Exam on Information Field Theory

Name: ______________________ Matriculation number: ____________

- The exam consists of **five exercises**. Please do check if you received all of them.
- In total one may gain 90 points. However it is likely to achieve the top grade with less points.
- The working time is **90 minutes**.
- **No aids** are allowed during the exam.
- In order to gain the full amount of possible points we strictly advise you to use conditional probabilities.

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**Grade**

Recommendation: If you run into problems with one exercise, jump to the next one. **GOOD LUCK!**
Question 1

Mark whether the following statements are always true (T) or not (F).

a) T F Rational beliefs can be described as probabilities. ___/1

b) T F A statement and its negation have different truth values. ___/1

c) T F A statement and its negation have different probabilities. ___/1

d) T F A rational mind expects a fair coin which has shown head ten times in a row to show tail with the next toss with higher probability than head. ___/2

e) T F A rational mind expects a dubious coin which has shown head ten times in a row to show head with the next toss with higher probability than tail. ___/2

f) T F The Wiener filter is a linear operation on the data. ___/1

g) T F A statistically homogeneous field on $\mathbb{R}^u$ has a Fourier power spectrum $P(\vec{k})$ with $P(\vec{k}) = P(k)$, where $k = |\vec{k}|$. ___/1

h) Assume a signal $s \in \mathbb{R}$ and different estimates $m$ for its value.

1) T F The posterior mean $m$ minimizes $\langle |s - m| \rangle_{P(s|d)}$. ___/1

2) T F The posterior median $m$ minimizes $\langle |s - m| \rangle_{P(s|d)}$. ___/1

3) T F The posterior median $m$ minimizes $\langle -\delta (s - m) \rangle_{P(s|d)}$. ___/1

4) T F The maximum a posteriori $m$ minimizes $\langle -\delta (s - m) \rangle_{P(s|d)}$. ___/1

5) T F The maximum a posteriori $m$ minimizes $\langle |s - m|^2 \rangle_{P(s|d)}$. ___/1

i) Assume the Hamiltonian

$$H(d, s) = -j^\dagger s + \frac{1}{2} s^\dagger D^{-1} s + \lambda^\dagger s^4$$

(1)

(remember that $\lambda^\dagger s^4 = \int dx \lambda(x) s^4(x)$) and the convention that indices that appear more than once in one term are integrated over.

1) T F $x \quad \quad \quad = -\frac{1}{2} D_{xy} \lambda_y D_{yz} D_{xz} j_z$ ___/2

2) T F $x \quad \quad \quad = +\frac{1}{2} D_{xy} \lambda_y (D_{yz})^3 \lambda_z D_{zu} j_u$ ___/2

3) T F $x \quad \quad \quad = +\frac{1}{2} D_{xy} \lambda_y D_{yy} D_{yz} \lambda_z D_{zz} D_{zu}$ ___/2

Question 2

A signal of known shape $r_x$ in $u$-dimensional position space ($x \in \mathbb{R}^u$) but completely unknown amplitude $s$ ($s \in \mathbb{R}$ or $s \in \mathbb{C}$) is contained in noisy data $d_x$ over the $x$-space. The additive noise is Gaussian with known covariance $N = \langle nn^\dagger \rangle_{(n)}$. ___/A
a) Derive the matched filter, which is the signal mean given the above information (3 points). ___/D

b) Assume the noise covariance to be diagonal in position-space and write down the matched filter (2 points). ___/A

c) Assume the noise covariance to be diagonal in Fourier-space and write down Fourier-space version of the matched filter (1 point). ___/A
Question 3

Consider a real-valued signal field \( s \) with a Gaussian prior,

\[ P(s) = \mathcal{G}(s, S), \quad (2) \]

that is observed with an instrument that exhibits an almost linear response,

\[ d = R(s + rs^2) + n. \quad (3) \]

Here, \( R \) is a linear operator, \( r \in \mathbb{R} \) with \( |r| \ll 1 \) is a small parameter that determines the strength of the nonlinearity in the instrumental response, \( s^2 \) denotes the local squaring of the signal field, i.e., \( (s^2)_x = (s_x)^2 \), and \( n \) is additive Gaussian noise, i.e.,

\[ P(n) = \mathcal{G}(n, N). \quad (4) \]

a) Consider first the case of an exactly linear response, i.e., \( r = 0 \). Derive the Hamiltonian

\[ H(d, s) = -\log (P(d, s)) \quad (5) \]

for this problem. You may drop all terms that do not depend on \( s \).

