2} \zeta_L \rangle$ into 2 steps :

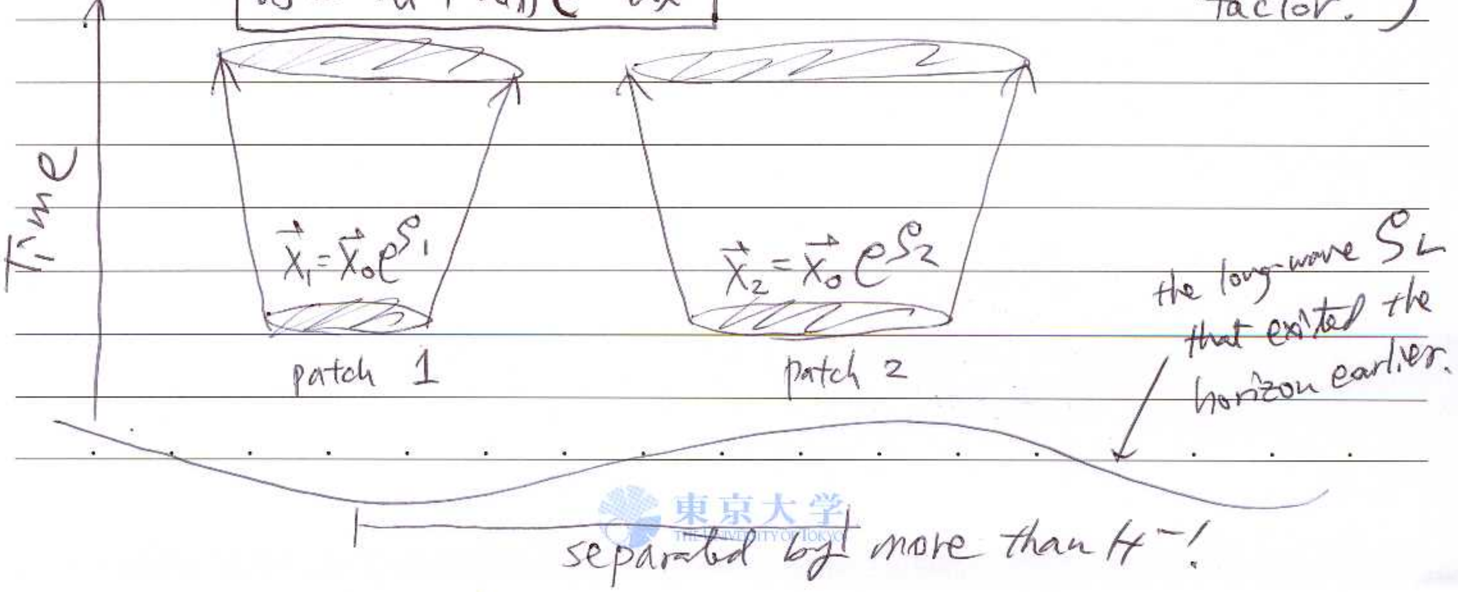
(i) Calculation of $\langle \zeta_s^{o2} \rangle$ in the presence of ζ_L
(ii) Calculation of $\langle \langle \zeta_s^{o2} \rangle_{\text{with } \zeta_L} \zeta_L \rangle$.

Basically, $\langle \zeta_s^{o2} \zeta_L \rangle \neq 0$ suggests that the power spectrum of ζ_s **somehow** gets modified when $n_s \neq 1$. How, and why?

