Exam on Information Field Theory

Name: ____________________ Matriculation number: ______________

- The exam consists of **four exercises**. Please do check if you received all of them.
- In total one may gain 48 points. However it is likely that the top grade is achieved with less points.
- The working time is **90 minutes**.
- **No aids** are allowed during the exam.
- In order to gain the full amount of possible points we strictly advise you to use conditional probabilities.

Question	**Points**
1	____/15
2	____/15
3	____/12
4	____/6
Bonus	
Total	____/48

Grade	

By signing below I agree, that my matriculation number including my achieved grade will be published on the lectures' homepage. The password to open the file will be **IFT-SS2016**.

Recommendation: If you run into problems with one exercise, jump to the next one.
GOOD LUCK!

Question 1

___/15

The task is to perform an uncertainty renormalized inference via the Gibbs energy principle. Consider the data model

$$d = Rs + n \tag{1}$$

with the instrument response R, the additive Gaussian noise, i.e. $n \hookleftarrow \mathcal{G}(n, N)$. The prior for the signal s is

$$P(s) \propto \exp - \left(\frac{1}{2} s^\dagger S^{-1} s + \frac{1}{4!} \lambda^\dagger s^4 \right) . \tag{2}$$

a) Derive the Hamiltonian $H(d, s)$ for this problem and introduce the information propagator $D_0^{-1} = S^{-1} + R^\dagger N^{-1} R$ as well as the information source $j = R^\dagger N^{-1} d$. ___/2

b) Give a recursion relation of the type ___/3

$$m_{\text{MAP}} = f(m_{\text{MAP}}) \tag{3}$$

and the calculate the inverse Hessian $(\partial^2 H/\partial s \partial s^\dagger)^{-1}$. Is this D_0?

c) Calculate the Gibbs free energy ___/5

$$\begin{aligned} G(m, D) &= U(m, D) - \mathcal{T} S(m, D) \\ &= \langle H(d, s) \rangle_{\mathcal{G}(s-m,D)} - \mathcal{T} S(m, D) \end{aligned} \tag{4}$$

with

$$S(m, D) = \frac{1}{2} \mathrm{Tr} \left[1 + \ln(2\pi D) \right] \tag{5}$$

and $\mathcal{T} = 1$ being the temperature.

d) Give a recursion relation for the Gibbs free energy of the type ___/5

$$m = f(m) \, . \tag{6}$$

and the uncertainty dispersion D. Is $m = m_{\text{MAP}}$ and is this $D = D_0$ or the inverse Hessian?

Question 2

___/15

The CMB is the relic radiation from the time of (re)combination 380 000 years after the Big Bang. Its variations in temperature with direction reflect the variations in the gravitational potential φ at this time. The simplest inflationary scenarios predict these variations to be a nearly Gaussian field. Measuring deviations from Gaussianity in this field is within the focus of contemporary research in cosmology.

The *local type* of non-Gaussianities can be modeled according to

$$\varphi = \phi + f\left(\phi^2 - \left\langle \phi^2 \right\rangle_{(\phi)}\right) = \phi + f\left(\phi^2 - \hat{\Phi}\right), \tag{7}$$

where

$$\phi \hookleftarrow \mathcal{G}(\phi, \Phi) \tag{8}$$

is an auxilliary Gaussian field and the degree of non-Gaussianity of the primordial gravitational potential φ is quantified by the parameter f, which shall be assumed here to be position-independent (i.e. one number).

During cosmic recombination, the variations in the gravitational field are transformed into temperature variations in the electromagnetic radiation. This radiation has been measured in recent times by satellites, balloons, and ground based observatories. These two processes (imprinting onto temperature variations and measurement) can be combined in a linear response operator R. Additionaly, these measurements contain a Gaussian noise contribution, which is not correllated with the signal, i.e.

$$d = R\varphi + n, \;\; n \hookleftarrow \mathcal{G}(n, N)\,. \tag{9}$$

Further assume uncorrelated and homogeneous noise in the position basis, i.e.

$$N_{\hat{n}\hat{n}'} = \delta_{\hat{n}\hat{n}'}\sigma^2, \tag{10}$$

and unit response,

$$R_{\hat{n}\hat{n}'} = \delta_{\hat{n}\hat{n}'}. \tag{11}$$

a) Derive an expression for the $H\left[d, \phi|f\right]$, bring into the form ___/6

$$H\left[d, \phi|f\right] = H_0 - j^\dagger\phi + \frac{1}{2}\phi^\dagger D^{-1}\phi + \frac{1}{3!}\lambda^{(3)\dagger}\phi^3 + \frac{1}{4!}\lambda^{(4)\dagger}\phi^4 \tag{12}$$

and identify the terms j, D, $\lambda^{(3)}$, and $\lambda^{(4)}$, as well as the f-dependent part of H_0. Show that $\lambda^{(3)} \propto f$ and $\lambda^{(4)} \propto f^2$.

