Exercise sheet 8

Exercise 8 - 1

Imagine a bar of length 1, which is broken into two pieces at a random position along its length.

a) What is the average length of the smaller of the two pieces? (2 points)

b) Calculate the average ratio of the length of the smaller to the lenght of the bigger piece. (2 points)

c) Calculate the average ratio of the length of the bigger to the one of the smaller piece. (2 points)

Exercise 8 - 2

Assume that a quantity y is linearly dependent on a quantity x, i.e., $y(x) = a + bx$. Assume further that the quantity y has been measured at $m-1$ different positions $(x_i)_i,\ i = 1, \ldots, m-1$, subject to additive uncorrelated Gaussian noise, i.e.,

$$d_i = y(x_i) + n_i, \quad n \hookleftarrow \mathcal{G}(n, N), \quad N_{ij} = \delta_{ij}\sigma_i^2. \tag{1}$$

Assuming a Gaussian prior for the parameters a and b, i.e.,

$$s = \begin{pmatrix} a \\ b \end{pmatrix} \hookleftarrow \mathcal{G}(s, S), \quad S = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \tag{2}$$

a linear fit can be performed using Wiener filter theory.

You have enough money left to finance one additional measurement with uncertainty σ_m. How should you choose the position x_m for that measurement to gain optimal knowledge about the parameter a?

Hint: Use the quadratic loss function $\mathcal{L}(s, x_m) = (a - m_a)^2$ and the formalism of risk minimization. Here, m_a is the Wiener filter estimate after the m measurements (4 points).

Exercise 8 - 3

The goal of this exercise is to evaluate the integral

$$I(x) = \int_{-\infty}^{\infty} \mathrm{d}k \frac{e^{-ikx}}{k^2 + m^2} \tag{3}$$

a) For $k \in \mathbb{C}$, the integrand can diverge. Identify the poles of the integrand and sketch them in the complex plane, sketch the integration path of $k \in (-\infty, \infty)$ as well (2 points).

b) Draw a contour line that closes the integral, connecting $k = \infty$ to $k = -\infty$. The integral of the added contour line should be 0. Illustrate why it is 0 by comparing orders in k (2 points).

Hint: You may assume $x > 0$.

c) The closed integral encloses one pole. Identify its order and evaluate the integral $I(x)$ using the residue theorem,

$$\int_\Gamma f = (2\pi\, i) \sum_{a \in R_f} \mathrm{ind}_\Gamma(a)\ \mathrm{Res}_a f, \tag{4}$$

where Γ is a closed contour, R_f are the poles of f, and $\mathrm{ind}_\Gamma(a)$ is the number of counter-clockwise windings of Γ around a, and the residue formula,

$$\mathrm{Res}_a f = \frac{1}{(n-1)!} \lim_{z \to a} \frac{\partial^{n-1}}{\partial z^{n-1}} \left[(z-a)^n f(z) \right], \tag{5}$$

where n is the order of the pole a (2 points).

Hint: A clockwise winding is a negative counter-clockwise winding.

This exercise sheet will be discussed during the exercises.
Group 01, Wednesday 16:00 - 18:00, online,
Group 02, Thursday, 10:00 - 12:00, online,
Group 03, Thursday, 16:00 - 18:00, online,
Please hand in your solution by Monday, June 14th 2021, noon online:
https://moodle.lmu.de/course/view.php?id=14933