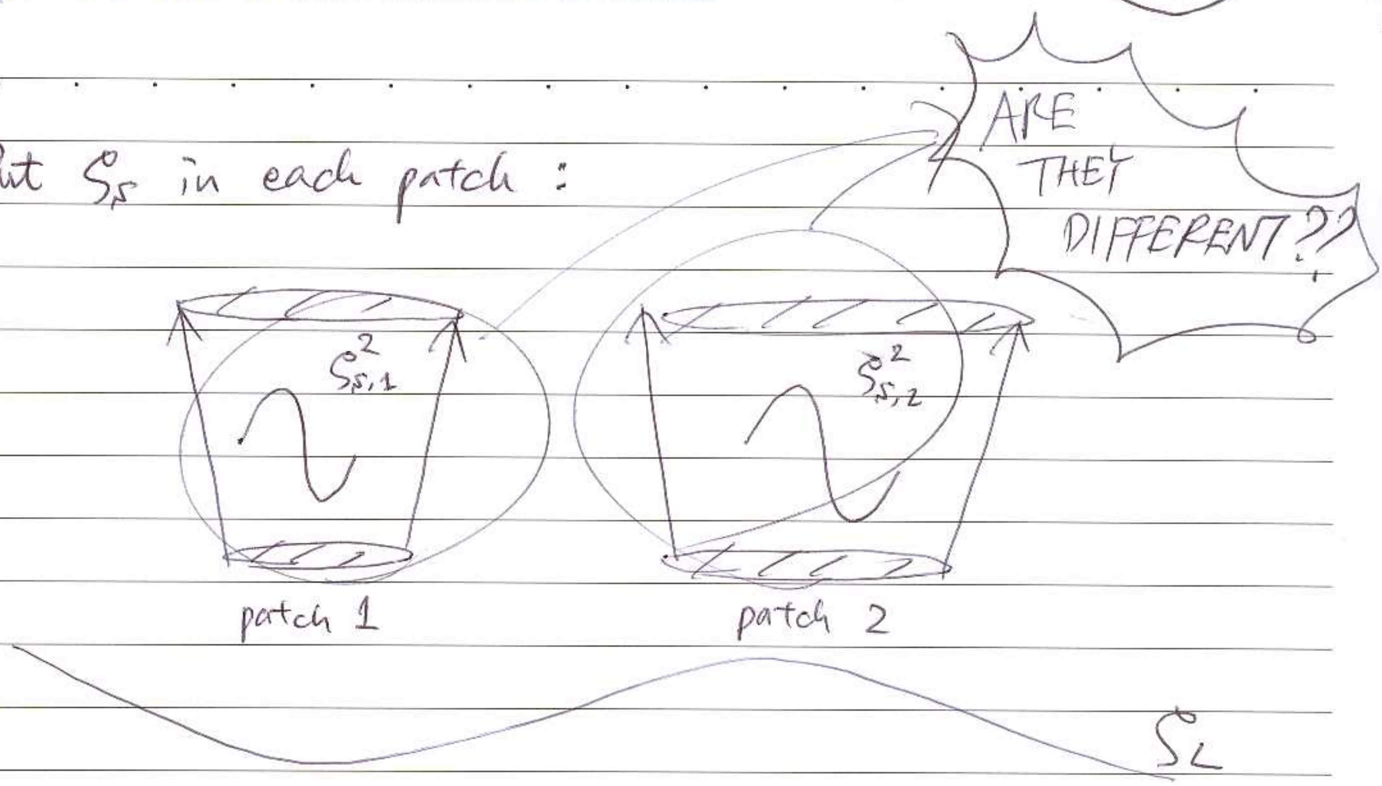
NOTE :
The formalism given in Ganc & Komatsu, arXiv: 1006.5457, provides the means to calculate $\langle \zeta_s^{o2} \rangle_{\text{with } \zeta_L^o}$ using the so-called "in-in formalism" of quantum fluctuations. This is a technical calculation, but provides more confidence in the following classical argument.

The reason : "ζ_L rescales the coordinates."
(or changes the expansion factor.)

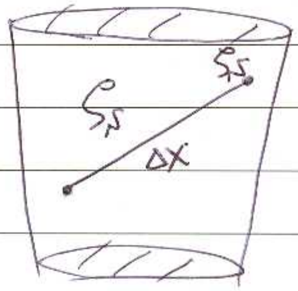
$$\boxed{ds^2 = -dt^2 + \bar{a}^2(t)\, e^{2\zeta}\, d\vec{x}^2}$$



Put ζ_s in each patch :



Let us calculate the real-space correlation function of ζ_s.



$$\langle \zeta_s^2 \rangle_{\zeta_L}(\Delta x) = \langle \zeta_s^2 \rangle_0 (\Delta x) + \zeta_L \times \left[\frac{d}{d\zeta_L} \langle \zeta_s^2 \rangle_{\zeta_L} \right] \Bigg|_{\zeta_L \to 0}$$

