Exercise sheet 7

Exercise 7 - 1

Exercise 7- 2: Experimental constraints Consider the potential $V(\vec{r})$ which is symmetric with respect to the radial distance $r = |\vec{r}|$,

$$V(r) = a\sqrt{r} + b. \tag{1}$$

This potential is parametrized by the unknown numbers $a, b \in \mathbb{R}$ and can be measured at strictly positive radii, i.e., $r > 0$. Furthermore, only a single data point $d \in \mathbb{R}$ can be obtained,

$$d = V(r) + n, \tag{2}$$

where the noise n is assumed to obey a Gaussian statistic $\mathcal{P}(n) = \mathcal{G}(n, N)$. The noise variance $N = N(r)$, however, depends on the measurement position,

$$N(r) = r^2 + 3. \tag{3}$$

a) Find an expression for the information entropy $S\left[\mathcal{P}(s|d)\right]$ for a Gaussian posterior

$$\mathcal{P}(s|d) = \mathcal{G}(s - m, D) \ , \tag{4}$$

with mean m and covariance D (2 points).

b) Consider the signal $s = \begin{pmatrix} a \\ b \end{pmatrix}$, for which a Gaussian prior $\mathcal{P}(s) = \mathcal{G}(s, \mathbb{1})$ can be assumed. (2 points)

- Write Eq. (2) in the form $d = R\,s + n$ and give R explicitly.
- Work out an expression for the joint probability $\mathcal{P}(d, s)$ and calculate the corresponding Hamiltonian $H(d, s) = -\log \mathcal{P}(d, s) = \frac{1}{2} s^\dagger D^{-1} s - j^\dagger s + H_0$. You may drop H_0.
- Identify the information source j and the inverse information propagator D^{-1}.

c) You verified in **a)** that information entropy $S = S(D)$ is a monotonically increasing function of $|D|$. Find the best position $\tilde{r}$ to estimate both, a and b, by minimizing $|D|$ from **b)** with respect to r (1 point).

d) Now, consider the signal $s = a$ for which b becomes a nuisance parameter. (1 point)

- Work out an expression for the joint probability $\mathcal{P}(d, a)$, and calculate the corresponding Hamiltonian $H(d, a) = -\log \mathcal{P}(d, a) = \frac{1}{2} D^{-1} a^2 - j a + H_0$. You may drop H_0.
- Identify the information source j and the information propagator D.

e) Find the best position $\tilde{r}_a$ to estimate a irrespectively of b, by minimizing D from **d)** with respect to r (1 point).

f) Guess at which radius $\tilde{r}_b$ one should measure in order to obtain the most certain estimate for the parameter b. No justification required (1 point).

PD Dr. Torsten Enßlin
Margret Westerkamp
Johannes Harth-Kitzerow

Exercise 7 - 2

The goal of this exercise is to evaluate the integral

$$I(x) = \int_{-\infty}^{\infty} \mathrm{d}k \frac{e^{-ikx}}{k^2 + m^2} \tag{5}$$

a) For $k \in \mathbb{C}$, the integrand can diverge. Identify the poles of the integrand and sketch them in the complex plane, sketch the integration path of $k \in (-\infty, \infty)$ as well (2 points).

b) Draw a contour line that closes the integral, connecting $k = \infty$ to $k = -\infty$. The integral of the added contour line should be 0. Illustrate why it is 0 by comparing orders in k (2 points).

Hint: You may assume $x > 0$.

c) The closed integral encloses one pole. Identify its order and evaluate the integral $I(x)$ using the residue theorem,

$$\int_\Gamma f = (2\pi\, i) \sum_{a \in R_f} \mathrm{ind}_\Gamma(a)\ \mathrm{Res}_a f, \tag{6}$$

where Γ is a closed contour, R_f are the poles of f, and $\mathrm{ind}_\Gamma(a)$ is the number of counter-clockwise windings of Γ around a, and the residue formula,

$$\mathrm{Res}_a f = \frac{1}{(n-1)!} \lim_{z \to a} \frac{\partial^{n-1}}{\partial z^{n-1}} \left[(z-a)^n f(z) \right], \tag{7}$$

where n is the order of the pole a (2 points).

Hint: A clockwise winding is a negative counter-clockwise winding.

This exercise sheet will be discussed during the exercises.
group A, Wednesday 16:00 - 18:00 Theresienstr. 39 - room B 101,
group B, Thursday, 10:00 - 12:00, Geschw.-Scholl-Pl. 1 (B) - room B 015
group C, Thursday, 16:00 - 18:00, Theresienstr. 37 - room A 449
backup slot, Wednesday, 8:00 - 10:00, Theresienstr. 37 - A 450
(to be used whenever Thursday is a vacation day)
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