Exercise sheet 7

Exercise 7 - 1

Consider the potential $V(\vec{r})$ which is symmetric with respect to the radial distance $r = |\vec{r}|$,

$$V(r) = a\sqrt{r} + b. \tag{1}$$

This potential is parametrized by the unknown numbers $a, b \in \mathbb{R}$ and can be measured at strictly positive radii, i.e., $r > 0$. Furthermore, only a single data point $d \in \mathbb{R}$ can be obtained,

$$d = V(r) + n, \tag{2}$$

where the noise n is assumed to obey a Gaussian statistic $\mathcal{P}(n) = \mathcal{G}(n, N)$. The noise variance $N = N(r)$, however, depends on the measurement position,

$$N(r) = r^2 + 3. \tag{3}$$

a) Find an expression for the information entropy $S\left[\mathcal{P}(s|d)\right]$ for a Gaussian posterior

$$\mathcal{P}(s|d) = \mathcal{G}(s - m, D)\ , \tag{4}$$

with mean m and covariance D (2 points).

b) Consider the signal $s = \begin{pmatrix} a \\ b \end{pmatrix}$, for which a Gaussian prior $\mathcal{P}(s) = \mathcal{G}(s, \mathbb{1})$ can be assumed. (2 points)

- Write Eq. (2) in the form $d = R\,s + n$ and give R explicitly.
- Work out an expression for the joint probability $\mathcal{P}(d, s)$ and calculate the corresponding Hamiltonian $H(d, s) = -\log \mathcal{P}(d, s) = \frac{1}{2} s^\dagger D^{-1} s - j^\dagger s + H_0$. You may drop H_0.
- Identify the information source j and the inverse information propagator D^{-1}.

c) You verified in **a)** that information entropy $S = S(D)$ is a monotonically increasing function of $|D|$. Find the best position $\tilde{r}$ to estimate both, a and b, by minimizing $|D|$ from **b)** with respect to r (1 point).

d) Now, consider the signal $s = a$ for which b becomes a nuisance parameter. (1 point)

- Work out an expression for the joint probability $\mathcal{P}(d, a)$, and calculate the corresponding Hamiltonian $H(d, a) = -\log \mathcal{P}(d, a) = \frac{1}{2} D^{-1} a^2 - ja + H_0$. You may drop H_0.
- Identify the information source j and the information propagator D.

e) Find the best position $\tilde{r}_a$ to estimate a irrespectively of b, by minimizing D from **d)** with respect to r (1 point).

f) Guess at which radius $\tilde{r}_b$ one should measure in order to obtain the most certain estimate for the parameter b. No justification required (1 point).

Exercise 7 - 2

You are interested in three numbers, $s = (s_1, s_2, s_3) \in \mathbb{R}^3$. Your measurement device, however, only measures three differences between the numbers, according to

$$d_1 = s_1 - s_2 + n_1 \tag{5}$$
$$d_2 = s_2 - s_3 + n_2 \tag{6}$$
$$d_3 = s_3 - s_1 + n_3 \tag{7}$$

with some noise vector $n \in \mathbb{R}^3$. Assume a Gaussian prior $\mathcal{P}(s) = \mathcal{G}(s, S)$ for s and a Gaussian PDF for the noise, $\mathcal{P}(n) = \mathcal{G}(n, N)$, with $N_{ij} = \sigma^2 \delta_{ij}$.

a) Assume that the prior is degenerate, i.e., $S^{-1} \equiv 0$. Write down the response matrix, try to give the posterior $\mathcal{P}(s|d)$, and explain why this is problematic (2 points).

b) Now assume that $S_{ij} = \sigma^2 \delta_{ij}$. Work out the posterior $\mathcal{P}(s|d)$ in this case (1 point).

Note: Using a computer algebra system, e.g., Sage (`http://www.sagemath.org/`), for the matrix operations is okay.

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Exercise 7 - 3

You have conducted a measurement of a quantity at n positions $\{x_i\}_i$, yielding n data points $\{(x_i, d_i)\}_i$. Now you want to fit some function to these data points. To this end, you write the function as a linear combination of m basis functions $\{f_j(x)\}_j$, i.e.,

$$f(x) = \sum_{j=1}^{m} s_j f_j(x). \tag{8}$$

If, for example, you were to fit a second order polynomial, you could choose the monomials as basis functions, i.e., $f(x) = s_2 x^2 + s_1 x + s_0$.
The fitting process now comes down to determining the coefficients $\{s_j\}_j$, allowing for some Gaussian and independent measurement error, i.e.,

$$d_i = \sum_{j=1}^{m} s_j f_j(x_i) + n_i. \tag{9}$$

Assume that you do not know anything about the coefficients a priori, i.e., $S^{-1} \equiv 0$, where $S_{ik} = \langle s_i s_k \rangle_{\mathcal{P}(s)}$.

a) Write down the response matrix for this problem (1 point).

b) For a given set of m basis functions, how many data points n are at least necessary for the calculation of the posterior mean of the coefficients (2 points)?

c) Now let's make a linear fit. Assuming $N_{ik} = \langle n_i n_k \rangle_{\mathcal{P}(n)} = \eta^{-1} \delta_{ik}$, choose two basis functions and work out the explicit formula for the posterior mean of the two coefficients (3 points).

This exercise sheet will be discussed during the exercises.
Group 01, Wednesday 18:00 - 20:00, Theresienstr. 37, A 449,
Group 02, Thursday, 10:00 - 12:00, Theresienstr. 37, A 249,
Group 03, Thursday, 16:00 - 18:00, Theresienstr. 37, A 449

https://wwwmpa.mpa-garching.mpg.de/ ensslin/lectures/lectures.html