$\uparrow$ in the absence of ζ_L

__KEY__ $\quad \dfrac{d}{d\zeta_L} \longrightarrow \dfrac{d}{d\ln \Delta x}$ because $ds^2 = -dt^2 + a^2(t)\underline{e^{2\zeta} d\vec{x}^2}$

Thus, $$\langle \zeta_s^2 \rangle_{\zeta_L}(\Delta x) = \langle \zeta_s^2 \rangle_0 (\Delta x) + \zeta_L \left[\frac{d}{d\ln \Delta x} \langle \zeta_s^2 \rangle_{\zeta_L} \right] \Bigg|_{\zeta_L \to 0}$$

$$= \langle \zeta_s^2 \rangle_0 (\Delta x) + \underline{\underline{\zeta_L (1-n_s)}} \langle \zeta_s^2 \rangle_0 (\Delta x).$$

next page!!

We found $\langle \zeta_s^{o2} \rangle_{\zeta_L} = \langle \zeta_s^{o2} \rangle_0 + \zeta_L (1-n_s) \langle \zeta_s^{o2} \rangle_0$.

Now, correlate this with ζ_L!

$$\left\langle \langle \zeta_s^{o2} \rangle_{\zeta_L} \zeta_L \right\rangle = (1-n_s) \langle \zeta_L^{o2} \rangle \langle \zeta_s^{o2} \rangle_0 .$$

In Fourier space,

$$B_\zeta(k_s, k_s, k_L \to 0) = (1-n_s) P_\zeta(k_L) P_\zeta(k_s)$$

Q.E.D.

Notice that we did not have to assume any details about inflation, except to say that the existence of ζ_L modifies $\langle \zeta_s^{o2} \rangle$.

In fact, this is satisfied only for single-field:

for multiple fields, $\langle \zeta_s^{o2} \rangle$ can be modified by ζ_L as well as by other things.

That's why this is the theorem for **all** single-field models, and can be used to rule them out!!

The local (f_{NL}) model

It is useful to parametrize the primordial bispectrum in the following form:

"LOCAL FORM"

$$\boxed{B_\zeta^{local}(k_1,k_2,k_3) = \left(\frac{6}{5}f_{NL}^{local}\right)\left[P_\zeta(k_1)P_\zeta(k_2)+P_\zeta(k_2)P_\zeta(k_3)+P_\zeta(k_3)P_\zeta(k_1)\right]}$$

This is called the "local" form, as this bispectrum follows from

$$\zeta(\vec{x}) = \zeta_{Gaussian}(\vec{x}) + \frac{3}{5}f_{NL}\,\zeta^2_{Gaussian}(\vec{x})$$

(where $\zeta_{Gaussian}$ is a Gaussian random field)

That is to say, both sides of this equation are evaluated at the same spatial position; hence the term, "local."

However, it is **very important** remember that $\zeta = \zeta_G + \left(\frac{3}{5}f_{NL}\right)\zeta_G^2$ is **not** the only way to obtain the local-form bispectrum. We will come back to this later.

Now, since $P_\zeta(k) \propto 1/k^{n_s-4} \approx 1/k^3$, it is clear that the local form bispectrum is maximized when one of k's is very small. Indeed, it is maximized in the squeezed limit!!

$$B_\zeta^{local}(k_1,k_1,k_3\to 0) = \frac{12}{5}f_{NL}^{local}\,P_\zeta(k_1)P_\zeta(k_3)$$

Therefore, single-field models give $f_{NL}^{local} = \frac{5}{12}(1-n_s) \approx 0.013$ for $n_s = 0.968$

The current best limit (from CMB) is :

$$f_{NL}^{local} = 32 \pm 21 \quad (68\% \,CL)$$

or

$$-10 < f_{NL}^{local} < +74 \quad (95\% \,CL)$$

I.e., no evidence for $f_{NL}^{local} \neq 0$, and thus single-field models are consistent with the data. (E. Komatsu et al., arXiv:1001.4538)

The Planck satellite is expected to reduce the error bar by a factor of four, $\sigma f_{NL}^{local} \simeq 5$ (Komatsu & Spergel, PRD, 63, 063002 (2001))

Therefore, **if** the central value, $f_{NL}^{local} \simeq 30$, persists (it may not!), then $f_{NL}^{local} > 0$ would be detected at the **6-σ** level. We will see!!

... Who knows, they might have seen it already ...