b) Show that the posterior probability density in the case with \( r = 0 \) is of Gaussian form, i.e.,

\[ P(s|d) = \mathcal{G}(s - m_0, D), \]

denote its mean and covariance,

\[ m_0 = \langle s \rangle_{P(s|d)} \quad \text{and} \quad D = \langle (s - m_0)(s - m_0)^\dagger \rangle_{P(s|d)}, \quad (6) \]

as a function of \( d, S, N, \) and \( R \).

c) Now consider the case with small but non-zero \( r \). Calculate the Hamiltonian in this case and write it in the form

\[ H(s, d) = H_0 - j^\dagger s + \frac{1}{2} s^\dagger D^{-1} s + \sum_{k=2}^{\infty} \frac{1}{k!} \Lambda^{(k)} s_{x_1} s_{x_2} \cdots s_{x_k}, \quad (7) \]

where only the coefficients \( \Lambda^{(k)} \) depend on \( r \) and we use the convention that repeated indices are integrated over. Give expressions for \( j, D, \) and all non-zero \( \Lambda^{(k)} \). You do not need to calculate \( H_0 \).

d) Write down the diagrammatic expansion of the partition function \( \log (Z(d)) \) up to linear order in \( r \).

e) Find the diagrammatic expressions for the posterior mean and covariance,

\[ m_r = \langle s \rangle_{P(s|d, r)} \quad \text{and} \quad \langle (s - m_r)(s - m_r)^\dagger \rangle_{P(s|d, r)}, \quad (8) \]

up to first order in \( r \).
Question 4

Consider a damped harmonic oscillator that is exposed to external fluctuations. The oscillator’s spatial displacement, \( x = x(t) \), is a function of time as described by the following differential equation,

\[
(\alpha \ddot{x} - \sigma \xi) + \dot{x} + \nu x = 0 \quad \text{with} \quad \alpha, \sigma, \nu \in \mathbb{R}. \tag{9}
\]

Here, the external fluctuations, \( \xi = \xi(t) \), are Gaussian white noise; i.e., \( \xi \sim \mathcal{G}(\xi, \Xi) \) with \( \Xi = \Xi(t, t') = \delta(t - t') \).

a) Derive the Fourier representation \( \Xi(\omega, \omega') \) of the autocorrelation function \( \Xi(t, t') \) of the noise process that is driving the external fluctuations, and read off its Fourier power spectrum, \( P_\xi(\omega) \). ___/2

b) Calculate the Fourier power spectrum, \( P_x(\omega) \), of the spatial displacement. ___/5

**Hint:** Your result should be of the form,

\[
P_x(\omega) \equiv \frac{A\sigma^2}{B^2 + (1 - C\alpha)\omega^2 + D\omega^4} \quad \text{with} \quad A, B, C, D \in \mathbb{R}, \tag{10}
\]

which you are allowed to use in c) and d) instead of your own result if need be.

Assume a direct measurement of the displacement, \( d = d(t) \), that is subject to additive noise, \( n = n(t) \), of the following form,

\[
d = x + n. \tag{11}
\]

The noise is Gaussian white noise; i.e., \( n \sim \mathcal{G}(n, N) \) with \( N(t, t') = \sigma^2 \delta(t - t') \), where \( \sigma \) is identical to the one used in equations (9) and (10), since the noise is also caused by the external fluctuations.

c) Calculate the Fourier representation of the a posteriori mean, \( m(\omega) = \langle x(\omega) \rangle_{(x|d)} \), and covariance, \( D(\omega, \omega') = \langle [x(\omega) - m(\omega)][x(\omega') - m(\omega')]^* \rangle_{(x|d)} \), assuming a Wiener filter. ___/5

Let the external fluctuations dominate the displacement. In consequence, the inertia force of the oscillator becomes negligible; i.e., the system is sufficiently described by the limit \( \alpha \to 0 \).

d) Find the (complex) poles of the Fourier power spectrum calculated in b) in the case of dominating external fluctuations and calculate the corresponding autocorrelation function, \( X(t, t') = \langle x(t)x(t') \rangle_{(x)} \). ___/8
Question 5

The decay of a known amount $m_0$ of a radioactive isotope with unknown decay rate $\nu$ is measured. The amount left after the time $t \geq 0$ is

$$m_t = m_0 e^{-\nu t} ,$$

(12)

where $\nu$ is our signal. Assume a flat prior, $P(\nu) = \text{const}$. On average a fraction $\kappa$ of the decays are registered in the time interval from $t = n\Delta$ to $t = (n + 1)\Delta$ for all $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots \}$ and stored in a data vector $d = (d_0, d_1, d_2, \ldots)$.

