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Exercise sheet 10

Exercise 10 - 1

Let $s : \mathbb{R} \to \mathbb{R}$ be a continuous process that follows the following stochastic differential equation: $\mathrm{d}s_t/\mathrm{d}t = a\,\xi_t - b\,s_t$. Here, a and b are non-negative constants and ξ is a stationary, white random field of unit variance, i.e., $\langle \xi_t \xi_r \rangle_{(\xi)} = \delta(t-r)$. Let $S = \left\langle s s^\dagger \right\rangle_{(s)}$ be the signal covariance.

a) Assume for the moment $s_0 = 0$, $b = 0$, and $t, t' \geq 0$. Calculate $S_{tt'}$. Use S_{tt} to argue why this so-called Wiener process is a frequently used model for diffusive motion of a particle (2 points).

b) Calculate the signal power spectrum for any a and b being non-negative. Try to explain with words why the spectral normalization and the appearing characteristic frequency depend on a and b the way they do (2 points).

Hint: Transform the differential equation to Fourier space.

Exercise 10 - 2

Consider a field $\varphi \equiv \varphi_{x,t}$ with a time domain and a one-dimensional spatial domain, following the stochastic differential equation

$$\partial_t \varphi = \kappa \Delta_x \varphi + \xi \tag{1}$$

with independent Gaussian noise contribution ξ of unit variance and constant κ.

a) Calculate the auto-correlation $\langle \varphi^*_{(\omega,k)} \varphi_{(\omega',k')} \rangle$ in its full harmonic domain (temporal and spatial Fourier basis) (2 points).

b) Perform the inverse Fourier transformation in the time domain and give the expression of this auto-correlation in time - spatial frequency domain $\langle \varphi^*_{(t,k)} \varphi_{(t',k')} \rangle$ using the residue theorem. (2 points)

Exercise 10 - 3

You are observing a region of the sky with a photodetector. On average a number of photons λ makes it's way from the sky into your detector per time interval.
Hint: The PDF for n photon counts per time interval is given by the Poisson distribution:
$\mathcal{P}(n|\lambda) = \frac{\lambda^n \mathrm{e}^{-\lambda}}{n!}$

a) Calculate the expectation value and the standard deviation for the photon counts per time interval (3 Points).

b) Use the PDF to calculate the expectation value $\left\langle \frac{n!}{(n-q)!} \theta(n-q) \right\rangle_{(n|\lambda)}$ with

$$\theta(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & \text{else} \end{cases} \tag{2}$$

for any $q \in \{0, 1, \ldots, n\}$, where n denotes the photon counts per time interval (2 Points).

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Exercise 10 - 4

Optional, but highly recommended:

- Install the NIFTy package from `https://gitlab.mpcdf.mpg.de/ift/NIFTy`.
- Run experiments with the Wiener filter demo `getting_started_1.py` in demos folder.
- Modify signal or noise covariances, change response operator and dimensionality.