☆ We will learn how to measure f_{NL}^{local} from the CMB data on the 3rd day.

2.2 Second-order non-Gaussianity

In terms of the curvature perturbation during the matter era, Φ, the temperature anisotropy on large scales (Sachs-Wolfe effect) is given by

$$\frac{\Delta T}{T} = -\frac{1}{3}\Phi$$

in the **linear order**. Moreover, Φ is related to the primordial curvature perturbation during inflation, ζ, by (on large scales)

$$\Phi = \frac{3}{5}\zeta$$

in the **linear** order.

However, the second-order effects (both in gravity and hydrodynamics) induce non-linear effects such that

$$\frac{\Delta T}{T} = -\frac{1}{3}\Phi + \mathcal{O}(\Phi^2)$$

and

$$\Phi = \frac{3}{5}\zeta + \mathcal{O}(\zeta^2)$$

These effects are expected to yield $f_{NL} \sim \mathcal{O}(1)$. But, are they in the local form?

Now, let us define f_{NL}^{local} using Φ during matter era:

$$\Phi = \Phi_{Gaussian} + f_{NL}^{local} \Phi_{Gaussian}^2$$

(This definition agrees with what we used for ζ.)

(i) Very large scales [Sachs-Wolfe Limit]

Dropping non-local terms, the second-order Sachs-Wolfe effect gives

$$1 + 4\frac{\Delta T}{T} = e^{-\frac{4}{3}\Phi^{(1)}}$$ non-perturbative expression.

$$\boxed{\frac{\Delta T}{T} = -\frac{1}{3}\left[\Phi^{(1)} - \frac{2}{3}\left(\Phi^{(1)}\right)^2\right]}$$

2nd order SW effect

where $\Phi^{(1)}$ is the 1st order (here assumed to be a Gaussian) perturbation during the matter era.

Refs: Bartolo, Matarrese & Riotto, JCAP, 0606, 024 (2006)
Boubekeur et al., JCAP, 0908, 029 (2009)
★ Pitrou, Uzan & Bernardeau, arXiv:1003.0481
Note that $\Phi^{(1)}$ in our notation is $-\Phi^{(1)}$ in these refs.
we followed this paper here.

Therefore, $f_{NL}^{local} = -\frac{2}{3}$ can be generated from the 2nd-order SW effect.

(Boubekeur et al. found $f_{NL}^{local} = -\frac{1}{6}$, due to a different definition of $\Delta T/T$. $\left(\frac{\Delta T}{T}\right)_{theirs} = e^{-\frac{1}{3}\Phi^{(1)}} - 1$
$= -\frac{1}{3}\left[\Phi^{(1)} - \frac{1}{6}\left(\Phi^{(1)}\right)^2\right]$)

(ii) Small scales [Newtonian Limit]

In the small scales, where the wavelength is deep inside the horizon, the second-order $\delta T/T$ is given by

$$\frac{\Delta T}{T} \simeq \left(\frac{\Delta T}{T}\right)^{(1)} + \frac{1}{2} R_* \Phi^{(2)}$$

where R_* is $3\rho_b/4\rho_r$ at $z_* = 1090$. For the standard value of $\Omega_b h^2 = 0.023$, $R_* = 0.67$.

in the matter era

In the sub-horizon limit, $\Phi^{(2)}$ can be solved analytically:

$$\frac{1}{2}\Phi^{(2)}(\vec{k}, \eta) = \frac{1}{6}\int \frac{d^3k'}{(2\pi)^3} d^3k'' \, \delta^D(\vec{k}' + \vec{k}'' - \vec{k}) \times \left(\frac{k'k''\eta}{k}\right)^2 F_2^{(s)}(\vec{k}', \vec{k}'') \Phi^{(1)}(\vec{k}') \Phi^{(1)}(\vec{k}'')$$

where η is the conformal time, and

$$F_2^{(s)}(\vec{k}_1, \vec{k}_2) \equiv \frac{5}{7} + \frac{\vec{k}_1 \cdot \vec{k}_2}{2k_1 k_2}\left(\frac{k_1}{k_2} + \frac{k_2}{k_1}\right) + \frac{2}{7}\left(\frac{\vec{k}_1 \cdot \vec{k}_2}{k_1 k_2}\right)^2.$$

NOTE!

