Information Field Theory
Summer term 2021 *PD Dr. Torsten Enßlin*

Exercise sheet 9

Exercise 9 - 1

Assume that you are measuring a field ψ with symmetric statistics, i.e.

$$\mathcal{P}(\psi) = \mathcal{P}(-\psi)\ \forall \psi, \tag{1}$$

with a perfect instrument, i.e.

$$d = \psi. \tag{2}$$

You are interested in the power of the field, i.e.

$$s = \psi^2. \tag{3}$$

a) Calculate the signal response and the noise using the definition

$$R(s) = \langle d \rangle_{(d|s=\psi_0^2)} \tag{4}$$

of the signal response (1 point).

b) Do the same for a new data set d' that is the square of the old data set, $d' = d^2$ (1 point).

Exercise 9 - 2

A signal is observed by an instrument with Gaussian point spread function (PSF), so that $d = R\,s + n$ with $R_{xy} = \mathcal{G}(x-y, B)$. Assume $B_{ij} = l_i^2\,\delta_{ij}$ and find the Fourier transformed PSF (2 points).
Hint: Express R in terms of $r := x - y$.

Exercise 9 - 3

Consider a random field s that is statistically homogeneous and isotropic,

$$\langle s(x)s(y)\rangle_{\mathcal{P}(s)} = C(|x-y|), \quad \text{with } x \in \mathbb{R}. \tag{5}$$

a) Show that the Fourier transformed autocorrelation function has the form

$$\langle s(k)s(q)^*\rangle_{\mathcal{P}(s)} = (2\pi)\delta(k-q)\, f(|k|) \tag{6}$$

(note that f depends only on the absolute of k). (2 points)

b) Show that if s follows Gaussian statistics all Fourier components are independent, i.e. show that

$$\mathcal{P}(s(k)|s(q)) = \mathcal{P}(s(k)) \quad \text{for} \quad k \neq q. \tag{7}$$

(3 points)

Hint: You are allowed to drop normalization factors and to be sloppy in case you discretize integrals.

Exercise 9 - 4

Given a field $s : \mathcal{S}^2 \to \mathbb{C}$ on the two-dimensional sphere, assume that it is statistically homogeneous and isotropic, i.e., $S(\hat{n}, \hat{n}') = \langle s(\hat{n}) s^*(\hat{n}') \rangle = S(\hat{n} \cdot \hat{n}')$, where $\hat{n}$ and $\hat{n}'$ are unit vectors that give directions or, equivalently, points on $\mathcal{S}^2$. Prove that the covariance matrix S is diagonal in the basis given by the spherical harmonic functions and its entries are independent of m, i.e.,

$$S_{(\ell m)(\ell' m')} := \langle s_{\ell m} s^*_{\ell' m'} \rangle = \delta_{\ell \ell'} \delta_{m m'} C_\ell. \tag{8}$$

(3 points)
<u>Hint</u>: Use the following properties of the spherical harmonic functions $Y_{\ell m}$ and the Legendre polynomials P_ℓ:

$$s(\hat{n}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} s_{\ell m} Y_{\ell m}(\hat{n}), \qquad s_{\ell m} = \int_{\mathcal{S}^2} \mathrm{d}\Omega\; s(\hat{n}) Y^*_{\ell m}(\hat{n}) \tag{9}$$

$$\int_{\mathcal{S}^2} \mathrm{d}\Omega\; Y^*_{\ell m}(\hat{n}) Y_{\ell' m'}(\hat{n}) = \delta_{\ell \ell'} \delta_{m m'} \tag{10}$$

$$P_\ell(\hat{n} \cdot \hat{n}') = \frac{4\pi}{2\ell + 1} \sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\hat{n}) Y_{\ell m}(\hat{n}') \tag{11}$$

Exercise 9 - 5

Imaging devices can probe continuous fields, such as physical flux or matter densities, by pixel averaged measurements. Assume the value d_i in the ith pixel of the obtained image satisfies

$$d_i = \int_\Omega \mathrm{d}x\; R_i(x) s(x) + n_i \qquad \text{with} \qquad i \in I = \{1, \dots, u\}, \tag{12}$$

where $R : I \times \Omega \to \mathbb{R}$ denotes the instrument response function, $s : \Omega \to \mathbb{R}$ the signal field, and $n \in \mathbb{R}^u$ the noise vector. Both, signal and noise, are *a priori* assumed to follow Gaussian distributions,

$$s \curvearrowleft \mathcal{G}(s - t, S) \qquad \text{and} \qquad n \curvearrowleft \mathcal{G}(n - r, N), \tag{13}$$

with kown non-zero means, $t : \Omega \to \mathbb{R}$ and $r \in \mathbb{R}^u$, as well as kown covariances, $S : \Omega \times \Omega \to \mathbb{R}$ and $N \in \mathcal{M}_{u \times u}(\mathbb{R})$.

a) Derive an expression for the likelihood, $P(d|s)$. Which quantity needs to be marginalized over? (1 point)

b) Compute the full information Hamiltonian (including constant terms); i.e., $H(d, s) = -\log P(d, s)$. Identify the information propagator, D and the information source, j. (2 points)

c) Derive an expression for the posterior mean field, $m = \langle s \rangle_{(s|d)}$, in terms of the given image, d. To do so use the maximum a posterior Ansatz. (2 points)

d) Say the field $\widetilde{m}$ was inferred from a modified data set $\widetilde{d}$ applying a Wiener filter that solely uses S, R, and N. Given the following relation between the image data d and $\widetilde{d}$,

$$\widetilde{d}_i = d_i - \int_\Omega \mathrm{d}x\; R_i(x) t(x) - r_i; \tag{14}$$

Find the relation between the posterior mean field m derived in **c)** and the field $\widetilde{m}$. (2 points)

This exercise sheet will be discussed during the exercises.
Group 01, Wednesday 16:00 - 18:00, online,
Group 02, Thursday, 10:00 - 12:00, online,
Group 03, Thursday, 16:00 - 18:00, online,
Please hand in your solution by Monday, June 21st 2021, noon online:
https://moodle.lmu.de/course/view.php?id=14933