Exercise sheet 10

Exercise 10 - 1

Let $s : \mathbb{R} \to \mathbb{R}$ be a continuous process that follows the following stochastic differential equation: $\mathrm{d}s_t/\mathrm{d}t = a\,\xi_t - b\,s_t$. Here, a and b are non-negative constants and ξ is a stationary, white random field of unit variance, i.e., $\langle \xi_t \xi_r \rangle_{(\xi)} = \delta(t-r)$. Let $S = \langle s s^\dagger \rangle_{(s)}$ be the signal covariance.

a) Assume for the moment $s_0 = 0$, $b = 0$, and $t, t' \geq 0$. Calculate $S_{tt'}$. Use S_{tt} to argue why this so-called Wiener process is a frequently used model for diffusive motion of a particle (2 points).

b) Calculate the signal power spectrum for any a and b being non-negative. Try to explain with words why the spectral normalization and the appearing characteristic frequency depend on a and b the way they do (2 points).

Hint: Transform the differential equation to Fourier space.

Exercise 10 - 2

Consider a field $\varphi \equiv \varphi_{x,t}$ with a time domain and a one-dimensional spatial domain, following the stochastic differential equation

$$\partial_t \varphi = \kappa \Delta_x \varphi + \xi \tag{1}$$

with independent Gaussian noise contribution ξ of unit variance and constant κ.

a) Calculate the auto-correlation $\langle \varphi^*_{(\omega,k)} \varphi_{(\omega',k')} \rangle$ in its full harmonic domain (temporal and spatial Fourier basis) (2 points).

b) Perform the inverse Fourier transformation in the time domain and give the expression of this auto-correlation in time - spatial frequency domain $\langle \varphi^*_{(t,k)} \varphi_{(t',k')} \rangle$ using the residue theorem. (2 points)

Exercise 10 - 3

Be s_t a stock price as a function of time t, which is believed to follow the Black-Scholes model

$$\frac{1}{s_t} \frac{\mathrm{d}s_t}{\mathrm{d}t} = \mu + \sigma\,\xi_t \tag{2}$$

with known and temporary constant drift rate μ and volatility σ. The stochastic field ξ is a white noise Gaussian random field with unit dispersion.

a) Find a variable $x_t = x(s_t)$ which follows a drift free Wiener process ($\dot{x}_t = \sigma\,\xi_t$). Give its probability distribution $\mathcal{P}(x)$ (1 point).

b) Use $\mathcal{P}(x)$ to calculate the expected (actually the average) future stock price $\bar{s}_t = \langle s_t \rangle_{(s_t|s_{t_0})}$. Why does this depend on σ^2 (2 points)?

c) At time $t_0 = 0$ a bank has given the guarantee to a customer (for a fee, of course) that she can buy one of these stocks at a specified later time t for at most the fixed price K, independent of its actual price s_t then. Calculate the expected (average) amount of money L the bank will need to invest at "stroke" time t to fulfill its obligation from this "European call option" (2 points).

d) Calculate now the minimal price C the bank has to ask for this call option in order not to loose any money on average. Consider that the bank has access to a risk-free interest rate r, which amplifies any amont of money C received at time $t_0 = 0$ to $C\,\mathrm{e}^{r\,t}$ at a later time t. Find the most compact notation for this Black-Scholes option pricing formula by (i) using the property $1 - \mathrm{erf}(x) = \mathrm{erf}(-x)$ for $\mathrm{erf}(x) = \int_{-\infty}^{x} \mathcal{G}(x, 1)$ and (ii) expressing the current stock price through $S = \bar{s}_t\,\mathrm{e}^{-r\,t}$, the expected future value of the stock for the bank at present time (a good brain twister, isn't it?). Compare your result to the famous Black-Scholes formula

$$C = S\,\mathrm{erf}(d_1) - K\,\mathrm{e}^{-rt}\,\mathrm{erf}(d_2). \tag{3}$$

Argue why in the Black-Scholes world $S = s_{t_0}$ is assumed (2 points).

e) **[Only for future stock traders]** Calculate the so-called "Greeks" (e.g., see Wikipedia), the derivatives of the option price C with respect to the various model parameters (μ, σ^2, t, r, K, and S).

Exercise 10 - 4

You are observing a region of the sky with a photodetector. On average a number of photons λ makes it's way from the sky into your detector per time interval.

a) What is the PDF for the photon counts per time interval (1 Point)?

b) Calculate the expectation value and the standard deviation for the photon counts per time interval (3 Points).

c) Use the PDF to calculate the expectation value $\left\langle \frac{n!}{(n-q)!}\theta(n-q) \right\rangle_{(n|\lambda)}$ with

$$\theta(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & \text{else} \end{cases} \tag{4}$$

for any $q \in \{0, 1, \ldots, n\}$, where n denotes the photon counts per time interval (2 Points).

This exercise sheet will be discussed during the exercises.
Group 01, Wednesday 16:00 - 18:00, online,
Group 02, Thursday, 10:00 - 12:00, online,
Group 03, Thursday, 16:00 - 18:00, online,
Please hand in your solution by Monday, June 28th 2021, noon online:
https://moodle.lmu.de/course/view.php?id=14933