

Exercise sheet 8

Exercise 8 - 1

Imagine a bar of length 1, which is broken into two pieces at a random position along its length.

- a) What is the average length of the smaller of the two pieces? (2 points)
- b) Calculate the average ratio of the length of the smaller to the length of the bigger piece. (2 points)
- c) Calculate the average ratio of the length of the bigger to the one of the smaller piece. (2 points)

Exercise 8 - 2

Assume that a quantity y is linearly dependent on a quantity x , i.e., $y(x) = a + bx$. Assume further that the quantity y has been measured at $m - 1$ different positions $(x_i)_i$, $i = 1, \dots, m - 1$, subject to additive uncorrelated Gaussian noise, i.e.,

$$d_i = y(x_i) + n_i, \quad n \leftrightarrow \mathcal{G}(n, N), \quad N_{ij} = \delta_{ij} \sigma_i^2. \quad (1)$$

Assuming a Gaussian prior for the parameters a and b , i.e.,

$$s = \begin{pmatrix} a \\ b \end{pmatrix} \leftrightarrow \mathcal{G}(s, S), \quad S = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \quad (2)$$

a linear fit can be performed using Wiener filter theory.

You have enough money left to finance one additional measurement with uncertainty σ_m . How should you choose the position x_m for that measurement to gain optimal knowledge about the parameter a ?

Hint: Use the quadratic loss function $\mathcal{L}(s, x_m) = (a - m_a)^2$ and the formalism of risk minimization. Here, m_a is the Wiener filter estimate after the m measurements (4 points).

Exercise 8 - 3

The goal of this exercise is to evaluate the integral

$$I(x) = \int_{-\infty}^{\infty} dk \frac{e^{-ikx}}{k^2 + m^2} \quad (3)$$

- a) For $k \in \mathbb{C}$, the integrand can diverge. Identify the poles of the integrand and sketch them in the complex plane, sketch the integration path of $k \in (-\infty, \infty)$ as well (2 points).
- b) Draw a contour line that closes the integral, connecting $k = \infty$ to $k = -\infty$. The integral of the added contour line should be 0. Illustrate why it is 0 by comparing orders in k (2 points).

Hint: You may assume $x > 0$.

- c) The closed integral encloses one pole. Identify its order and evaluate the integral $I(x)$ using the residue theorem,

$$\int_{\Gamma} f = (2\pi i) \sum_{a \in R_f} \text{ind}_{\Gamma}(a) \text{Res}_a f, \quad (4)$$

where Γ is a closed contour, R_f are the poles of f , and $\text{ind}_{\Gamma}(a)$ is the number of counter-clockwise windings of Γ around a , and the residue formula,

$$\text{Res}_a f = \frac{1}{(n-1)!} \lim_{z \rightarrow a} \frac{\partial^{n-1}}{\partial z^{n-1}} [(z-a)^n f(z)], \quad (5)$$

where n is the order of the pole a (2 points).

Hint: A clockwise winding is a negative counter-clockwise winding.

This exercise sheet will be discussed during the exercises.

Group 01, Wednesday 18:00 - 20:00, Theresienstr. 39, B 101,

Group 03, Thursday, 16:00 - 18:00, Theresienstr. 37, A 449,

<https://wwwmpa.mpa-garching.mpg.de/~enssln/lectures/lectures.html>