Details are not important, but it is sufficient to note that $F_2^{(s)}$ vanishes in the squeezed limit, $\vec{k}_1 = -\vec{k}_2$.

Therefore, this effect cannot produce $f_{NL}^{local} \gtrsim 1$.

Refs:
Bartolo & Riotto, JCAP 03, 017 (2009)

(iii) Intermediate scales?

★ As we have seen, $f_{NL}^{local} > 1$ is not possible in both large (all wavelengths are super horizon at $z = 1090$) and small (all wave lengths are sub horizon at $z = 1090$) scales!!

Nevertheless, the numerical calculations done by Pitrou, Uzan & Bernardeau, arXiv:1003.0481 show $f_{NL}^{local} \sim 5$ (!)

$\boxed{\text{precisely where and how this comes about is still not completely clear, but it is very important to understand this.}}$

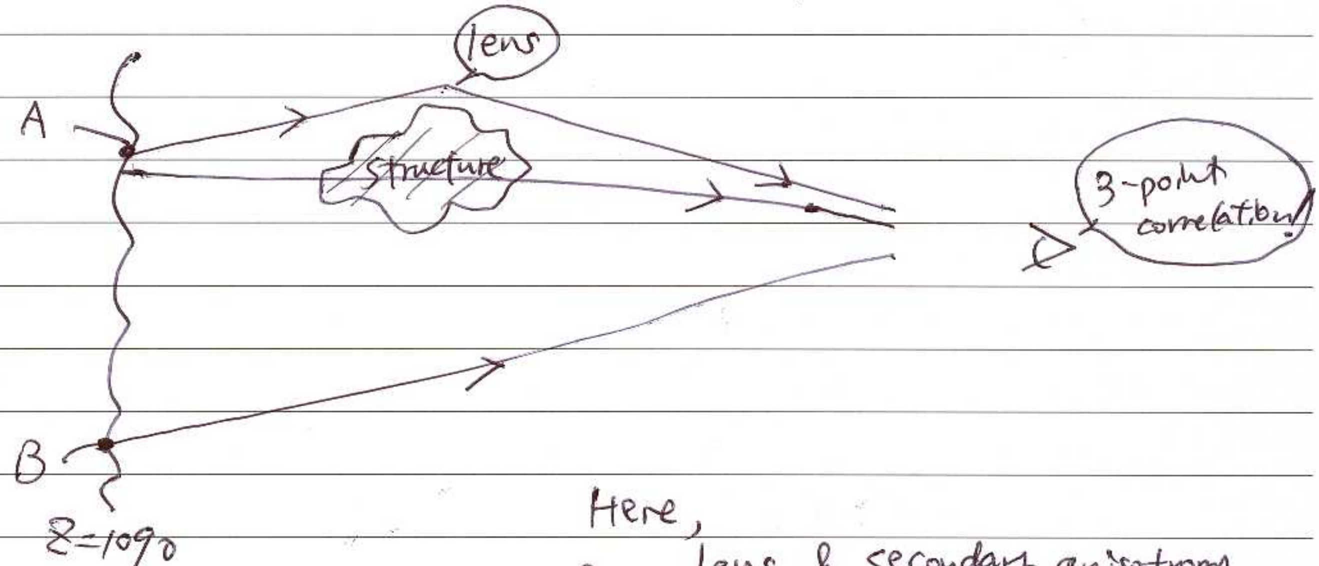
Cutting Edge!

2.3 Secondary non-Gaussianity

There are many secondary sources of microwave anisotropies:

- Integrated Sachs-Wolfe effect (ISW), caused by $\partial\Phi/\partial t$
- Sunyaev-Zel'dovich effect, caused by either inverse Compton scattering of hot electrons in clusters (thermal SZ) or the bulk motion of gas (kinetic SZ)
- Gravitational lensing

Of these, it turns out that the bispectrum containing the gravitational lensing gives a large signal in the squeezed triangles. Schematically:



Here,

- lens & secondary anisotropy caused by "structure" are correlated, and
- primary CMB of A & B are correlated.

$\Rightarrow$ non-zero 3-point correlation.

