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Exercise sheet 2

Exercise 2 - 1

Given two statements A and B, label the statement $A \Rightarrow B$, i.e., "If A is true, then B is true.", with C. Show the strong syllogisms $P(B|AC) = 1$ and $P(A|\overline{B}C) = 0$ (2 points).

Exercise 2 - 2

Any Boolean circuit can be constructed using the NOT, OR, and AND gates. In this question we explore the family of universal gates.

Universality of NOT, OR, and AND: Any Boolean circuit with inputs $I_1 \dots I_n$ and outputs $O_1 \dots O_m$ can be constructed out of single output circuits with output O_i (henceforth called O).

We describe a basis of Boolean circuits out of which any generic Boolean circuit can be constructed: each element of the basis is such that one of the outputs is 1 whereas all the others are 0.

For example, a basis of one input gates is:

$$\overline{I_1} = \begin{array}{c|c} I_1 & O \\ \hline 0 & 1 \\ 1 & 0 \end{array} \quad \text{and} \quad I_1 = \begin{array}{c|c} I_1 & O \\ \hline 0 & 0 \\ 1 & 1 \end{array}. \tag{1}$$

a) Complete the table for the two input basis gates using AND and NOT as logical connectors:

$$\overline{I_1}\,\overline{I_2} = \begin{array}{cc|c} I_1 & I_2 & O \\ \hline 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{array} \qquad \overline{I_1}\,I_2 = \begin{array}{cc|c} I_1 & I_2 & O \\ \hline 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{array}. \tag{2}$$

(1 point)

b) Write down the truth table for "$I_1 \Leftrightarrow I_2$". Express this statement in terms of the above basis using OR to connect the truth tables. (2 points)

c) Consider the three input logical gate:

$$\begin{array}{ccc|c} I_1 & I_2 & I_3 & O \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{array}, \tag{3}$$

write an expression for this gate using the symbols I_1, I_2, and I_3 and using the AND, OR, and NOT gates. Can this argument be generalised to arbitrary n-input logic gates? (3 points)

Universal Logic Gates: We now demonstrate the possible universal binary (two-input) logic gates. A universal gate is one out of which all logical circuits can be constructed — in effect, the AND, OR, and NOT gates have to be constructed out of the universal gates. Note that with unary (single-input) gates one can never construct an OR or an AND gate.

d) Construct the AND gate out of OR and NOT gates. Construct the OR gate out of AND and NOT gates. (2 points)

e) Show that the NAND (↑), which is AND followed by NOT, and NOR (↓), which is OR followed by NOT, gates are universal. (2 points)

f) Are there other universal gates? List them. (3 points)

Exercise 2 - 3

Assume that loosing a fraction x of your budget hits you as $l(x) = x/(1-x)$, being your personal loss function.

a) Up to which fraction $y < x < 1$ of your budget should you invest to insure against the risk of losing the budget fraction $x = x_E$ by an event E occurring with probability $p = P(E)$ (2 points) ?

b) An insurance company asks you to pay a budget fraction $z = \alpha p x$ to insure your loss under E with $\alpha > 1$ (to ensure that the company makes profit on average). Under which conditions should you take their offer (1 point)?

c) Show that a tiny monetary loss $x \ll 1$ should never be insured (1 point).

d) The insurance company knows the true p_E perfectly well, whereas the beliefs of its customers on it are uniformly distributed in $p \in [0, 1]$ What is the expected profit per potential costumer as a function of the insurance price z (2 points) ?

Exercise 2 - 4

What is Intelligent?

1. A rock.
2. A human.
3. An ant colony.
4. Language.

a) Which of the above would you classify as an intelligent system. Why? (2 point)

b) Which of the above is a physical system? (1 points)

c) Draw an "intelligence spectrum" line with intelligent systems on one end and non-intelligent systems on the other. Mark a few systems on this intelligence spectrum. (2 points)