Exercise sheet 6

Exercise 6 - 1

Let $P(s) = \mathcal{G}(s, S)$ with $s = (s_1, ..., s_n)^t$ be a real multivariate zero-centered Gaussian with Covariance $\langle ss^t \rangle = S$. We would like to fit another Gaussian distribution $P'(s) = \mathcal{G}(s, S')$ to it that has a diagonal covariance matrix $S'_{ij} = \delta_{ij}\sigma_i$. Here δ_{ij} denotes the Kronecker delta.

a) What is the optimal approximating Gaussian $P'(s)$ to $P(s)$, as parameterized by σ, obtained through minimizing the loss

$$\sigma = \text{arg min}_\sigma \text{KL}(P(s), P'(s)) \ ?$$

(2 points)

b) What is the least updating fit of $P'(s)$ to $P(s)$, as obtained through

$$\sigma = \text{arg min}_\sigma \text{KL}(P'(s), P(s)) \ ?$$

(2 points)

c) Let

$$S = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0.2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} .$$

Visualize the Gaussian $\mathcal{G}(s, S)$ as well as the two different ways to fit a diagonal Gaussian to it introduced above. To visualize them, use a computer plotting samples of each of the three distributions. You can draw a sample from a multivariate Gaussian distribution by applying the square root of its covariance matrix to a white noise sample. (optional)

Exercise 6 - 2

The numbers quantifying the degree of industrialization $\tilde{\iota}$ of a society, its fertility rate $\tilde{f}$, and the stork population $\tilde{s}$ on its territory are random variables assumed to belong to a joint three-dimensional Gaussian distribution. Consider the fluctuations around the respective mean values $\iota = \tilde{\iota} - \langle \tilde{\iota} \rangle_{(\tilde{\iota})}$, $f = \tilde{f} - \langle \tilde{f} \rangle_{(\tilde{f})}$, and $s = \tilde{s} - \langle \tilde{s} \rangle_{(\tilde{s})}$. It is known that both the stork index s and the fertility index f are anticorrelated with the degree of industrialization. The normalized correlation coefficients are $c_{s\iota} = -0.85$ and $c_{f\iota} = -0.70$, where

$$c_{ab} = \frac{\langle ab \rangle_{(a,b)}}{\sqrt{\langle aa \rangle_{(a)} \langle bb \rangle_{(b)}}}. \tag{1}$$

Assume further that there is no direct correlation between f and s, i.e., $\mathcal{P}(s|f, \iota) = \mathcal{P}(s|\iota)$ and $\mathcal{P}(f|s, \iota) = \mathcal{P}(f|\iota)$. Derive an expression for $\langle sf \rangle_{(s,f,\iota)}$. Use this to calculate the normalized correlation coefficient c_{sf} (2 points).

Exercise 6 - 3

Assume the measurement of a signal s, which yields the data d, leads to

$$\mathcal{P}(s) = \mathcal{G}(s, S)$$
$$\mathcal{P}(s|d) = \mathcal{G}(s - m, D).$$

a) Calculate the amount of information in terms of entropy one gains via the measurement (3 points).

b) Now assume that your signal prior in the above measurement was flat. How much information in terms of entropy does one gain via the measurement? Explain the result (1 point).

Exercise 6 - 4

Consider a zero mean Gaussian random field s^x, $x \in \mathbb{R}$, with Covariance S.

a) Given a single, noisy measurement of s at location $x = 0$ of the form:

$$d = s^0 + n = \int \mathrm{d}x \; \delta(0 - x) \; s^x + n \equiv Rs + n, \quad n \hookleftarrow \mathcal{G}(n, \sigma^2) \;, \tag{2}$$

calculate the posterior distribution of s given d. (2 points)

b) How much information did you gain via the measurement? State your result in terms of S, d and σ! (4 points)

c) What is the least informative value that you could have gotten for d? Discuss your result! (1 point)

This exercise sheet will be discussed during the exercises.
Group 01, Wednesday 16:00 - 18:00, online,
Group 02, Thursday, 10:00 - 12:00, online,
Group 03, Thursday, 16:00 - 18:00, online,
Please hand in your solution by Monday, May 24th 2021, noon online:
https://moodle.lmu.de/course/view.php?id=14933