12 Information Field Theory

Bayes theorem:

$$\mathcal{P}(s|d) = \frac{\mathcal{P}(d,\, s)}{\mathcal{P}(d)} = \frac{e^{-\mathcal{H}(d,s)}}{\mathcal{Z}(d)}$$

$$\mathcal{Z}(d) = \int \mathcal{D}s\, \mathcal{P}(d,\, s) = \int \mathcal{D}s\, e^{-\mathcal{H}(d,s)}$$

Moment generating function:

$$\mathcal{Z}(d,\, J) = \int \mathcal{D}s\, e^{-\mathcal{H}(d,s)+J^\dagger s}$$

Moments:

$$\langle s^{x_1} \ldots s^{x_n} \rangle_{(s|d)} = \frac{1}{\mathcal{Z}} \frac{\delta^n \mathcal{Z}(d,\, J)}{\delta J_{x_1} \ldots \delta J_{x_n}} \bigg|_{J=0}$$

Cumulants:

$$\langle s^{x_1} \ldots s^{x_n} \rangle^{\mathrm{c}}_{(s|d)} = \frac{\delta^n \ln \mathcal{Z}(d,\, J)}{\delta J_{x_1} \ldots \delta J_{x_n}} \bigg|_{J=0}$$

Cumulants

$$\begin{aligned}
\mathcal{Z}(d,\,J) &= \int \mathcal{D}s\, e^{-\mathcal{H}(d,s)+J^\dagger s} \\
\langle s^{x_1} \ldots s^{x_n} \rangle^{\mathrm{c}}_{(s|d)} &= \left.\frac{\delta^n \ln \mathcal{Z}(d,\,J)}{\delta J_{x_1} \ldots \delta J_{x_n}}\right|_{J=0} \\
\langle s^{x_1} \rangle^{\mathrm{c}}_{(s|d)} &= \frac{1}{\mathcal{Z}}\frac{\delta}{\delta J_{x_1}}\mathcal{Z} = \langle s^{x_1} \rangle_{(s|d)} = \overline{s}_{x_1} \\
\langle s^{x_1} s^{x_2} \rangle^{\mathrm{c}}_{(s|d)} &= \frac{\delta}{\delta J_{x_2}}\left[\frac{1}{\mathcal{Z}}\frac{\delta}{\delta J_{x_1}}\mathcal{Z}\right] \\
&= \left.\frac{1}{\mathcal{Z}}\frac{\delta^2 \mathcal{Z}}{\delta J_{x_1}\delta J_{x_2}} - \frac{1}{\mathcal{Z}^2}\frac{\delta \mathcal{Z}}{\delta J_{x_1}}\frac{\delta \mathcal{Z}}{\delta J_{x_2}}\right|_{J=0} \\
&= \langle s^{x_1} s^{x_2} \rangle_{(s|d)} - \langle s^{x_1} \rangle_{(s|d)} \langle s^{x_2} \rangle_{(s|d)} \\
&= \langle (s-\overline{s})^{x_1} (s-\overline{s})^{x_2} \rangle_{(s|d)}
\end{aligned}$$

Cumulants of Gaussian Distribution

- $\mathcal{P}(s|d) = \mathcal{G}(s-m,\, D)$
- $\mathcal{H}(s|d) \widehat{=} \frac{1}{2}\,(s-m)^{\dagger} D (s-m)$

$$
\begin{aligned}
\langle s \rangle_{(s|d)} &= m \\
\langle s s^{\dagger} \rangle^{\mathrm{c}}_{(s|d)} &= D \\
\langle s s^{\dagger} \rangle_{(s|d)} &= D + m\, m^{\dagger} \\
\langle s^{x_1} \ldots s^{x_n} \rangle^{\mathrm{c}}_{(s|d)} &= 0 \text{ for } n \geq 3
\end{aligned}
$$

