NON-LINEAR DYNAMICS AND CHAOS

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ABSTRACT

We explore several basic aspects of chaos, chaotic systems, and non-linear dynamics through three different setups. The logistic map, a canonical one-dimensional system exhibiting surprisingly complex and aperiodic behavior, is modeled as a function of its chaotic parameter. We consider maps of its phase space, and the progression through period-doubling bifurcations to the onset of chaos. The Feigenbaum ratio of successive bifurcation periods is estimated at \( \approx 4.674 \), in good agreement with the accepted value. The Lyapunov exponent, governing the exponential growth of small perturbations in chaotic systems, is calculated and its fractal structure compared to the corresponding bifurcation diagram for the logistic map. Using a non-linear p-n junction circuit we analyze the return maps and power spectrums of the resulting time series at various types of system behavior. Similarly, an electronic analog to a ball bouncing on a vertically driven table provides insight into real-world applications of chaotic motion. For both systems we calculate the fractal information dimension and compare with theoretical behavior for dissipative versus Hamiltonian systems.

Subject headings: non-linear dynamics; non-linear dynamical systems; fractal dimension; chaos; strange attractors; logistic map

1. INTRODUCTION

The investigation of chaos and chaotic systems is a relatively new area of research, with strong ties to the fields of physics and applied mathematics. Non-linear dynamics is the study of systems whose equations of motion contain non-linear dependence on their state variables. In fact, while chaos frequently occurs in non-linear systems, it can never occur in linear systems. Several basic examples, such as the driven damped harmonic oscillator, exhibit surprising behavior as they pass into the chaotic regime.

Theoretical simulation as well as direct experiment allow us to explore non-linear dynamics and chaos theory. We discuss the defining characteristics of chaotic systems in and the connection to fractal theory in §2. We model the logistic map, a simple one-dimensional system exhibiting chaotic behavior, in §3. In §4 and §5 we examine two chaotic circuits, one based on the non-linear relation between current and voltage, the other an electronic analog of a ball bouncing on a sinusoidally driven table. For all three systems we examine the period-doubling, bifurcation route to chaos. Where able we estimate the Lyapunov exponent, Feigenbaum ratio, and the information dimension of the attractors. We present a theoretical bifurcation diagram of the logistic map, as well as experimentally measured return maps of the two circuit systems and the power spectrum of the resultant time series at various chaotic periods. Finally, we comment on the differences between these three systems and the various characteristics of chaotic behavior that they exhibit.

2. CHAOTIC SYSTEMS

The complexity inherent in chaotic systems gives rise to the term chaos, which does not indicate complete disorder, as in everyday usage. Rather, chaos is the apparently random behavior of a system which is in fact deterministic. This means that the system has no inherent randomness or noise, and that the irregular behavior arises from its nonlinearity. Given a specific initial condition for a chaotic system, its behavior for all future time is well-defined and predictable. Several important characteristics serve to define a chaotic system. First, such systems are highly sensitive to initial conditions - two states with an arbitrarily small initial separation may have widely different final states, and perturbations from initial conditions grow exponentially. The phase space of a chaotic system is topologically mixing, such that any subset of phase space will eventually overlap with any other given subset. Phase space may contain structures such as regions of stability and points or regions of accumulation which are “strange” or otherwise quite complex. Finally, orbits of a chaotic system in phase space are by definition aperiodic.

The sensitivity to initial conditions is perhaps the most well known quality of chaos. It has been popularized by the so called “butterfly effect”, whereby a butterfly flapping its wings on one side of the globe may as a consequence cause a tornado halfway across the world. Similarly, the gravitational force of a player’s blinking eyelids on the billiard balls of a pool game introduces an exponentially growing uncertainty that prevents any accurate determination of the future state of the moving billiard balls after only a relatively short timescale.

