Exercise sheet 1

Exercise 1 - 1

Weak Syllogism
Given three statements A, B, and C, label the statement "$A \Rightarrow BC$", i.e. "if A is true, B and C are true" with I. Show

a) $P(A|BI) \geq P(A|I)$ (1 point).

b) $P(A|B+C,I) \geq P(A|I)$.
Note that a comma binds the arguments of a probability function as a logical "and", i.e., $P(A|B+C,I) = P(A|(B+C)I)$. (1 point)

Exercise 1 - 2

Happy Birthday!
A number of persons, k, meet. Assume that the probability of a person to have their birthday is the same for every day of the year. Assume further that the number of days per year is always 365.

a) How high is the probability that the birthday of at least q of these people is on the first of January (2 Points)?

b) How high is the probability of at least two persons in the room having their birthday on the same day (1 Point)?

c) For which k is this probability larger than 50% (1 Point)?

Exercise 1 - 3

Weather in Markovia
You are traveling to the beautiful country of Markovia. Your travel guide tells you that the weather w_i in Markovia on a particular day i is sunny, $w_i = \mathrm{s}$, for 80% of all days or it is cloudy, $w_i = \mathrm{c}$, for 20% of all days. There are no other weather conditions in Markovia and the weather changes only during nights. The probability for a weather change is 10% if it is sunny,

$$P(w_{i+1} = \mathrm{c}|w_i = \mathrm{s}) = 0.1, \tag{1}$$

and 40% if it is cloudy,

$$P(w_{i+1} = \mathrm{s}|w_i = \mathrm{c}) = 0.4, \tag{2}$$

irrespective of what it has been on earlier days, $P(w_{i+1}|w_i, w_{i-1}, w_{i-2}, \ldots) = P(w_{i+1}|w_i)$.

a) You arrive on a sunny day, $w_i = \mathrm{s}$, in Markovia. Calculate the probability that it was cloudy there the day before, $P(w_{i-1} = \mathrm{c}|w_i = \mathrm{s})$ Hint: Use Bayes-Theorem. (2 points).

b) What is the total probability for a weather change $P(w_{i+1} \neq w_i)$ in Markovia during an arbitrary night (1 point)?

c) The Markovian weather forecast for some day i predicts a sunshine probability of

$$p_i = P(w_i = \mathrm{s}\,|\text{forecast}). \tag{3}$$

What is the sunshine probability there for the following day,

$$p_{i+1} = P(w_{i+1} = \mathrm{s}\,|\text{forecast})? \tag{4}$$

(1 point)

d) Verify or correct the travel guide's statement on the frequency of 80% sunny and 20% cloudy days in Markovia (1 point).

<u>Hint</u>: Your result of question **c)** might be useful for this.

e) Implement an algorithm to simulate the weather in Markovia and verify your results numerically (optional).

Exercise 1 - 4

During World War II, Allied intelligence bureaus made sustained efforts to estimate the amount of German tank production. The statistical Ansatz to this problem is known as the German Tank Problem.

An intelligence bureau wants to estimate the number of opposing tanks n. Intelligence reports contain the serial numbers of spotted tanks. The data available to the data analyst are the number of spotted tanks k and the highest spotted serial number m. The intelligence bureau assumes that the serial numbers range from 1 to n and that they are randomly and uniformly distributed among the tanks in action.

<u>Hint</u>: You are allowed to use both, binomial coefficients and factorials, as you prefer.

a) What is the likelihood of the highest observed serial number being m, assuming n tanks are in action and k have been spotted?

If you could not solve exercise **a)** you may use $P(m|n,k) \propto \frac{\binom{m-2}{k-2}+\binom{m-3}{k-3}}{\binom{n}{k}}$ (note, that this is not the correct result) for $1 \leq m \leq a$ and $P(m|n,k) = 0$ elsewhere. (3 points).

b) Assuming a flat prior between the number of observed tanks k and some maximum number Ω,

$$P(n) = \begin{cases} \frac{1}{\Omega} & \text{for} \quad 1 \leq n < \Omega \\ 0 & \text{otherwise} \end{cases}, \tag{5}$$

derive the posterior $P(n|m,k)$ and take the limit $\Omega \to \infty$ (2 points) .

<u>Hint</u>: $\sum_{j=i}^{M} \frac{1}{\binom{j}{k}} = \frac{k}{k-1}\left(\frac{1}{\binom{i-1}{k-1}} - \frac{1}{\binom{M}{k-1}}\right) \iff \sum_{j=i}^{M} \frac{(j-k)!}{j!} = \frac{1}{k-1}\left(\frac{(i-k)!}{(i-1)!} - \frac{(M-k+1)!}{M!}\right)$ for $k \geq 2$

c) Calculate the posterior mean. What is the minimal amount of spotted tanks in order for the posterior mean to be finite (2 points)? Plot the posterior for the number n of tanks given $k = 30$, $m = 250$, and $\Omega = 10000$ (optional).

This exercise sheet will be discussed during the exercises.
Group 01, Wednesday 18:00 - 20:00, Theresienstr. 37, A 449,
Group 02, Thursday, 10:00 - 12:00, Theresienstr. 37, A 249,

https://wwwmpa.mpa-garching.mpg.de/ ensslin/lectures/lectures.html