12.2 Free Theory

- linear response: $d = Rs + n$
- independent Gaussian signal and noise: $\mathcal{P}(s,\, n) = \mathcal{G}(s,\, S)\, \mathcal{G}(n,\, N)$
- signal covariance: $S = \langle ss^{\dagger} \rangle_{(s,n)}$
- noise covariance: $N = \langle nn^{\dagger} \rangle_{(s,n)}$

$$
\begin{aligned}
\mathcal{P}(d,\, s) &= \mathcal{G}(s,\, S)\, \mathcal{G}(n = d - Rs,\, N) \\
\mathcal{H}(d,\, s) &= \frac{1}{2}(d - Rs)^{\dagger} N^{-1} (d - Rs) + \frac{1}{2} s^{\dagger} S^{-1} s + \frac{1}{2} \ln(|2\pi S||2\pi N|) \\
&= \frac{1}{2} s^{\dagger} (S^{-1} + R^{\dagger} N^{-1} R) s + s^{\dagger} R^{\dagger} N^{-1} d + \mathcal{H}_0 \\
&= \frac{1}{2} s^{\dagger} D^{-1} s + s^{\dagger} j + \mathcal{H}_0
\end{aligned}
$$

12.2 Free Theory

Generating function:

$$\begin{aligned}
\mathcal{Z}(d,\,J) &= \int \mathcal{D}s\, e^{-\mathcal{H}(d,\,s)+J^\dagger s} \\
&= \int \mathcal{D}s\ \exp\left(-\frac{1}{2}s^\dagger D^{-1}s + (J+j)^\dagger s - \mathcal{H}_0\right) \\
&= \int \mathcal{D}s\ \exp\left[-\frac{1}{2}\left(s^\dagger D^{-1}s - 2j'^\dagger D\,D^{-1}s + j'^\dagger D\,D^{-1}\underbrace{D\,j'}_{=m'}\right) + \frac{1}{2}j'^\dagger D\,j' - \mathcal{H}_0\right] \\
&= \int \mathcal{D}s\ \exp\left[-\frac{1}{2}\left((s-m')^\dagger D^{-1}(s-m')\right) + \frac{1}{2}j'^\dagger D\,j' - \mathcal{H}_0\right] \\
&= |2\pi D|^{1/2}\exp\left(+\frac{1}{2}j'^\dagger D\,j' - \mathcal{H}_0\right) \\
\Rightarrow \ln \mathcal{Z}(J) &= \frac{1}{2}(J+j)^\dagger D\,(J+j) + \frac{1}{2}\ln|2\pi D| - \mathcal{H}_0 \\
\ln \mathcal{Z}(j) &= \frac{1}{2}j^\dagger D\,j + \frac{1}{2}\ln|2\pi D| - \mathcal{H}_0
\end{aligned}$$

12.2 Free Theory

$$\ln \mathcal{Z}(j) = \frac{1}{2} j^\dagger D j + \frac{1}{2} \ln |2\pi D| - \mathcal{H}_0$$

Cumulants:

$$\langle s \rangle^c_{(s|d)} = m = \frac{\delta \ln \mathcal{Z}(j)}{\delta j} = Dj$$

$$\langle s s^\dagger \rangle^c_{(s|d)} = \langle (s - \bar{s})(s - \bar{s})^\dagger \rangle = \frac{\delta^2 \ln \mathcal{Z}(j)}{\delta j \delta j^\dagger} = D$$

$$\langle s^{x_1} \ldots s^{x_n} \rangle^c_{(s|d)} = \frac{\delta^n \ln \mathcal{Z}(j)}{\delta j_{x_1} \ldots \delta j_{x_n}} = \frac{\delta^{n-2}}{\delta j_{x_3} \ldots \delta j_{x_n}} D^{x_1 x_2} = 0$$

12.3 Interacting Field Theory

$$\mathcal{H}(d,\,s) \quad = \quad \underbrace{\frac{1}{2}s^{\dagger}D^{-1}s - j^{\dagger}s + \mathcal{H}_0}_{=\mathcal{H}_{\mathcal{G}}(d,s)} + \underbrace{\sum_{n=0}^{\infty}\frac{1}{n!}\Lambda^{(n)}_{x_1\ldots x_n}s^{x_1}\ldots s^{x_n}}_{=\mathcal{H}_{\mathrm{int}}(d,s)}$$

Shift of field variables: $s \to \varphi = s - t$

- $\mathcal{H}_0' = \mathcal{H}_0 - j^{\dagger}t + \frac{1}{2}t^{\dagger}D^{-1}t$
- $j' = j - D^{-1}t$
- $\Lambda'^{(m)}_{x_1\ldots x_m} = \sum_{n=0}^{\infty}\frac{1}{n!}\Lambda^{(m+n)}_{x_1\ldots x_{m+n}}t^{x_{m+1}}\ldots t^{x_{m+n}}$

$$\begin{aligned}\Rightarrow \mathcal{H}(d,\,\varphi|t) &= \mathcal{H}(d,\,s = t+\varphi)\\ &= \frac{1}{2}\varphi^{\dagger}D^{-1}\varphi - j'^{\dagger}\varphi + \mathcal{H}_0' + \sum_{n=0}^{\infty}\frac{1}{n!}\Lambda'^{(n)}_{x_1\ldots x_n}\varphi^{x_1}\ldots\varphi^{x_n}\end{aligned}$$