2.1. Fractal Dimension

The phase space structures of chaotic systems, such as “strange” attractors, are aptly described by fractals. In general, a fractal is a complex geometric shape with details down to the smallest spatial scales. Often, fractals are self-similar, whereby a subset of a fractal, and any subset thereof, may resemble the fractal as a whole. Consequently, a measure of the area of a fractal is often
found empirically that \( \epsilon \) tractor inside some \( \epsilon \) this number for many different points dimension \( d \) values of \( r \) section, in phase space, of a given periodic orbit with a map or a return map. The \( \text{Poincaré} \) the evolution of the system, we can plot either a \( \text{Poincare} \) map, or a return map. The \( \text{Poincaré} \) map is the intersection, in phase space, of a given periodic orbit with a lower dimensional subspace, transverse to the flow. For example, the subspace may be a one-dimensional line through some band of a two-dimensional attractor. For an orbit whose initial conditions place it on the subspace, we can observe the point at which it successively crosses the subspace. Alternatively, a return map plots the state variables at some later time as a function of those same variables at an earlier time. In the case of the logistic map, we can plot \( x_{n+1} \) as a function of \( x_n \). We do this for several different values of the chaotic parameter \( r \) in Figure (1).

The limiting fractal is a generalization of the two-dimensional Cantor set. Although it can be shown that the Sierpinski Carpet has zero area, it is still useful to make some kind of determination of its dimension. One approach is to cover the set in boxes of size \( \epsilon \), determining the minimum number \( N(\epsilon) \) needed to do so. The box dimension \( d \) is then a measure of how \( N(\epsilon) \) depends on \( \epsilon \), and can be written

\[
d = \lim_{\epsilon \to 0} \frac{\ln N(\epsilon)}{\ln(1/\epsilon)} .
\]

Fractals are characterized by a non-integer box dimension. For example, we find that the Sierpinski Carpet has \( d = \ln 8/\ln 3 \simeq 1.89 \). In practice, the box dimension is difficult to calculate numerically and has several drawbacks. An alternative is the information, or correlation, dimension. If we fix a point \( r \) on an attractor in phase space, we count the number of points of the attractor inside some \( \epsilon \)-neighborhood of \( r \). The average of this number for many different points \( r \) is \( C(\epsilon) \), and it is found empirically that

\[
C(\epsilon) \propto \epsilon^d ,
\]

where \( d \) is the correlation dimension. This measure of dimension is sensitive to the density of points, unlike the box dimension. We estimate the fractal nature of both the PN junction and bouncing ball systems by numerically estimating \( d \) below.

3. LOGISTIC MAP

The canonical one-dimensional chaotic system is the so called \textit{logistic map}. The system is described by a single state variable \( x \in [0,1] \), whose evolution is governed by a discrete-time recursion relation:

\[
x_{n+1} = rx_n (1-x_n) .
\]

Here, \( r \) is called the \textit{chaotic parameter}, \( 0 < r < 4 \). For values of \( r \) in this range, the system exhibits drastically different long-term behavior (as \( n \to \infty \)). To visualize the evolution of the system, we can plot either a Poincaré map or a return map. The \textit{Poincaré map} is the intersection, in phase space, of a given periodic orbit with a lower dimensional subspace, transverse to the flow. For example, the subspace may be a one-dimensional line through some band of a two-dimensional attractor. For an orbit whose initial conditions place it on the subspace, we can observe the point at which it successively crosses the subspace. Alternatively, a return map plots the state variables at some later time as a function of those same variables at an earlier time. In the case of the logistic map, we can plot \( x_{n+1} \) as a function of \( x_n \). We do this for several different values of the chaotic parameter \( r \) in Figure (2).

We find that, if \( r < 1 \), then \( x \to 0 \) as \( n \to \infty \), independent of the starting position \( x_0 \). In such a situation \( x = 0 \) is called an \textit{attractor} of the system. In this case the attractor is a single point, although we will see chaotic systems in general exhibit much more complex attractors. The basin of attraction for \( x = 0 \), the set of all points that end up there as \( n \to \infty \), is in this case \( \forall x \in [0,1] \).

Increasing the chaotic parameter such that \( r > 1 \), we find that the system in general converges uniformly to some final point \( x_f \in [0,1] \). The point \( x_f \) is determined by \( r \) and is independent of the starting point \( x_0 \). For even higher values of the chaotic parameter a new behavior emerges. Instead of converging to a single point, the system oscillates between two or more stable points in the limit \( n \to \infty \). This behavior of a stable cycle between two endpoints is shown in the lower-left panel of Figure (2). The process of moving from one point attractor to a two point attractor, a two point attractor to a four point attractor, and so on, is known as a \textit{period doubling bifurcation}.

