Exercise sheet 12

Exercise 12 - 1

Consider a real-valued signal field s with a Gaussian prior,

$$\mathcal{P}(s) = \mathcal{G}(s, S), \tag{1}$$

that is observed with an instrument that exhibits an almost linear response,

$$d = R\left(s + rs^2\right) + n. \tag{2}$$

Here, R is a linear operator, $r \in \mathbb{R}$ with $|r| \ll 1$ is a small parameter that determines the strength of the nonlinearity in the instrumental response, s^2 denotes the local squaring of the signal field, i.e., $\left(s^2\right)_x = \left(s_x\right)^2$, and n is additive Gaussian noise, i.e.,

$$\mathcal{P}(n) = \mathcal{G}(n, N). \tag{3}$$

a) Consider first the case of an exactly linear response, i.e., $r = 0$. Derive the Hamiltonian

$$H(d, s) = -\log\left(\mathcal{P}(d, s)\right) \tag{4}$$

for this problem. You may drop all terms that do not depend on s (1 point) .

b) Show that the posterior probability density in the case with $r = 0$ is of Gaussian form, i.e., $\mathcal{P}(s|d) = \mathcal{G}(s - m_0, D)$, and derive expressions for its mean and covariance,

$$m_0 = \langle s \rangle_{\mathcal{P}(s|d)} \qquad \text{and} \qquad D = \left\langle (s - m_0)(s - m_0)^\dagger \right\rangle_{\mathcal{P}(s|d)}, \tag{5}$$

as a function of d, S, N, and R (2 points).

c) Now consider the case with small but non-zero r. Calculate the Hamiltonian in this case and write it in the form

$$H(s, d) = H_0 - j^\dagger s + \frac{1}{2} s^\dagger D^{-1} s + \sum_{k=2}^{\infty} \frac{1}{k!} \Lambda^{(k)}_{x_1 x_2 \cdots x_k} s_{x_1} s_{x_2} \cdots s_{x_k}, \tag{6}$$

where only the coefficients $\Lambda^{(k)}$ depend on r and we use the convention that repeated indices are integrated over. Give expressions for j, D, and all non-zero $\Lambda^{(k)}$. You do not need to calculate H_0 (3 points).

Exercise 12 - 2

This exercise is a continuation of exercise $12 - 1$. Given is a Hamiltonian of the form

$$H(s, d) = H_0 - j^\dagger s + \frac{1}{2} s^\dagger D^{-1} s + \sum_{k=2}^{4} \frac{1}{k!} \Lambda^{(k)}_{x_1 x_2 \cdots x_k} s_{x_1} s_{x_2} \cdots s_{x_k}, \tag{7}$$

where $\Lambda^{(2)} \propto r$, $\Lambda^{(3)} \propto r$ and $\Lambda^{(4)} \propto r^2$. Here r is assumed to be a very small parameter $|r| \ll 1$.

d) Write down the diagrammatic expansion of the partition function $\log\left(Z(d)\right)$ up to linear order in r. (1 point)

e) Find the diagrammatic expressions for the posterior mean and covariance,

$$m_r = \langle s \rangle_{\mathcal{P}(s|d)} \qquad \text{and} \qquad \left\langle (s - m_r)(s - m_r)^\dagger \right\rangle_{\mathcal{P}(s|d)}, \tag{8}$$

up to first order in r. (1 point)

Exercise 12 - 3

Non-Gaussianities in the Cosmic Microwave Background

The CMB is the relic radiation from the time of (re)combination 380 000 years after the Big Bang. Its variations in temperature with direction reflect the variations in the gravitational potential φ at this time. The simplest inflationary scenarios predict these variations to be a nearly Gaussian field. Measuring deviations from Gaussianity in this field is within the focus of contemporary research in cosmology.

The *local type* of non-Gaussianities can be modeled according to

$$\varphi = \phi + f_{\mathrm{nl}} \left(\phi^2 - \left\langle \phi^2 \right\rangle_{(\phi)} \right) = \phi + f_{\mathrm{nl}} \left(\phi^2 - \hat{\Phi} \right), \tag{9}$$

where

$$\phi \hookleftarrow \mathcal{G}(\phi, \Phi) \tag{10}$$

is an auxilliary Gaussian field and the degree of non-Gaussianity of the primordial gravitational potential φ is quantified by the parameter f_{nl}, which shall be assumed here to be position-independent (i.e. one number).

During cosmic recombination, the variations in the gravitational field are transformed into temperature variations in the electromagnetic radiation. This radiation has been measured in recent times by satellites, balloons, and ground based observatories. These two processes (imprinting onto temperature variations and measurement) can be combined in a linear response operator R. Additionaly, these measurements contain a Gaussian noise contribution, which is not correllated with the signal, i.e.

$$d = R\varphi + n, \ n \hookleftarrow \mathcal{G}(n, N). \tag{11}$$

a) Derive an expression for $H\left[d, \phi, f_{\mathrm{nl}}\right] = -\log\left(\mathcal{P}(d, \phi, f_{\mathrm{nl}})\right)$, assuming a flat prior for f_{nl}.
(2 points)

Assume now uncorrelated and homogeneous noise, i.e.

$$N_{ij} = \delta_{ij}\sigma^2, \tag{12}$$

and unit response,

$$R_{ij} = \delta_{ij}. \tag{13}$$

b) Bring this Hamiltonian into the form

$$H\left[d, \phi | f_{\mathrm{nl}}\right] = H_0 - j^\dagger \phi + \frac{1}{2}\phi^\dagger D^{-1}\phi + \frac{1}{3!}\lambda^{(3)\dagger}\phi^3 + \frac{1}{4!}\lambda^{(4)\dagger}\phi^4 \tag{14}$$

and identify the terms j, D, $\lambda^{(3)}$, and $\lambda^{(4)}$, as well as the f_{nl}-dependent part of H_0.
(3 points)

c) Write down the diagrammatic expansion for the logarithm of the partition function

$$\log Z_{f_{\mathrm{nl}}}(d) = \log \mathcal{P}(d|f_{\mathrm{nl}}) = \log \int \mathcal{D}\phi \ \mathcal{P}(d, \phi | f_{\mathrm{nl}}), \tag{15}$$

up to second order in f_{nl}, i.e., use all diagrams that contain terms of lower than third order. You do not need to formulate the results algebraically, nor do you need to care about terms that are constant in ϕ and f_{nl}.
(2 points)

d) Write down the diagrammatic version of the expectation value $\langle\phi\rangle_{\mathcal{P}(\phi|d,f_{\mathrm{nl}})}$ up to first order in f_{nl}. Read off the corresponding algebraic formula for the expectation value in terms of j, D, $\lambda^{(3)}$, and $\lambda^{(4)}$.
(2 points)

e) Write down the diagrammatic version of the dispersion $\left\langle \left(\phi - \langle\phi\rangle_{\mathcal{P}(\phi|d,f_{\mathrm{nl}})}\right)\left(\phi - \langle\phi\rangle_{\mathcal{P}(\phi|d,f_{\mathrm{nl}})}\right)^{\dagger}\right\rangle_{\mathcal{P}(\phi|d,f_{\mathrm{nl}})}$ up to first order in f_{nl}. Read off the corresponding algebraic formula for the dispersion in terms of j, D, $\lambda^{(3)}$, and $\lambda^{(4)}$.
(2 points)

This exercise sheet will be discussed during the exercises.
Please hand in your solution by Monday, July 12th 2021, noon online:
https://moodle.lmu.de/course/view.php?id=14933