Calculation of the lens-secondary coupling

The observed CMB in the direction of $\hat{n}$ is given by

$$\Delta T(\hat{n}) = \Delta T^{\text{primary}}(\hat{n} + \vec{\partial}\phi) + \Delta T^{\text{secondary}}(\hat{n})$$

Here, $\Delta T^{\text{primary}}$ is the unlensed primary CMB, and $\Delta T^{\text{secondary}}(\hat{n})$ is the (yet to be specified) secondary anisotropy.

Moreover, $\phi(\hat{n})$ is the so-called "lensing potential" given by

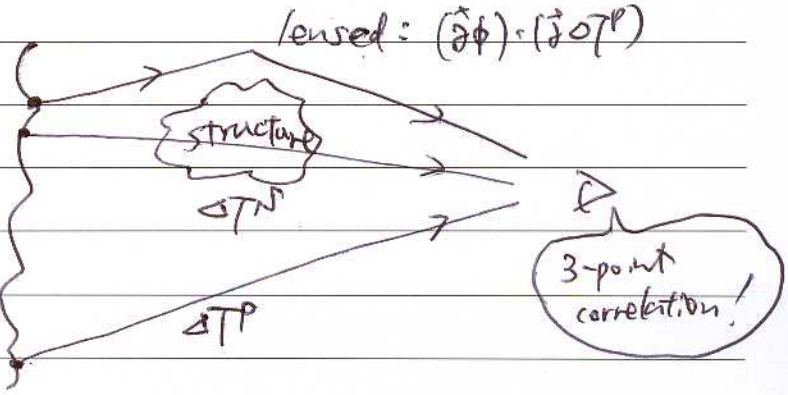
$$\phi(\hat{n}) = -2\int_0^{r_*} dr \, \frac{r_* - r}{r r_*} \Phi(r, \hat{n} r)$$

where r_* is the comoving distance out to $z = 1090$.

Expanding in $\vec{\partial}\phi$, we obtain

$$\boxed{\Delta T(\hat{n}) = \Delta T^{P}(\hat{n}) + [(\vec{\partial}\phi)\cdot(\vec{\partial}\Delta T^{P})](\hat{n}) + \Delta T^{S}(\hat{n})}$$

correlated! correlated!



☆ What would "structures" be?

- ISW effect
- thermal SZ effect
- kinetic SZ effect
- point sources, tracing the large scale structure
⋮

Of these, the lensing-ISW effect is found to be the most dominant contribution. The bispectrum is given by

$$b_{l_1 l_2 l_3}^{\text{lens-ISW}} = \frac{l_1(l_1+1) - l_2(l_2+1) + l_3(l_3+1)}{2} C_{l_1}^{P} C_{l_3}^{\phi\text{-ISW}} + (5\text{ perm})$$

Here, $C_{l_1}^{P}$ falls off like $1/l_1^2$, and $C_{l_3}^{\phi\text{-ISW}}$ falls off faster than $1/l_3^3$. Therefore, $b_{l_1 l_2 l_3}$ peaks in the squeezed limit, $\underline{l_3 \ll l_1 \simeq l_2}$.

☆ For the Planck sattelite the ISW-lens coupling is expected to give $\boxed{f_{NL}^{local} = 9.3}$, if not subtracted properly!!

Refs: Goldberg & Spergel, PRD, 59, 103002 (1999)
Verde & Spergel, PRD, 65, 043007 (2002)
Serra & Cooray, PRD, 77, 107305 (2008)
Hanson et al., PRD, 80, 083004 (2009)

3. Measuring non-Gaussianity from the cosmological data.

Follow Komatsu, arXiv: 1003.6097.

Also talk about the trispectrum, as well as multi-field, δN formalism & multi-field consistency condition, if time permits.

Multi-field Case

The locality demands that $\zeta(\vec{x})$ should be given by

$$\zeta(\vec{x}) = F\left[\delta\varphi_1(\vec{x}), \delta\varphi_2(\vec{x}), \cdots, \delta\varphi_N(\vec{x})\right]$$

<u>If</u> F is a smooth function [see 0903.3407 for non-smooth case],

we can expand this form and obtain:

$$\zeta(\vec{x}) = \frac{\partial F}{\partial \varphi_1}\delta\varphi_1(\vec{x}) + \frac{1}{2}\frac{\partial^2 F}{\partial \varphi_1^2}\delta\varphi_1^2(\vec{x}) + \cdots$$
$$+ \frac{\partial F}{\partial \varphi_2}\delta\varphi_2(\vec{x}) + \frac{1}{2}\frac{\partial^2 F}{\partial \varphi_2^2}\delta\varphi_2^2(\vec{x}) + \cdots$$
$$+ \cdots$$
$$+ \frac{\partial F}{\partial \varphi_N}\delta\varphi_N(\vec{x}) + \frac{1}{2}\frac{\partial^2 F}{\partial \varphi_N^2}\delta\varphi_N^2(\vec{x}) + \cdots$$

What determines F ??