Slowly increasing the chaotic parameter we can see that bifurcations continue to occur, with ever increasing frequency, until \( r \simeq 3.56 \), and which point the system becomes chaotic. Instead of bouncing between a finite number of attractor points, the return map (lower-middle panel) exhibits aperiodic behavior, never returning to the same point twice. This chaotic behavior continues with increasing \( r \), except at some values for which windows of stability with stable limit cycles appear, as in the lower-right panel. To more quantitatively trace the final state behavior of the system, we can generate a bifurcation diagram, which shows all possible end states as a function of the chaotic parameter. The result is shown in Figure (3).

Here, the bifurcation route to chaos via a period-doubling cascade is clearly evident. The diagram confirms that successive bifurcations occur after smaller and smaller windows in \( r \), where the ratio between these bi-
Fig. 2.— Six return maps plotting $x_{n+1}$ as a function of $x_n$ for different values of the chaotic parameter $r$ and the initial point $x_0$. The first two maps take $r = 0.9$, and $x_0 = 0.5$, $x_0 = 0.8$, respectively. The upper-right exhibits uniform convergence at $r = 2.7$, $x_0 = 0.2$. The lower-left is a stable limit cycle of period one at $r = 3.36$, $x_0 = 0.2$. Center bottom shows fully chaotic behavior when $r = 3.9$ and $x_0 = 0.5$, and lower-right is a high order, stable limit cycle window after the onset of chaos, at $r = 3.83$ and $x_0 = 0.2$. The quadratic map and the line $y = x$ are shown in all six return maps as dashed lines.

The period doubling ratio is equal to the Feigenbaum constant $\delta$. Indeed, it has been shown that all one-dimensional systems with a single hump, like the logistic map, have this same period doubling ratio of $\approx 4.669$. Finally, several other striking features of the bifurcation diagram can be seen. For instance, bifurcation envelopes can still be seen tracing through the chaotic regions, and an odd, decreasing, number of stable limit cycles can be seen in the windows with increasing $r$.

3.1. Liapunov Exponent

We mentioned earlier that given a small perturbation to some initial state, the two resulting states will diverge exponentially in time. This is a manifestation of the sensitive dependence of chaotic systems on initial conditions. Suppose $r(t)$ is a point in phase space, and $r(t) + \delta(t)$ is a nearby point, the two separated by some initial length $|\delta_0| \ll 1$. It is found that

$$|\delta(t)| \approx |\delta_0|e^{\lambda t}, \quad (4)$$

where $\lambda$ is called the Liapunov exponent. If the long-term behavior is periodic (non-chaotic), $\lambda < 0$. If the long-term motion is non-periodic (chaotic), then $\lambda > 0$. In general, an $n$-dimensional system will have $n$ Liapunov exponents, where each denotes the length of a principal axis of some ellipsoid of phase space. The largest Liapunov exponent effectively controls the size of the ellipsoid, and the $\lambda$ in (4) is actually this value.

For a one-dimensional system such as the logistic map, the Liapunov exponent can be calculated as

$$\lambda = \lim_{n \to \infty} \left( \frac{1}{n} \sum_{i=0}^{n-1} \ln|f'(x_i)| \right) \quad (5)$$

where $f'$ denotes the first derivative of the logistic map function. Taking $n$ sufficiently large to ensure convergence, we can calculate the Liapunov exponent as a function of chaotic parameter $r$, as shown in Figure (4).

We note, by comparison with the bifurcation diagram, that points of period-doubling bifurcation correspond to local maxima in the Liapunov graph. By measuring four successive points, we estimate the first three Feigenbaum ratios in the limiting sequence as $\delta = 4.738, 4.721, 4.674, \ldots$. These appear to be converging to the accepted value of $\delta = 4.6692\ldots$, as expected for the logistic map.

4. PN JUNCTION

Having examined some of the basic ideas of chaotic dynamics in our simulations of the logistic map, we proceed with two experimental setups exhibiting chaotic behavior. The first is a driven damped oscillator circuit with a non-linear response. Here damped means dissipative, and so the system is non-Hamiltonian. A schematic diagram of the circuit is shown in Figure (5), which consists of a PN-junction connected in series with an inductor, resistor, and sinusoidally driving voltage oscillator.
Fig. 3.— Bifurcation diagram of the logistic map from \( r = 2.8 \) to \( r = 4.0 \), showing all possible final states \( f(x) \) at that point. We generate by calculating 1400 iterations per value of \( r \), discarding the first 400 as pre-iterates, and plotting the remaining values. Starting positions \( x_0 \) are selected randomly from a normal distribution between 0 and 1. Period doubling bifurcations, the chaotic regime, and windows of stability are all visible.