12.4 Diagrammatic Perturbation Theory

$$\mathcal{H}(d,\, s) \quad = \quad \underbrace{\frac{1}{2} s^{\dagger} D^{-1} s - j^{\dagger} s + \mathcal{H}_0}_{=\mathcal{H}_{\mathcal{G}}(d,s)} + \underbrace{\sum_{n=0}^{\infty} \frac{1}{n!} \Lambda^{(n)}_{x_1 \ldots x_n} s^{x_1} \ldots s^{x_n}}_{=\mathcal{H}_{\text{int}}(d,s)}$$

Partition function:

$$\begin{aligned}
\mathcal{Z} \quad &= \quad \int \mathcal{D}s\, e^{-\mathcal{H}(d,s)} = \int \mathcal{D}s\, e^{-\mathcal{H}_{\mathcal{G}}(d,s)}\, e^{-\mathcal{H}_{\text{int}}(d,s)} \\
&= \quad \int \mathcal{D}s\, e^{-\mathcal{H}_{\mathcal{G}}(d,s)} \sum_{m=0}^{\infty} \frac{1}{m!} \left(- \sum_{n=0}^{\infty} \frac{1}{n!} \Lambda^{(n)}_{x_1 \ldots x_n} s^{x_1} \, \ldots \, s^{x_n} \right)^m \\
&\propto \quad \int \mathcal{D}s\, \mathcal{G}(s - m, D) \sum_{m=0}^{\infty} \frac{1}{m!} \left(- \sum_{n=0}^{\infty} \frac{1}{n!} \Lambda^{(n)}_{x_1 \ldots x_n} s^{x_1} \, \ldots \, s^{x_n} \right)^m
\end{aligned}$$

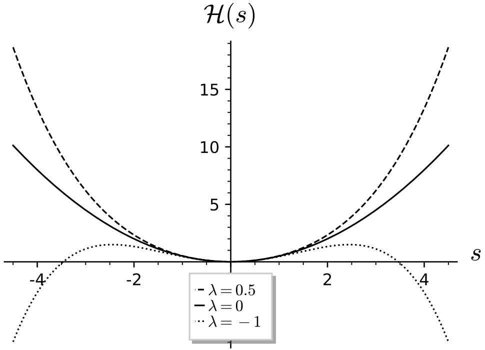
Local and Position Independent Anhormonic Interaction

$$
\begin{aligned}
\Lambda^{(4)}_{x_1 \ldots x_4} &= \delta(x_1 - x_2)\delta(x_1 - x_3)\delta(x_1 - x_4)\,\lambda \\
\Rightarrow \mathcal{H}_{\text{int}} &= \frac{\lambda}{4!}\int dx_1\, dx_2\, dx_3\, dx_4\, \delta(x_1 - x_2)\,\delta(x_1 - x_3)\,\delta(x_1 - x_4)\, s^{x_1} s^{x_2} s^{x_3} s^{x_4} \\
&= \frac{\lambda}{4!}\int dx_1 \delta^{x_1}_{x_2}\delta^{x_1}_{x_3}\delta^{x_1}_{x_4}\, s^{x_1} s^{x_2} s^{x_3} s^{x_4} \\
&= \frac{\lambda}{4!}\int dx_1 (s^{x_1})^4 \\
\mathcal{Z} &= \int \mathcal{D}s e^{-\mathcal{H}_{\mathcal{G}}(d,\, s)} \sum_{m=0}^{\infty} \frac{1}{m!}\left(-\sum_{n=0}^{\infty}\frac{1}{n!}\Lambda^{(n)}_{x_1 \ldots x_n} s^{x_1} \ldots\, s^{x_n}\right)^m \\
&= \int \mathcal{D}s\, e^{-\mathcal{H}_{\mathcal{G}}} \sum_{m=0}^{\infty} \frac{1}{m!}\left[-\frac{\lambda}{4!}\int dx\, (s^x)^4\right]^m
\end{aligned}
$$

