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Exercise sheet 8

Exercise 8 - 1

Consider a random field s that is statistically homogeneous and isotropic,

$$\langle s(x)s(y)\rangle_{\mathcal{P}(s)} = C(|x-y|), \quad \text{with } x \in \mathbb{R}. \tag{1}$$

a) Show that the Fourier transformed autocorrelation function has the form

$$\langle s(k)s(q)^*\rangle_{\mathcal{P}(s)} = (2\pi)\delta(k-q)\, f(|k|) \tag{2}$$

(note that f depends only on the absolute of k). (2 points)

b) Show that if s follows Gaussian statistics all Fourier components are independent, i.e. show that

$$\mathcal{P}(s(k)|s(q)) = \mathcal{P}(s(k)) \quad \text{for} \quad k \neq q. \tag{3}$$

(3 points)

Hint: You are allowed to drop normalization factors and to be sloppy in case you discretize integrals.

Exercise 8 - 2

A signal of known shape r_x in u-dimensional position space ($x \in \mathbb{R}^u$) but completely unknown amplitude s ($s \in \mathbb{R}$ or $s \in \mathbb{C}$) is contained in noisy data d_x over the x-space. The additive noise is Gaussian with known covariance $N = \left\langle nn^\dagger \right\rangle_{(n)}$.

a) Derive the matched filter, which is the signal mean given the above information. (3 points)

b) Assume the noise covariance to be diagonal in Fourier-space and write down the Fourier-space version of the matched filter. (1 point)

Exercise 8 - 3

Imaging devices can probe continuous fields, such as physical flux or matter densities, by pixel averaged measurements. Assume the value d_i in the ith pixel of the obtained image satisfies

$$d_i = \int_\Omega \mathrm{d}x\, R_i(x)s(x) + n_i \qquad \text{with} \qquad i \in I = \{1, \ldots, u\}, \tag{4}$$

where $R : I \times \Omega \to \mathbb{R}$ denotes the instrument response function, $s : \Omega \to \mathbb{R}$ the signal field, and $n \in \mathbb{R}^u$ the noise vector. Both, signal and noise, are *a priori* assumed to follow Gaussian distributions,

$$s \curvearrowleft \mathcal{G}(s-t, S) \qquad \text{and} \qquad n \curvearrowleft \mathcal{G}(n-r, N), \tag{5}$$

with kown non-zero means, $t : \Omega \to \mathbb{R}$ and $r \in \mathbb{R}^u$, as well as kown covariances, $S : \Omega \times \Omega \to \mathbb{R}$ and $N \in \mathcal{M}_{u\times u}(\mathbb{R})$.

a) Derive an expression for the likelihood, $P(d|s)$. Which quantity needs to be marginalized over (1 point)?

b) Compute the full information Hamiltonian (including constant terms); i.e., $H(d, s) = -\log P(d, s)$. Identify the information propagator, D, the information source, j, and the interaction terms (third or higher order in the signal), Λ. What are the implications for the posterior, $P(s|d)$ (2 point)?

c) Derive an expression for the posterior mean field, $m = \langle s \rangle_{(s|d)}$, in terms of the given image, d. To do so use the maximum a posterior Ansatz (2 points).

d) Calculate $\widetilde{m} = \langle s \rangle_{(s|\widetilde{d})}$ given a modified data set $\widetilde{d}$ defined as:

$$\widetilde{d}_i = d_i - \int_\Omega \mathrm{d}x\, R_i(x)t(x) - r_i. \tag{6}$$

Your result should only depend on S, R, N, and $\widetilde{d}$. Find the relation between the posterior mean field m derived in **c)** and the field $\widetilde{m}$ (2 points).

Exercise 8 - 4

Optional, but highly recommended:

- Install the NIFTy package from `https://gitlab.mpcdf.mpg.de/ift/NIFTy`.
- Run experiments with the Wiener filter demo `getting_started_1.py`.
- Modify signal or noise covariances, change response operator and dimensionality.

[Installation under Python 3 is necessary.]

This exercise sheet will be discussed in the tutorials.
group A, Wednesday 16:00 - 18:00 Theresienstr. 39 - room B 101,
group B, Thursday, 10:00 - 12:00, Geschw.-Scholl-Pl. 1 (B) - room B 015
group C, Thursday, 16:00 - 18:00, Theresienstr. 37 - room A 449
backup slot, Wednesday, 8:00 - 10:00, Theresienstr. 37 - A 450
(to be used whenever Thursday is a vacation day)
www.mpa-garching.mpg.de/∼ensslin/lectures