Fig. 4.— Liapunov exponent \( \lambda \) as a function of chaotic parameter \( r \). A high degree of fractal self-similarity is evident as \( \lambda \) approaches zero, the expected dividing point between the non-chaotic and chaotic regimes.

The diode has a non-linear current voltage relation, given by

\[
I_d(V_d) = I_0 \exp \left( eV_d/kT \right) - 1,
\]

(6)

where \( V_d \) is the voltage across the diode. It also has a non-linear capacitance, which can be written

\[
C(V_d) = \begin{cases} 
C_0 \exp \left( eV_d/kT \right) & V_d > 0 \\
\frac{C_0}{\sqrt{1-\exp(eV_d/kT)}} & V_d \leq 0
\end{cases}
\]

(7)

Alternatively, in order to apply the tools we developed when studying the logistic map, we can rewrite (6) and (7) as the equations of motion for the circuit, thereby moving from the time domain into the phase space of two variables, \( I \) and \( V_d \):

\[
\dot{I} = \frac{V_0 - RI - V_d}{L}
\]

(8)
\[ \dot{V}_d = \frac{I - I_d(V_d)}{C(V_d)} \]  

We use a computer to sample \( V_s(t) \) across the resistor, which is proportional to the time-varying current \( I(t) \). Plotted as a function of \( V_{05} \cos \theta \), we are able to observe a two-dimensional projection of the system’s path in phase space. The computer also controls the driving voltage and frequency, which are in effect the chaotic parameters. As in our theoretical simulation, we observe period-doubling bifurcations as the driving voltage is modulated between 0 and 10 V. Fixing the driving frequency at 47 kHz, and considering the projection of the \( (I, V) \) space on the oscilloscope, we see a single stable orbit at low voltages. The cycle bifurcates into period two, four, and eight orbits before becoming jittery and finally band-chaotic. Unfortunately, the voltage source was insufficient to make the system completely chaotic or reach any windows of stability. At four critical chaotic parameter values we recorded 8192 samples at 30 kHz using the ADC card on the computer. From this time series the return map and power spectrum were constructed at each value, and are shown in Figures (10)-(11), in the appendix.

Several interesting features are evident in these figures. First, as the driving voltage is increased, harmonic content at integer multiples of the driving frequency (i.e. \( 2\omega_0, 3\omega_0, \ldots \)) emerges. After the first bifurcation, the system produces subharmonic content at \( \omega_0/2 \) and similar. Indeed, we expect subharmonic content at \( \omega_0/2^n \), \( n \in \mathbb{Z}^+ \), up until the onset of chaos.

As the orbits become chaotic, the peaks of the power spectrum subside into the background noise. It is important to note that this “noise” is not e.g. thermal or statistical. Rather, it is a consequence of the non-linearity of the system. Previous work has found that the presence of noise in such a driven PN-junction circuit effectively broadens the lowest subharmonics (Perez, 1982) until they are no longer evident. In our case, the resolution of our frequency spectrum is limited by the sampling equipment, although the qualitatively expected behavior is present.

As with the logistic map, we would like to construct a quantitative bifurcation diagram for the circuit. In order to do this, we need to sample synchronously with the voltage source. This requires the use of the strobe pulse generator, which is unfortunately not working at present. Alternatively, we can use internal timing on the ADC, by which we can observe only the envelope containing the final states. For example, during a stable period two orbit, instead of two discrete lines tracing through the chaotic parameter, the state space between these two lines is also filled in. With this caveat in mind, we can still observe the expected bifurcation behavior - the result is plotted in Figure (6).

Finally, we would like to consider the information dimension previously discussed in §2.1. Plotting \( \log_2 C(N, \epsilon) \) vs. \( 2\log_2 \epsilon \), and finding the best-fit slope to the linear region, \( d \) is then given by twice this value. The region of interest of this plot is shown for the four previous driving voltages of \( A = 4.31, A = 5.84, A = 7.84, \) and \( A = 8.80 \) in Figure (7).