δN Formalism

Salopek & Bond, PRD, 42, 3936 (1990)
Sasaki & Stewart, Prog. Theor. Phys. 95, 71 (1996) [astro-ph/9507001]
Lyth, Malik & Sasaki, JCAP, 05, 004 (2005) [astro-ph/0411220]

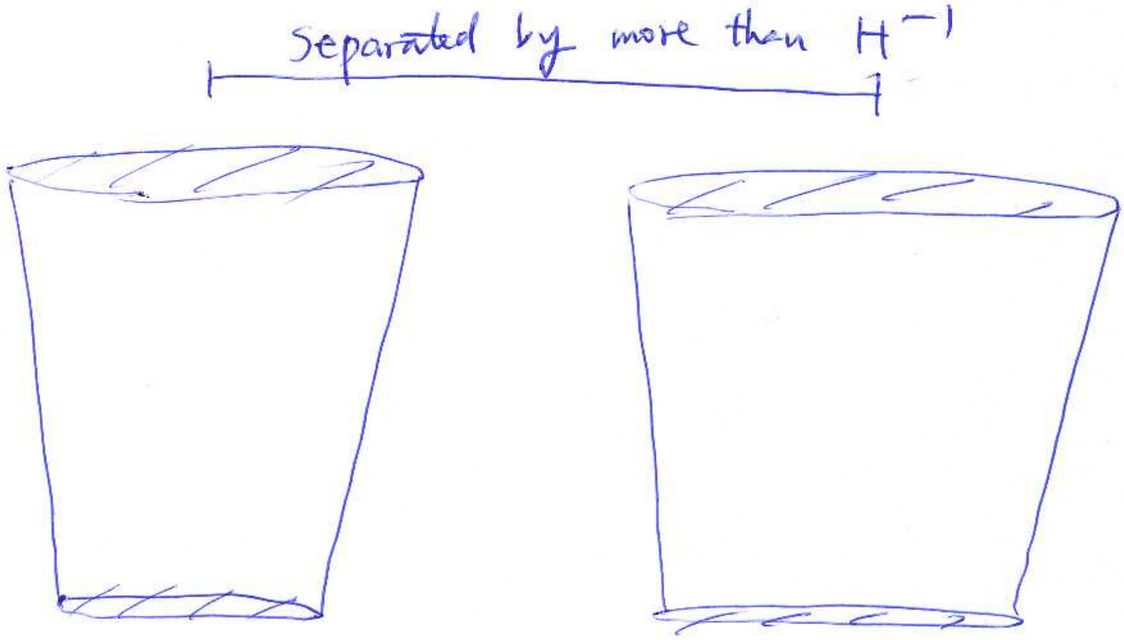
have shown :

$$F = N \; (\equiv \# \text{ of e-foldings of expansion})$$

$$= \int_{t_{\text{horizon-crossing}}}^{t_{\text{today}}} H \, dt$$

$$= \ln \left[\frac{a(t_{\text{today}})}{a(t_{\text{horizon-crossing}})} \right]$$

Why so? Let's go back to this picture:



$$ds^2 = -dt^2 + a^2(t)\, e^{2\zeta(\vec{x})} (d\vec{x})^2$$

$$= -dt^2 + [\tilde{a}(\vec{x},t)]^2 (d\vec{x})^2$$

where $\underline{\tilde{a}(\vec{x},t) = a(t)\, e^{\zeta(\vec{x})}}$

Therefore, we can interpret $\boxed{\ln \tilde{a}}$ as the curvature perturbation:

$$\boxed{\ln \tilde{a} = \zeta + \ln a(t)}$$

Then, only thing we have to care about is "How much has each horizon patch expanded relative to the others?"