Local and Position Independent Anhormonic Interaction

Visualization of

$$\mathcal{H}(s) \;=\; \frac{1}{2}s^2 + \frac{\lambda}{4!}s^4$$



Local and Position Independent Anhormonic Interaction

Asymptotic expansion:

$$\mathcal{Z} = \int \mathcal{D}s\, e^{-\mathcal{H}_\mathcal{G}} \overbrace{\sum_{n=0}^{\infty} \frac{1}{n!} \left[-\frac{\lambda}{4!} \int dx\, (s^x)^4 \right]^n}^{\exp(-\mathcal{H}_\text{int})}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathcal{D}s\, \underbrace{\frac{1}{\mathcal{Z}_\mathcal{G}} e^{-\mathcal{H}_\mathcal{G}}}_{\mathcal{G}} \left[-\frac{\lambda}{4!} \int dx\, (s^x)^4 \right]^n \mathcal{Z}_\mathcal{G}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle \left[-\frac{\lambda}{4!} \int dx\, (s^x)^4 \right]^n \right\rangle_\mathcal{G} \mathcal{Z}_\mathcal{G}(j)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left[-\frac{\lambda}{4!} \int dx\, \frac{\delta^4}{\delta j_x^4} \right]^n \int \mathcal{D}s\, e^{-\frac{1}{2}s^\dagger D^{-1} s + j^\dagger s - \mathcal{H}_0}$$

$$= \exp\left(-\frac{\lambda}{4!} \int dx\, \frac{\delta^4}{\delta j_x^4} \right) \mathcal{Z}_\mathcal{G}(j) = \exp\left[-\mathcal{H}_\text{int}\left(\frac{\delta}{\delta j} \right) \right] \mathcal{Z}_\mathcal{G}(j)$$

Local and Position Independent Anhormonic Interaction

Gaussian partition function:

$$\mathcal{Z}_{\mathcal{G}}(j) \quad = \quad \int \mathcal{D}s\, e^{-\frac{1}{2}s^\dagger D^{-1}s + j^\dagger s - \mathcal{H}_0} = e^{-\mathcal{H}_0}|2\pi D|^{1/2} e^{+\frac{1}{2}j^\dagger D j} = \mathcal{Z}_{\mathcal{G}}(0)\, e^{+\frac{1}{2}j^\dagger D j}$$

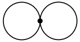
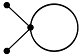
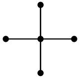
Expansion of $\mathcal{Z}$ around λ: $\mathcal{Z}(j) = \exp\left[-\mathcal{H}_{\text{int}}\left(\frac{\delta}{\delta j}\right)\right] \mathcal{Z}_{\mathcal{G}}(j)$

$$\begin{aligned}
\mathcal{Z}(j) \quad &= \quad \left(1 - \frac{\lambda}{4!}\int dx\, \frac{\delta^4}{\delta j_x^4} + \mathcal{O}(\lambda^2)\right) e^{\frac{1}{2}j^\dagger D j}\, \mathcal{Z}_{\mathcal{G}}(0) \\
&= \quad \mathcal{Z}_{\mathcal{G}}(j) - \frac{\lambda}{4!}\mathcal{Z}_{\mathcal{G}}(0) \int dx\, \frac{\delta^4}{\delta j_x^4} e^{\frac{1}{2}j_y D^{yz} j_z} + \mathcal{O}(\lambda^2) \\
&= \quad \mathcal{Z}_{\mathcal{G}}(j) - \frac{\lambda}{4!}\mathcal{Z}_{\mathcal{G}}(0) \int dx\, \frac{\delta^3}{\delta j_x^3} D^{xz} j_z e^{\frac{1}{2}j^\dagger D j} + \mathcal{O}(\lambda^2) \\
&= \quad \mathcal{Z}_{\mathcal{G}}(j) - \frac{\lambda}{4!}\mathcal{Z}_{\mathcal{G}}(0) \underbrace{\int dx\, \frac{\delta^2}{\delta j_x^2} \left[D^{xx} + (D^{xz} j_z)^2\right] e^{\frac{1}{2}j^\dagger D j}}_{=A} + \mathcal{O}(\lambda^2)
\end{aligned}$$