Regions before and after those shown in Figure (7) are excluded as they are both straight lines. The curve saturates at large \( \epsilon \) since the neighborhood encloses the entire attractor, while as \( \epsilon \rightarrow 0 \), the neighborhood can contain only the single point it is centered on. The dimension results are given in Table (1) for each of the four chaotic parameters \( A \), along with the qualitative description of the attractor type at that point. We note that the information dimension \( d \) increases monotonically with \( A \),

![Fig. 6. — Experimentally measured bifurcation diagram of the PN-junction circuit over the full range of driving voltages. Due to the lack of the external timing provided by the strobe pulse generator, the sampling is out of phase with the voltage source and exhibits several undesired side-effects.](image)

![Fig. 7. — Information dimension calculation for the PN-junction circuit. Plotting a logarithmic measure of the bulk of the attractor as a function of the logarithmic set size \( \epsilon \) used as a measurement neighborhood, we can calculate \( d \) as twice the slope of the relation in the linear regime.](image)

<table>
<thead>
<tr>
<th>Parameter ( A )</th>
<th>Dimension ( d )</th>
<th>Attractor Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.31</td>
<td>1.17</td>
<td>Period Two Cycle</td>
</tr>
<tr>
<td>5.84</td>
<td>1.35</td>
<td>Period Four Cycle</td>
</tr>
<tr>
<td>7.84</td>
<td>1.55</td>
<td>Period Eight Cycle</td>
</tr>
<tr>
<td>8.80</td>
<td>1.67</td>
<td>Band-Chaotic Regime</td>
</tr>
</tbody>
</table>
and is largest when the system is chaotic. We can compare this result with the second experimental circuit, discussed below.

5. BOUNCING BALL

Our second circuit simulates the vertical position $x(t)$ and the velocity $v(t)$ of a ball bounding on a sinusoidally driven table, under the influence of gravity. The equations of motion for such as system are well known, and can be formulated mechanically or via an electronic analog. If $v_N$ is the ball velocity immediately following the $N$th impact, and $\Phi_N$ is the phase of the table, the values immediately following the $(N+1)\text{th}$ collision are given by the coupled recursive relations

\begin{align}
\Phi_{N+1} &= \Phi_N + \omega t_N \\
v_{N+1} &= K (gt_N - v_N) + x_0 \omega (1 + K) (\Phi_N + \omega t_N)
\end{align}

where $K$ is the coefficient of restitution. The electrical analog is established using op-amps and a feedback current, similar to that shown in Figure (8). Here, $V_a(t)$ and $dV_a/dt$ are the equivalents of the ball position and velocity, respectively. More extensive details can be found in Zimmerman, 1991, or Mello, 1985.

For small driving amplitudes, we expect the ball to remain fixed on the table surface. As the oscillations increase, however, the position and velocity of the ball as a function of time become more complex, and eventually chaotic. The orbit in $(x, v)$ phase space begins as an ellipse, becoming jittery and exhibiting period-doubling bifurcations, just as in the PN-junction circuit. To consider the system’s behavior as a function of driving parameters, we again take representative samples and create the associated return maps and power spectrums of the sampled time series. The results are presented in Figures (12)-(14) in the appendix. The final plot in the sequence represents the fully chaotic regime, and as expected, the power spectrum shows no strong peaks, but rather a continuous chaotic “noise”. Frequency peaks for states after successive bifurcations are closer and closer together, until they finally merge into this noise appearance.

For low driving voltages, the system follows a single trajectory through phase space. Since the ball position mirrors the sinusoidal driving voltage, the return map is an ellipse. The period-doubling bifurcations, unlike the PN-junction, do not necessarily exhibit a continuous change in frequency. That is, the period two cycle has two different frequencies even slightly after its bifurcation from the stable ellipse. This may be due to the dissipative nature of the system. The level of dissipation of the system due to impacts between the ball and the table with a nonzero coefficient of restitution $K$ is governed by the resistance $R_f$ in the circuit. The system may be energy-conserving (Hamiltonian) for certain configurations. In such a case, we would expect qualitatively different behavior between the bouncing ball and say the logistic map. However, we believe that the experimental circuit models inelastic, though instantaneous, collisions. We do not vary the $K$ parameter from its fixed value.

