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Exercise sheet 2

Exercise 2 - 1

A number of persons, k, meet. Assume that the probability of a person to have his/her birthday is the same for every day of the year. Assume further that the number of days per year is always 365.

a) How high is the probability that the birthday of at least q of these people is on the first of January (2 Points)?

b) How high is the probability of at least two persons in the room having their birthday on the same day (1 Point)?

c) For which k is this probability larger than 50% (1 Point)?

Exercise 2 - 2: Product Rule

It was argued in the lecture that the plausibility of A and B must be a continuous and monotonic function f of the plausibility of B and of the plausibility of A given B, i.e.,

$$(AB|C) = f\left((B|C), (A|BC)\right). \tag{1}$$

Furthermore, this function needs to fulfill the equation

$$f\left(f(x,y), z\right) = f\left(x, f(y,z)\right). \tag{2}$$

It can be shown (Cox, 1946) that such a function can be written as $f(x,y) = \omega^{-1}\left(\omega(x)\omega(y)\right)$.

a) Verify that any such function does indeed fulfill Equation (2) (2 points).

b) Given three statements A, B, and C, consider the cases $C \Rightarrow A$ and $C \Rightarrow \overline{A}$. Derive the possible values of $\omega(A|C)$ in these two cases (2 points).

Hint: Use $\omega(AB|C) = \omega(B|C)$ in the first case, and $\omega(AB|C) = \omega(A|C) = \omega(A|BC)$ in the second case.

c) Use the product rule for plausibilities to derive Bayes' Theorem (1 point).

Exercise 2 - 3

Given three statements A, B, and C, label the statement "$A \Rightarrow BC$", i.e. "if A is true, B and C are true" with I. Show

a) $P(A|BI) \geq P(A|I)$ (1 point).

b) $P(A|B+C, I) \geq P(A|I)$.
Note that a comma binds the arguments of a probability function as a logical "and", i.e., $P(A|B+C, I) = P(A|(B+C)I)$. (1 point)

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Exercise 2 - 4

Consider the *generalized* sum rule:

$$P(A+B|C) = P(A|C) + P(B|C) - P(AB|C) \tag{3}$$

a) Prove the *generalized* sum rule using the sum rule and the product rule (2 points).

b) Prove or disprove the following statement (1 point):

$$P(C|A+B) = P(C|A) + P(C|B) - P(C|AB)\ \forall\ A, B, C \text{ with } \mathrm{A} \neq \overline{\mathrm{B}}. \tag{4}$$

c) Generalize the generalized sum rule to three statements, i.e., find an expression for $P(A+B+C|D)$ analogous to Equation (3) (2 points).

d) Given a set of propositions $\{A_1, \dots, A_n\}$ which are mutually exclusive, given C, i.e., $P(A_iA_j|C) = \delta_{ij}P(A_i|C)$, show that

$$\sum_{i=1}^{n} P(A_i|C) = P(A_1 + \dots + A_n|C). \tag{5}$$

(2 points)

e) Given a set of propositions $\{A_1, \dots, A_n\}$ which are mutually exclusive, given C, i.e., $P(A_iA_j|C) = \delta_{ij}P(A_i|C)$, show that

$$P(B|(A_1 + \dots + A_n)C) = \frac{\sum_{i=1}^{n} P(A_i|C)P(B|A_iC)}{\sum_{i=1}^{n} P(A_i|C)}, \tag{6}$$

i.e., the left hand side is a weighted average of the separate plausibilities $P(B|A_iC)$ (1 points).

f) Given a set of propositions $\Omega^{(n)} = \{A_1, \dots, A_n\}$, define $\Lambda_i^{(n)}$ as the set of all statements that arise from the combination of i $(1 \leq i \leq n)$ different elements of $\Omega^{(n)}$ with the AND operator; e.g., for $\Omega^{(n=3)}$: $\Lambda_1^{(3)} = \{A_1, A_2, A_3\}, \Lambda_2^{(3)} = \{A_1A_2, A_1A_3, A_2A_3\}, \Lambda_3^{(3)} = \{A_1A_2A_3\}$.
Show the sum rule

$$P(A_1 + \dots + A_n|C) = \sum_{i=1}^{n} \sum_{\pi \in \Lambda_i^{(n)}} (-1)^{i+1} P(\pi|C). \tag{7}$$

(2 points)

This exercise sheet will be discussed during the exercises.
Group 01, Wednesday 18:00 - 20:00, Theresienstr. 37, A 449,
Group 02, Thursday, 10:00 - 12:00, Theresienstr. 37, A 249,
Group 03, Thursday, 16:00 - 18:00, Theresienstr. 37, A 449

https://wwwmpa.mpa-garching.mpg.de/ ensslin/lectures/lectures.html