Local and Position Independent Anhormonic Interaction

$$
\begin{aligned}
A &= \int dx \, \frac{\delta^2}{\delta j_x^2} \left[D^{xx} + (D^{xz} j_z)^2 \right] e^{\frac{1}{2} j^\dagger D j} \\
&= \int dx \, \frac{\delta}{\delta j_x} \left[0 + 2(D^{xz} j_z) D^{xx} + (D^{xz} j_z) \left((D^{xz} j_z)^2 + D^{xx} \right) \right] e^{\frac{1}{2} j^\dagger D j} \\
&= \int dx \, \frac{\delta}{\delta j_x} \left[3(D^{xz} j_z) D^{xx} + (D^{xz} j_z)^3 \right] e^{\frac{1}{2} j^\dagger D j} \\
&= \int dx \, \left[3 D^{xx} D^{xx} + 3 (D^{xz} j_z)^2 D^{xx} + 3 (D^{xz} j_z)^2 D^{xx} + (D^{xz} j_z)^4 \right] e^{\frac{1}{2} j^\dagger D j} \\
\Rightarrow \mathcal{Z}(j) &= \mathcal{Z}_{\mathcal{G}}(j) - \lambda \int dx \left[\frac{1}{8} D^{xx} D^{xx} + \frac{1}{4} D^{xx} D^{xy} j_y D^{xz} j_z + \frac{1}{4!} (D^{xz} j_z)^4 \right] \mathcal{Z}_{\mathcal{G}}(j)
\end{aligned}
$$

Diagramatic representation:

$Z(j) = Z_{\mathcal{G}}(j) \Big[1 +$  $+$  $+$  $+ \mathcal{O}(\lambda^2) \Big]$

12.5 Feynman Rules

- $D^{xy} =$
- $j_y =$ 
- $-\lambda =$ 
- $-\Lambda^{(n)}_{x_1 \ldots x_n} =$ vertex with n ends
- all internal positions are intergrated over
- prefactor $= \frac{1}{\text{symmetry factor}}$
- symmetry factor = # permutations of half lines that leave the diagram invariant

Symmetry factor

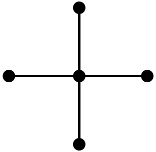
Wikipepia at https://en.wikipedia.org/wiki/Feynman_diagram#Symmetry_factors

The symmetry factor theorem gives the symmetry factor for a general diagram: the contribution of each Feynman diagram must be divided by the order of its group of automorphisms, the number of symmetries that it has.
An automorphism of a Feynman graph is a permutation M of the lines and a permutation N of the vertices with the following properties:

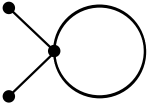
1. If a line l goes from vertex v to vertex v', then $M(l)$ goes from $N(v)$ to $N(v')$.
2. If the line is undirected, as it is for a real scalar field, then $M(l)$ can go from $N(v')$ to $N(v)$ too.
3. If a line l ends on an external line, $M(l)$ ends on the same external line. If there are different types of lines, $M(l)$ should preserve the type.

This theorem has an interpretation in terms of particle-paths: when identical particles are present, the integral over all intermediate particles must not double-count states that differ only by interchanging identical particles.

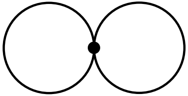
Examples

1. 

$$= -\frac{\lambda}{4!}\int dx\, D^{xz} j_z D^{xy} j_y D^{xv} j_v D^{xu} j_u = -\frac{1}{4!}\lambda^\dagger (Dj)^4$$

2. 

$$= -\frac{\lambda}{4}\int dx\, D^{xx} D^{xy} j_y D^{xz} j_z = -\frac{1}{4}\lambda^\dagger (Dj)^2 \mathrm{diag}(D) = -\frac{1}{4}\lambda^\dagger (Dj)^2 \hat{D}$$

3. 

$$= -\frac{\lambda}{8}\int dx\, D^{xx} D^{xx} = -\frac{1}{8}\lambda^\dagger \hat{D}^2$$

Connected and Disconnected Diagrams

- $\{C_i\}$: all connected diagrams
- $D = D(\{n_i\})$: disconnected diagram defined as n_i copies of C_i $\forall i$

Theorem: $\log \mathcal{Z}(j)$ =sum over all connected diagrams

Proof:

$$\begin{aligned}
\mathcal{Z}(j) &= \sum_{\{n\}} D(\{n\}) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \dots D(\{n\}) \\
&= \prod_{i=1}^{\infty} \left(\sum_{n_i=0}^{\infty} \frac{(C_i)^{n_i}}{n_i!} \right) = \prod_{i=1}^{\infty} \exp(C_i) = \exp(\sum_i C_i) \\
\Rightarrow \ln \mathcal{Z}(j) &= \sum_i C_i
\end{aligned}$$

End