Unfortunately, the issues with the strobe pulse generator prevented us from measuring a bifurcation diagram for the bouncing ball circuit. However, we were still able to calculate the information dimension as before. The results are given in Table (2).

![Diagram of the bouncing ball electronic circuit.](image)

![Information dimension calculation for the bouncing ball circuit.](image)

As with the PN-junction, there seems to be a general correlation between high dimension and chaotic behavior. This makes sense, as $d$ is in some sense a measure of the bulk of the attractor as a function of the logarithmic set size $\epsilon$ used as a measurement neighborhood, we can calculate $d$ as twice the slope of the relation in the linear regime.

![TABLE 2 BOUNCING BALL INFORMATION DIMENSION](image)
system we known keeps phase space volumes invariant with time. Equivalently, a Hamiltonian system can have no attractors, and all orbits are inherently stable. That is, the equations of motion are time-reversible, and orbits return arbitrarily close to their initial points. Such a system with \( n \) degrees of freedom has a \( 2n \)-dimensional phase space, assuming no other external constraints, and can become chaotic only if \( n \) is strictly greater than one. Finally, Hamiltonian systems can only have certain types of fractal structures, and so have fundamentally different dimension than dissipative systems.

6. CONCLUSION

We have explored the defining characteristics and behavioral signatures of chaotic systems and non-linear dynamics through the logistic map, the PN-junction circuit, and the bouncing ball electronic analog. The fractal information dimension was calculated for both of the experimental circuits, and was found to be correlated with the state of the system. We estimated the Feigenbaum period-doubling bifurcation ratio using the logistic map simulation, and found it to be in agreement with published results. By sampling the return map and frequency content we were able to observe the bifurcating cascade of stable limit cycles and the onset of chaos. Plotting the bifurcation diagram allowed us to quantify the states available to each system as a function of their chaotic parameter(s). This lab introduces us to many of the basic ideas of chaos, a topic which is typically not approached at the undergraduate level, and for which there is a wealth of ongoing activity and research.

Acknowledgements. The author would like to thank his partner, Joey Cheung, our GSI Ben MacBride, lab director Don Orlando, and professors K. Luk and D. Budker for their invaluable assistance.

REFERENCES


Fig. 10.— (Left) Response of the PN-junction circuit at chaotic parameter $A = 4.31$ and the associated power spectrum. At this driving voltage, the system is stable in a period two limit cycle. (Right) Response of the PN-junction circuit at chaotic parameter $A = 5.84$ and the associated power spectrum. At this driving voltage, the system is stable in a period four limit cycle.
Fig. 11.— (Left) Response of the PN-junction circuit at chaotic parameter $A = 7.84$ and the associated power spectrum. At this driving voltage, the system is stable in a period eight limit cycle. (Right) Response of the PN-junction circuit at chaotic parameter $A = 8.80$ and the associated power spectrum. At this driving voltage, the system dominated by chaotic bands which have yet to fully merge into fully chaotic behavior.
Fig. 12.— (Left) Response of the bouncing ball circuit at chaotic parameter $A = 2.5$ and the associated power spectrum. At this driving voltage, the system is stable with an elliptical orbit in phase space. (Right) Response of the bouncing ball circuit at chaotic parameter $A = 3.9$ and the associated power spectrum. The system orbits in a deformed period one cycle.
Fig. 13.— (Left) Response of the bouncing ball circuit at chaotic parameter $A = 4.5$ and the associated power spectrum. At this driving voltage, the system has undergone the first bifurcation, and oscillates in a stable period two limit cycle. (Right) Response of the bouncing ball circuit at chaotic parameter $A = 4.87$ and the associated power spectrum. The system is in a stable period four limit cycle.
Fig. 14.— (Left) Response of the bouncing ball circuit at chaotic parameter $A = 6.3$ and the associated power spectrum. At this driving voltage, the system has passed through the initial chaotic regime and is in a window of stability, undergoing stable oscillations in a period two limit cycle. (Right) Response of the bouncing ball circuit at chaotic parameter $A = 8.7$ and the associated power spectrum. At this point, the system exhibits fully chaotic behavior.