# Exam about Information Field Theory

31.05.2016

Name: ____________________ Matrikelnummer: ____________________

- The exam consists of **seven exercises**. Please do check if you received all of them.
- There are more questions than you will be able to solve within the give time.
- Therefore jump to the next questions if you cannot solve one.
- The working time is **90 minutes**.
- **No aids** are allowed during the exam.
- In order to gain the full amount of possible points we strictly advise you to use conditional probabilities.

| **Question** | **Points** |
| --- | --- |
| 1 | ____/12 |
| 2 | ____/14 |
| 3 | ____/14 |
| 4 | ____/8 |
| 5 | ____/6 |
| 6 | ____/8 |
| 7 | ____/10 |
| Bonus | |
| Total | ____/72 |

| | |
| --- | --- |
| Grade | |

***GOOD LUCK!***

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**Question 1** ___/12

Let's play *White Wins*. This game is about drawing blindly from an urn filled with a certain number of black and white balls. After each trial, when the ball's color has been identified, the ball is put pack into the urn.

Background information $I$: The pupil version of this game is played with one single urn $u$ that is filled with an equal number of black and white balls; e.g., six each.

$$\lfloor \bullet\bullet\bullet\bullet\bullet\bullet\circ\circ\circ\circ\circ\circ \rfloor$$

**a)** Give the probability, $P(w_1|I)$, for drawing a white ball from the urn $u$. ___/1

**b)** Say you have drawn a white ball in the first trial, and put it back. Did you gain any knowledge on the next outcome after this first trail? What is the probability, $P(w_2|w_1, I)$, of drawing again a white ball from urn $u$ under this condition? ___/1

**c)** Calculate the probability, $P(w_2, w_1|I)$, for drawing two white balls from the urn $u$ sequentially, putting the first one back as in *b*). ___/1

Background information $J$: The student version of *White Wins* is played with a set of $N + 1$ urns, $\{u_n\}_{0 \leq n \leq N}$. Every urn $u_n$ is filled with $N - n$ black and $n$ white balls. $N = 3$ would then correspond to the same total numbers of balls as above, distributed on fours urns as:

$$\lfloor \bullet\bullet\bullet \rfloor \quad \lfloor \bullet\bullet\circ \rfloor \quad \lfloor \bullet\circ\circ \rfloor \quad \lfloor \circ\circ\circ \rfloor$$

However, during the game one will only draw balls from a single urn that is randomly chosen prior to any trial; i.e., $P(u_n|J) = \frac{1}{N+1}$. All except the chosen urn are put aside.

**d)** Calculate the probability, $P(w_1|J)$, for drawing a white ball from the chosen urn. Compare your result with the one obtained in **a)**. ___/3

**e)** Say you have drawn a white ball in the first trial, and put it back. Did you gain any knowledge in this first trail? Infer the probability, $P(u_n|w_1, J)$, for the urn that might have been chosen, in general and in the case of $N = 3$. ___/2

Note: Your result of question **d)** might be useful for this.

**f)** Calculate the probability, $P(w_2, w_1|J)$, for drawing two white balls from the chosen urn sequentially, putting the first one back. Give a result for $N = 3$ and $N \to \infty$. Compare your result with the one obtained in **c)**. ___/4

Hint: Remember the finite sums,

$$\sum_{n=0}^{N} n = \frac{N(N+1)}{2} \qquad \sum_{n=0}^{N} n^2 = \frac{N(N+1)(2N+1)}{6} \qquad \sum_{n=0}^{N} n^3 = \left(\frac{N(N+1)}{2}\right)^3.$$

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**Question 2** ___/14

Consider a biased coin with the probability $f$ of yielding head.

**a)** Calculate the probability of achieving exactly $k = 1$ heads after $n = 3$ tosses, given that $f = 1/3$. ___/2

Hint:

$$\binom{n}{k} = \binom{n}{n-k} = \frac{n!}{k!(n-k)!}$$

**b)** Now investigate the limit of a very large number of tosses, i.e. $n \to \infty$, while the expected number of heads $\lambda = nf$ remains finite. Under this condition, derive the probability distribution for yielding $k$-times head, given $\lambda$. How is the resulting probability distribution, given by ___/4

$$P(k|\lambda) = \frac{\lambda^k e^{-\lambda}}{k!},$$

called?

