Exam about Information Field Theory

12.06.2017

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- The exam consists of **six exercises**. Please do check if you received all of them.
- There are more questions than you will be able to solve within the give time.
- Therefore jump to the next questions if you cannot solve one.
- The working time is **90 minutes**.
- **No aids** are allowed during the exam.
- In order to gain the full amount of possible points we strictly advise you to use conditional probabilities.

Question	**Points**
1	____/6
2	____/6
3	____/10
4	____/12
5	____/8
6	____/6
Bonus	
Total	____/54

Grade	

GOOD LUCK!

Question 1 ___/6

You are interested in two numbers, $s = (s_1, s_2) \in \mathbb{R}^2$. Your measurement device, however, only measures sum and difference of the numbers, according to

$$\begin{aligned} d_1 &= s_1 + s_2 + n_1 \\ d_2 &= s_1 - s_2 + n_2 \end{aligned}$$

with some noise vector $n \in \mathbb{R}^2$. Assume a Gaussian prior $\mathcal{P}(s) = \mathcal{G}(s, S)$ for s and a Gaussian PDF for the noise, $\mathcal{P}(n) = \mathcal{G}(n, N)$, with $N_{ij} = \sigma^2 \delta_{ij}$.

a) Assume a flat prior. Write down the response matrix, and calculate the posterior $\mathcal{P}(s|d)$. Reminder: $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1} = \frac{1}{\alpha\delta - \beta\gamma} \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}$ ___/3

b) In the setting from above, now assume a generic response $R = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Calculate the mean of $\mathcal{P}(s|d)$. ___/3
Discuss in which circumstances (regarding R) the linear filter fails to provide a definite answer and give the underlying reason.

Question 2 ___/6

Assume the measurement of a signal s, which yields the data d, leads to.

$$\begin{aligned} \mathcal{P}(s) &= \mathcal{G}(s, S) \\ \mathcal{P}(s|d) &= \mathcal{G}(s - m, D) \end{aligned}$$

a) Calculate the amount of information in terms of entropy one gains via the measurement. ___/4

b) Now assume that your signal prior in the above measurement was flat. How much information in terms of entropy does one gain via the measurement? Explain the result. ___/2

Question 3 ___/10

A sequence of n coin tosses is performed and stored in a data vector $d^{(n)} = (d_1, \dots d_n) \in \{0, 1\}^n$. The coin produced a head with constant, but unknown frequency $f = P(d_i = 1|f) \in [0, 1]$.

a) How many bits of extra information on f are provided by the data vector $d^{(n)} = (1, \dots 1)$? ___/4

Hint: $\int_0^1 dx\, x^n (1-x)^m = \frac{n!\, m!}{(n+m+1)!}$ for $n, m \in \mathbb{N}$, $\int_0^1 dx\, x^n \ln x = -\frac{1}{(n+1)^2}$

b) After how many such sequential heads did one obtain 10 bits of information on f? An accuracy of 10% is sufficient. ___/2

Hint: If $n > 10$ you can use $\frac{n}{n+1} \approx 1$. Use $2^{1/\ln 2} = 2^{\ln e/\ln 2} = 2^{\log_2 e} = e \approx 2.7$.

c) How many bits on the outcome of the next toss is provided by a sequence of n heads? Provide also the asymptotic for $n \to \infty$! ___/4

Hint: It is helpful to guess the maximal amount of obtainable information before the detailed calculation is done.

Question 4 ___/12

Assume that for a PDF $p(x)$ only a set of constraints are known. From the principle of maximum entropy one derives that

$$p(x) = \exp\left(\lambda_0 + \sum_{i=1}^{n} \lambda_i f_i(x)\right) = \frac{\exp\left(\sum_{i=1}^{n} \lambda_i f_i(x)\right)}{Z} = \frac{e^{\mu^\dagger f(x)}}{Z(\mu)} \tag{1}$$

with λ_i such that the constraints are satisfied.

a) Assume a PDF $P(x \mid \lambda)$ with $x \in \mathbb{R}$. Show that no maximum entropy function exists if the only non-trivial constraint solely affects the first moment, $\langle x \rangle_{(x|\lambda)} = \lambda$. ___/3

b) Now take the above case and add the restriction that the support of x is only on $\mathbb{R}^+$. Derive the corresponding maximum entropy PDF. ___/5

c) Now assume that $\langle \ln x \rangle_{(x|\alpha)} = \alpha$, with $\alpha > 0$ and $x \in [1, \infty)$ is known. Calculate the PDF. ___/4

Question 5 ___/8

An unknown number m of atoms of a long living radioactive isotope with known decay rate ν is observed for a short time $t \ll \nu^{-1}$ with a Geiger counter that is known to register on average a fraction f of the decay events and actually registers n decays.

a) Calculate the posterior $P(m|n, I)$? ___/4

b) At a slightly later observing time $t_2 = t + \delta t > t$ you still have only n counts. Did you obtain thereby more information and if yes how much? ___/4

Question 6 ___/12

We want to calculate the probability that the apocalypse will happen in your lifetime. Let n denote the total amount of humans that will ever live.

a) Take one random human of all the humans that will ever have lived. Numbering all humans from first to last by the time they were born, what is the probability of this person being the d-th human (with $d \in \mathbb{N}$) given that there will be a total of n humans ever to live? ___/1

b) Calculate the posterior probability for n given that the random human of **a)** is the d-th human. For this, take the improper prior ___/4

$$P(n) \propto \frac{1}{n} \tag{2}$$

This prior is not normalizable, but for a Bayesian argumentation it will be sufficient if the posterior is normalizable.
Hint: You can approximate sums as integrals at your convenience.

c) You are yourself an independent sample of the human population. You are approximately the 10^{11}-th human (assuming you were born in 1990). If you grow to be 100 years old, then at the time of your death an approximative total of $1.2 \cdot 10^{11}$ humans will have been born according to todays extrapolations. What is the probability of the apocalypse happening in your lifetime, i.e. $P(n < 1,2 \cdot 10^{11} | d = 10^{11})$? ___/2
Hint: You may again approximate sums as integrals at your convenience.

d) We want to test how the choice of prior affected the above outcome. As in **c)**, calculate the posterior probability of the last human birthing in your life time, but this time assuming a uniform prior ___/5

$$P'(n) = \begin{cases} \frac{1}{x} & \text{for } n < x \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

where x is a fixed upper limit to n that we fix to $x = 10^{35} \approx (1.2)^{200} \cdot 10^{11}$.
Hint: You may once again approximate sums as integrals at your convenience.