Assume Poissonian statistics for the count rates with

$$P(d_n|\lambda_n) = \frac{\lambda_n^{d_n} e^{-\lambda_n}}{d_n!}$$

(13)

and the expected detection rate

$$\lambda_n(\nu) = \kappa \int_{n\Delta}^{(n+1)\Delta} \left( -\frac{d}{dt'} m_{t'} \right) dt' = \kappa (m_{n\Delta} - m_{(n+1)\Delta}).$$

(14)

a) Give the Hamiltonian $H(d, \nu)$, disregarding any terms that are not $\nu$-dependent. Hint:

$$\sum_{n=0}^{\infty} e^{-n\Delta\nu} = \sum_{n=0}^{\infty} (e^{-\Delta\nu})^n = \frac{1}{1 - e^{-\Delta\nu}}$$

(15)

b) Find the maximum a posteriori estimator for $\nu$ given $d$.

c) Derive the Gaussian approximation

$$P(\nu|d) \approx \mathcal{G}(\nu - \nu_{\max}, D)$$

(16)

of the posterior around its maximum, i.e. give $\nu_{\max}$ and $D$ using a saddle point approximation.
Question 6

Consider a real-valued signal field $s$ with a Gaussian prior,

$$\mathcal{P}(s) = \mathcal{G}(s, S),$$

which is observed with an instrument that exhibits a linear response,

$$d = R(s) + n = Rs + n.$$  

Here, $R$ is a linear operator and $n$ is additive Gaussian noise, i.e.,

$$\mathcal{P}(n|s) = \mathcal{G}(n, N).$$

a) Derive the Hamiltonian

$$H_{\mathcal{G}}(d, s) = -\log (\mathcal{P}(d, s))$$

for this problem and introduce the information propagator $D^{-1} = S^{-1} + R^*N^{-1}R$ as well as the information source $j = R^*N^{-1}d$. You may drop all terms that do not depend on $s$.

b) Calculate the moment generating functional (partition function),

$$Z_{\mathcal{G}}(J) = \int Ds \ e^{-H_{\mathcal{G}}(d, s) + Js}.$$  (17)

c) Now suppose that the assumption of a linear response was wrong and we have to include an interaction term,

$$H_{\text{int}}(s) = \frac{\lambda}{3!} \int dx \ s^3(x) + \mathcal{O}(s^4), \ \lambda \in \mathbb{R},$$

in the Hamiltonian (the $\mathcal{O}(s^4)$ terms ensure proper normalizability of the probability distribution function, but can be assumed to be negligible in this calculation). Thus, the correct Hamiltonian is given by

$$H(d, s) = H_{\mathcal{G}}(d, s) + H_{\text{int}}(s).$$

Show that the moment generating functional up to first order in $\lambda$ can be written as

$$Z(J) = \left\{ 1 - H_{\text{int}}(\frac{\delta}{\delta J}) + \ldots \right\} Z_{\mathcal{G}}(J).$$  (18)

d) Determine the generating functional, which is given by equation (17) and (18), up to first order in $\lambda$, either by a calculation or by Feynman diagrams. You should finally obtain

$$Z(J) = \exp \left\{ -\frac{\lambda}{2} \int dx \int dy \ D(x, x)D(x, y)(J + j)(y) \right.$$  

$$-\frac{\lambda}{6} \left( \int dx \int dy \ D(x, y)(J + j)(y) \right)^3 \right\} Z_{\mathcal{G}}(J),$$  (19)

after re-identifying $1 - x \approx e^{-x}$.

e) Finally calculate the mean and covariance of $s$ by the use of equation (19) and rewrite the analytical expressions in terms of Feynman diagrams.