Hint:

$$\frac{n!}{(n-k)!(n-\lambda)^k} \stackrel{n\to\infty}{\longrightarrow} 1$$

**c)** Calculate the expectation value of $k$ with the use of the probability distribution derived in **b)**. ___/2

**d)** Under the condition $\lambda \gg 1$, the probability distribution of **b)** makes a transition into a Gaussian distribution. Derive this Gaussian distribution $P(k|\lambda)$. For this purpose use the substitution ___/6

$$k = \lambda(1+\delta), \text{ with } \lambda \gg 1, \ \delta \ll 1,$$

and use Stirling's formula for $k!$:

$$k! \approx \sqrt{2\pi k}\left(\frac{k}{e}\right)^k.$$

Hint: Use $(1+\delta)^{\lambda(1+\delta)+1/2} \approx \exp\left(\lambda\delta + \lambda\delta^2/2 + \mathcal{O}(\delta^3)\right)$. Don't forget to reverse the substitution at the end of your calculation.

**Question 3** ___/14

A sports game in which two players play for a point in each round, the game is won by the first player who leads by two points. Player $A$ always has the probability $\theta$ to win a round. What are his chances?

**a)** State the three probabilities (as a function of $\theta$) that after two rounds player has $A$ won, has lost, or has to continue the game. Call them in the following $w$, $l$, and $c$, respectively. ___/3

**b)** State now the three probabilities (as a function of $w$, $l$, and $c$) that after (at most) four rounds player $A$ has won, has lost, or that the game still continues. ___/4

**c)** State now the three probabilities that after (at most) an infinite numbers of rounds player $A$ has won, lost, or that the game still continues. ___/3

**d)** Now you hear that player $A$ has won the game. What is the probability $P(n|W)$ (as a function of $\theta$) that the game ended with round $n$? If you were not able to solve c) use the following: ___/4

$$P(W|n \leq \infty) = \frac{\theta^2}{1 - 2\theta(1-\theta)} \qquad P(L|n \leq \infty) = \frac{(1-\theta)^2}{1 - 2\theta(1-\theta)} \qquad P(C|n \leq \infty) = 0$$

### Question 4 ___/8

Assume that loosing a fraction $x$ of your budget hits you as $l(x) = x/(1-x)$, being your personal loss function.

**a)** Up to which fraction $y < x < 1$ of your budget should you invest to insure against the risk of loosing the budget fraction $x = x_E$ by an event $E$ occurring with probability $p = P(E)$? ___/3

**b)** An insurance company asks you to pay a budget fraction $z = \alpha\, p\, x$ to insure your loss under $E$ with $\alpha > 1$ (to ensure that the company makes profit on average). Under which conditions should you take their offer? ___/1

**c)** Show that a tiny monetary loss $x \ll 1$ should never be insured. ___/1

**d)** The insurance company knows the true $p_E$ perfectly well, whereas the beliefs of its customers on it are uniformly distributed in $p \in [0,1]$. What is the expected profit per potential costumer as a function of the insurance price z? ___/3

### Question 5 ___/6

Assume the measurement of a signal $s$, which yields the data $d$, leads to

$$\mathcal{P}(s) = \mathcal{G}(s, S)$$
$$\mathcal{P}(s|d) = \mathcal{G}(s - m, D)\,.$$

**a)** Calculate the amount of information in terms of entropy one gains via the measurement. ___/4

**b)** Now assume that your signal prior in the above measurement was flat. How much information in terms of entropy does one gain via the measurement? Explain the result. ___/2

### Question 6 ___/10

A strictly positive quantity $x$ is bound from above, say by $x \leq 1$. You learn that its natural logarithm is typically the negative number $-l$.

**a)** How much information did you gain about $x$? ___/4

**b)** What is the least informative value for $l$ you could have gotten? ___/2

**c)** Use these insights to explain **Benford's law** $B$, under which a set of positive numbers exhibits the leading digit $d \in \{1, \ldots 9\}$ with a probability $P(d|B) = \log_{10}\left(1 + \frac{1}{d}\right)$. Such distributions are observed in many human made, measured, and even mathematical sets of numbers. Benford's law is used to identify potential tax frauds. ___/4

### Question 7 ___/8

An unknown number $m$ of a long living radioactive isotope with known decay rate $\nu$ is observed for a short time $t \ll \nu^{-1}$ with a Geiger counter that is known to registers on average a fraction $f$ of the decay events and actually registers $n$ decays.

**a)** What do you known about $m$? ___/4

**b)** At a slightly later observing time $t_2 = t + \delta t > t$ you still have only $n$ counts. Did you obtain thereby more information and if yes how much? ___/4