※ Single-field example

For single-field,

$$N = \int_{H.C.}^{NOW} H\,dt = \int_{\phi_{H.C.}}^{\phi_{NOW}} H \frac{dt}{d\phi}\,d\phi$$

$$= \int_{\bar{\phi}_{H.C.}+\delta\phi_{H.C.}}^{\bar{\phi}_{NOW}} \frac{H}{\dot{\phi}}\,d\phi$$

$$\simeq \underbrace{\int_{\bar{\phi}_{H.C.}}^{\bar{\phi}_{NOW}} \frac{H}{\dot{\phi}}\,d\phi}_{\bar{N}} + \left.\frac{H}{\dot{\phi}}\right|_{\phi=\bar{\phi}_{H.C.}} \delta\phi_{H.C.}$$

$$\therefore\ \delta N = \left.\frac{H}{\dot{\phi}}\right|_{\phi=\bar{\phi}_{H.C.}} \delta\phi_{H.C.} \Rightarrow \zeta = \left.\frac{H}{\dot{\phi}}\right|_{\phi=\bar{\phi}_{H.C.}} \delta\phi_{H.C.}$$

⇒ This is the famous result for ζ, obtained by

- Guth & Pi, PRL, 49, 1110 (1982)
- Hawking, PLB, 115, 295 (1982)
- Starobinsky, PLB, 117, 175 (1982)
- Bardeen, Steinhardt & Turner, PRD, 28, 679 (1983)

※ Multi-field generalization

Lyth & Rodriguez, PRL, 95, 121302 (2005) [astro-ph/0504045]

$$\zeta = N(\bar{\varphi}_1 + \delta\varphi_1, \bar{\varphi}_2 + \delta\varphi_2, + \cdots, \bar{\varphi}_N + \delta\varphi_N) - N(\bar{\varphi}_1, \bar{\varphi}_2, \cdots, \bar{\varphi}_N)$$

$$\simeq \sum_i \frac{\partial N}{\partial \varphi_i}\delta\varphi_i + \frac{1}{2}\sum_{ij}\frac{\partial^2 N}{\partial\varphi_i \partial\varphi_j}\delta\varphi_i\delta\varphi_j + \cdots$$

Now, let's remind us of the fact that :

$$\langle \delta\varphi(\vec{k})\,\delta\varphi^*(\vec{k}')\rangle = (2\pi)^3\delta(\vec{k}-\vec{k}')\,P_{\delta\varphi}(k)$$

where

$$P_{\delta\varphi}(k) = \left(\frac{H}{2\pi}\right)^2 \frac{2\pi^2}{k^3}$$

for a scale-invariant spectrum.

So - - -

For uncorrelated φ_i, i.e., $\langle \delta\varphi_i \delta\varphi_j \rangle \propto \delta_{ij}$,

$$P_\zeta(k) = \left(\frac{H}{2\pi}\right)^2 \frac{2\pi^2}{k^3}\left[\sum_i \left(\frac{\partial N}{\partial \varphi_i}\right)^2\right] + \cdots$$

and the bispectrum is :

$$B_\zeta(k_1,k_2,k_3) = \left(\frac{H}{2\pi}\right)^4 \left[\sum_{ij}\left(\frac{\partial N}{\partial \varphi_i}\right)\left(\frac{\partial N}{\partial \varphi_j}\right)\left(\frac{\partial^2 N}{\partial \varphi_i \partial \varphi_j}\right)\right] \times (2\pi^2)^2 \left[\frac{1}{k_1^3 k_2^3} + \frac{1}{k_2^3 k_3^3} + \frac{1}{k_3^3 k_1^3}\right] + \cdots$$

Therefore, REMARKABLY, we recover the local-form bispectrum :

$$B_\zeta(k_1,k_2,k_3) = \frac{\sum_{ij}\left(\frac{\partial N}{\partial \varphi_i}\right)\left(\frac{\partial N}{\partial \varphi_j}\right)\left(\frac{\partial^2 N}{\partial \varphi_i \partial \varphi_j}\right)}{\left[\sum_i \left(\partial N/\partial \varphi_i\right)^2\right]^2}\left[P_\zeta(k_1)P_\zeta(k_2) + \text{cyclic}\right]$$

$$= \frac{6}{5} f_{NL}.$$

$f_{NL} \gg 1$ is possible for many models !!