b) Write down the diagrammatic expansion for the logarithm of the partition function ___/6

$$\log Z_f(d) = \log \mathcal{P}(d|f) = \log \int \mathcal{D}\phi\; \mathcal{P}(d, \phi|f), \tag{13}$$

up to second order in f, i.e., use all diagrams that contain terms of lower than third order. You do not need to formulate the results algebraically, nor do you need to care about terms that are constant in ϕ and f.

c) Write down the diagrammatic version of the expectation value $\langle\phi\rangle_{\mathcal{P}(\phi|d,f)}$ up to first order in f. Read off the corresponding algebraic formula for the expectation value in terms of j, D, $\lambda^{(3)}$, and $\lambda^{(4)}$. __/3

Question 3 ___/12

A temporally varying radio signal $s = (s_t \in \mathbb{R})_t$ is detected with a radio receiver. The receiver amplifies the signal via a resonant circuit, for which the voltage $u = (u_t \in \mathbb{R})_t$ follows the equation of a signal-driven, but weakly damped oscillator:

$$\frac{\mathrm{d}^2 u_t}{\mathrm{d}t^2} + 2\eta \frac{\mathrm{d}u_t}{\mathrm{d}t} + u_t = s_t. \tag{14}$$

Here, problem specific and therefore dimensionless time units were chosen, so that the small damping constant $\eta \in (0, 1/\sqrt{2})$ is dimensionless. The receiver electronics adds some statistically homogeneous white Gaussian noise $n = (n_t \in \mathbb{R})_t$ with known variance $\langle n_t n_{t'} \rangle_{\mathcal{P}(n)} = \sigma^2 \delta(t - t')$ to the output data

$$d_t = u_t + n_t. \tag{15}$$

Please use the Fourier sign conventions $s_\omega = \int \mathrm{d}t\ \mathrm{e}^{-i\omega t} s_t$ and $s_t = \int \frac{\mathrm{d}\omega}{2\pi}\ \mathrm{e}^{+i\omega t} s_\omega$ or note explicitely the ones you use. Factors of 2π may be neglected in your calculation.

a) Work out the signal response of the voltage in the Fourier domain, i.e., give $R_{\omega\omega'}$ of ___/2

$$u_\omega = \int \frac{\mathrm{d}\omega'}{2\pi}\ R_{\omega\omega'} s_{\omega'}. \tag{16}$$

b) Work out the signal response of the voltage in the time domain, i.e., give $R_{tt'}$ of ___/4

$$u_t = \int \mathrm{d}t'\ R_{tt'} s_{t'}. \tag{17}$$

c) Suppose the signal is a statistically homogeneous Gaussian random field with known power spectrum $P_s(\omega) = \langle |s_\omega|^2 \rangle_{\mathcal{P}(s)}$. In order to assess how well the signal can be reconstructed at each frequency from the output data, calculate the uncertainty spectrum ___/2

$$P_D(\omega) = \left\langle \left| s_\omega - \langle s_\omega \rangle_{\mathcal{P}(s|d)} \right|^2 \right\rangle_{\mathcal{P}(s|d)}. \tag{18}$$

d) Now, suppose you want to design a matching transmitter to send an amplitude (modulated) signal to the receiver. The transmitter just produces (approximately) a fixed amount of signal power $p = P_s(\omega_*)$ at a single, tunable frequency $\omega_* \geq 0$. What is the optimal transmission frequency ω_* to permit the most accurate signal reconstruction on the receiver side? Assume that p as well as ω_* are known to the receiver. ___/4

Question 4

___/6

Given a field $s : \mathcal{S}^2 \to \mathbb{C}$ on the two-dimensional sphere, assume that it is statistically homogeneous and isotropic, i.e., $S(\hat{n}, \hat{n}') = \langle s(\hat{n}) s^*(\hat{n}') \rangle = S(\hat{n} \cdot \hat{n}')$, where $\hat{n}$ and $\hat{n}'$ are unit vectors that give directions or, equivalently, points on $\mathcal{S}^2$. Prove that the covariance matrix S is diagonal in the basis given by the spherical harmonic functions and its entries are independent of m, i.e.,

$$S_{(\ell m)(\ell' m')} := \langle s_{\ell m} s^*_{\ell' m'} \rangle = \delta_{\ell\ell'} \delta_{mm'} C_\ell. \tag{19}$$

Hint: Use the following properties of the spherical harmonic functions $Y_{\ell m}$ and the Legendre polynomials P_ℓ:

$$s(\hat{n}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} s_{\ell m} Y_{\ell m}(\hat{n}), \qquad s_{\ell m} = \int_{\mathcal{S}^2} \mathrm{d}\Omega \; s(\hat{n}) Y^*_{\ell m}(\hat{n}) \tag{20}$$

$$\int_{\mathcal{S}^2} \mathrm{d}\Omega \; Y^*_{\ell m}(\hat{n}) Y_{\ell' m'}(\hat{n}) = \delta_{\ell\ell'} \delta_{mm'} \tag{21}$$

$$P_\ell(\hat{n} \cdot \hat{n}') = \frac{4\pi}{2\ell + 1} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\hat{n}) Y^*_{\ell m}(\hat{n}') \tag{22}$$