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Exercise sheet 11

Exercise 11 - 1

Assume a linear measurement of some field. Assume further a log-normal model for this field and an additive Gaussian noise term, i.e.

$$d = Re^s + n, \; s \hookleftarrow \mathcal{G}(s, S), \; n \hookleftarrow \mathcal{G}(n, N). \tag{1}$$

a) Derive the information Hamiltonian $H(s, d)$ for this problem.
(2 points)

b) Give a recursion relation of the type

$$m_{\text{MAP}} = f(m_{\text{MAP}}) \tag{2}$$

for the *maximum a posteriori* solution m_{MAP} of the signal field s.
(1 point)

Exercise 11 - 2

A signal $s : \mathbb{R}^u \to \mathbb{R}$ with Gaussian statistics and known covariance $S = \left\langle ss^\dagger \right\rangle_{(s)}$ is measured via $d_x = s_x + n_x$. The noise follows Gaussian statistics and is homogeneous except for a slight enhancement in an area Ω, i.e., $N_{xy} = \left\langle n_x n_y \right\rangle_{(n)} = \delta(x - y)\,(1 + \epsilon\,\Theta_\Omega(x))\,\sigma^2$. Here, $\Theta_\Omega(x) = 1$ for $x \in \Omega$ and $\Theta_\Omega(x) = 0$ for $x \notin \Omega$. Consider the Wiener filter for this inference problem.

a) Calculate perturbatively to first order in ϵ the effect of the noise inhomogeneity on the real-space structure of the propagator (3 points).

Hint: The following relation for the differentiation of a regular matrix A depending on a parameter p might be useful:

$$\begin{aligned} &\frac{\partial}{\partial p}\left(A^{-1}A\right) = 0 \\ \Leftrightarrow\; &0 = \frac{\partial A^{-1}}{\partial p} A + A^{-1}\frac{\partial A}{\partial p} \\ \Leftrightarrow\; &\frac{\partial A^{-1}}{\partial p} = -A^{-1}\frac{\partial A}{\partial p}A^{-1} \end{aligned}$$

b) Calculate N in its Fourier representation for general Ω and for $\Omega = [-L, L]$ in the one-dimensional case (3 points).

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Exercise 11 - 3

An inference problem with an unknown prior distribution can be treated with an hierarchical Bayesian model, which estimates parameters of the prior distribution from the data itself.

You want to reconstruct the results of a random process, which produces identically distributed and independent samples s_i. You have u independent but noisy measurements of this random process which follow the likelihood

$$P(d|s) = \mathcal{G}(d-s, N), \qquad \text{with} \quad N_{ij} = \delta_{ij}\sigma^2 \tag{3}$$

You assume this process is well described by a Gaussian with zero mean,

$$P(s|p) = \mathcal{G}(s, S), \qquad \text{with} \quad S_{ij} = \delta_{ij}p, \tag{4}$$

but you have no clue about its variance p. To filter out the noise contribution to each data point d_i, an estimate of p is needed. The only information you have about p is that it is positive definite. A useful parametrization is therefore $\eta \equiv \ln(p)$. Since there is no further information about η, you assume $P(\eta) = const.$

a) Write down the joint probability distribution $P(d, s, \eta)$ and marginalize out s. Write down your solution using σ and η explicitly in order to get rid of all determinants (3 points).

Hint: You can drop all factors which do not depend on s or η, but remember that some of the normalizations are η-dependent.

b) Write down the Hamiltonian $\mathcal{H}(d, \eta) \equiv -\ln P(d, \eta) + const.$ and calculate its first derivative with respect to η (2 points).

c) Set the first derivative of the Hamiltonian to zero to derive the *maximum a posteriori* (MAP) solution for η (2 points).

Hint: It is useful to replace e^η with p and then solve for p.

d) Plug in the MAP estimator for η into the Wiener Filter formula for $\langle s \rangle_{\mathcal{P}(s|d,\eta)}$ to derive an η-independent estimator for s (1 point).

This exercise sheet will be discussed during the exercises.
Group 01, Wednesday 18:00 - 20:00, Theresienstr. 39, B 101,
Group 03, Thursday, 16:00 - 18:00, Theresienstr. 37, A 449,

https://wwwmpa.mpa-garching.mpg.de/ ensslin/lectures/lectures.html