Analysis of Wilberforce Pendulum

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Abstract

The report summarizes the detailed analysis of a tuned Wilberforce pendulum (Ref-[2]). In the initial sections the normal modes and coordinates are reproduced theoretically and numerically and are in agreement with Berg and Marshall(1990). In the subsequent sections we present a detailed stability analysis for damped and undamped systems using linearization technique. We find that origin is the only fixed point in both the situations and that too a universal sink. The critical values of b and k are also found numerically which determine the underdamped and overdamped situation. We also present our theoretical and numerical calculation for the damped Wilberforce pendulum. All the numerics have been performed using RK-4 scheme in MATLAB.

1 Analysis of the Damped Wilberforce Pendulum

We analyse the motion of the damped Wilberforce pendulum. The equations are no longer solvable analytically and hence we find the solution numerically using MATLAB. We will consider the simplistic case of damping which linearly depends on the velocity (and angular velocity). Since damping term is a non-conservative force, there is no potential associated with it. Hence it can not be incorporated as an energy function in the Lagrangian (L) of the system. But we can include it in Euler-Lagrange equation of motion, as follows-

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial D}{\partial q_j} = 0 \tag{1}$$

where $(q_i, \dot{q}_i)$ are the generalized coordinates, D is the term which incorporates the damping force and $i, j = 1, 2, 3, .., N$, and $i \neq j$.

Now introduce the damping term as follows-

$$Translational\ Motion : F_{damping} = -a_1\dot{z}$$

$$Rotational\ Motion : F_{damping} = -a_2\dot{\theta}$$

where $a_1$and a_2 are the damping constants.

Now the Euler-Lagrange E.O.Ms are as follows -

$$\ddot{z} + b\dot{z} + \omega_o^2 z + \frac{\epsilon}{2m}\theta = 0 \tag{2}$$

$$\ddot{\theta} + k\dot{\theta} + \omega_o^2\theta + \frac{\epsilon}{2I}z = 0 \tag{3}$$

where $b = \frac{a_1}{m}$ and $k = \frac{a_2}{I}$.

Writing z in terms of θ from equation (3) with $k = 0$ and simplifying equation (2), we get the following differential equation-

$$\frac{d^4\theta}{dt^4} + b\frac{d^3\theta}{dt^3} + 2\omega_o^2\frac{d^2\theta}{dt^2} + b\omega_o^2\frac{d\theta}{dt} + (\omega_o^4 - \frac{\epsilon^2}{4mI})\theta = 0 \tag{4}$$

Assuming periodic solution, $\theta(t) = Ae^{\alpha t}e^{i\omega t}$ from equation (4) we get-

$$g(\omega) = (\alpha+i\omega)^4+b\cdot(\alpha+i\omega)^3+2\omega_o^2\cdot(\alpha+i\omega)^2+b\omega_o^2\cdot(\alpha+i\omega)+(\omega_o^4-\frac{\epsilon^2}{4mI}) = 0 \tag{5}$$

Equation (14) will be satisfied only when-

$$Re(g(\omega)) = 0 \ \& \ Im(g(\omega)) = 0$$

where Re and Im denote the real and imaginary part of $g(\omega)$.

$$Re(g(\omega)) = 0 \Rightarrow \omega^4-(6\alpha^2+3\alpha b-2\omega_o^2)\cdot\omega^2+(\alpha^4+b\alpha^3-2\omega_o^2\alpha^2+b\omega_o^2\alpha+\omega_o^4-\frac{\epsilon^2}{4mI}) = 0 \tag{6}$$

$$Im(g(\omega)) = 0 \Rightarrow 4\alpha^3\omega + 3\alpha^2 b\omega - \alpha\cdot(4\omega^3 + 4\omega\omega_o^2) + b\omega_o^2\omega - b\omega^3 = 0 \tag{7}$$

Equations are too complex to solve simultaneously for $\omega$and α. Using Mathematica solution to equation (15) (which is a quadratic equation in ω^2) gives

$$\omega_{1,2}^2 = \frac{1}{2}\left[6\alpha^2 + 3b\alpha + 2\omega_o^2 \pm \sqrt{32\alpha^4 + 32\alpha^3 b + 16\alpha^3\omega_o^2 + 9\alpha^2 b^2 + 8\alpha b\omega_o^2 + \frac{\epsilon^2}{4mI}}\right] \tag{8}$$

But the equation (7) can not be solved for α in terms of constants. Please note that if we put $b = \alpha = 0$ which is the undamped case we exactly recover the normal modes for the undamped Wilberforce pendulum, confirming that our solution is correct. Now, we can guess that $\alpha < 0$ as the amplitudes in both z and θ should decrease with time in damped situation (see Figure 2). Also in

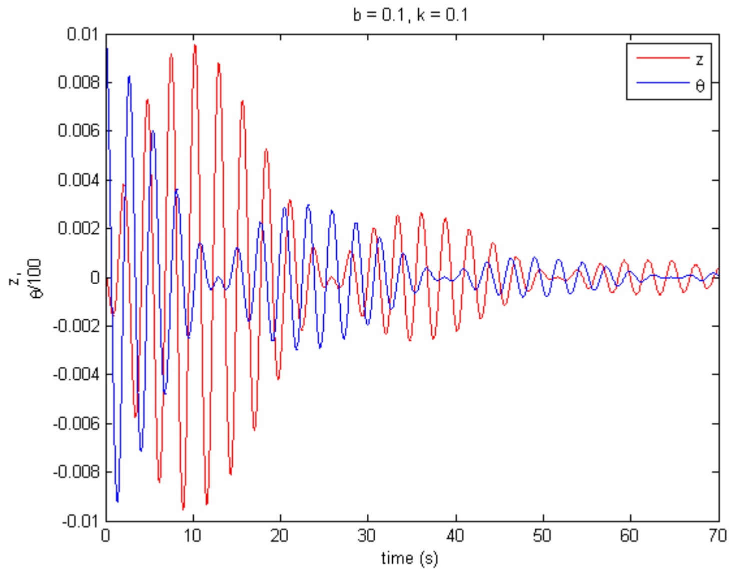

Figure 1: Solution for the coordinates z and θ (underdamped situation)

this case energy of the system will no longer be constant, in fact it will decrease exponentially (see Figure 3). The beat phenomena (underdamped) (see Figure 1) is observed in this case also but with small values b and k. Following table[1] summarizes the underdamped and overdamped situation.

$b(s^{-1})$	$k(s^{-1})$	Beat	Situation
0	0	Yes	Undamped
0.1	0.1	Yes	Underdamped
0.4	≥ 0	No	Overdamped
< 0.4	< 0.1	Yes	Underdamped
≥ 0	0.4	No	Overdamped
> 0.25	> 0.25	No	Overdamped

[1] These are some typical values for b and k.

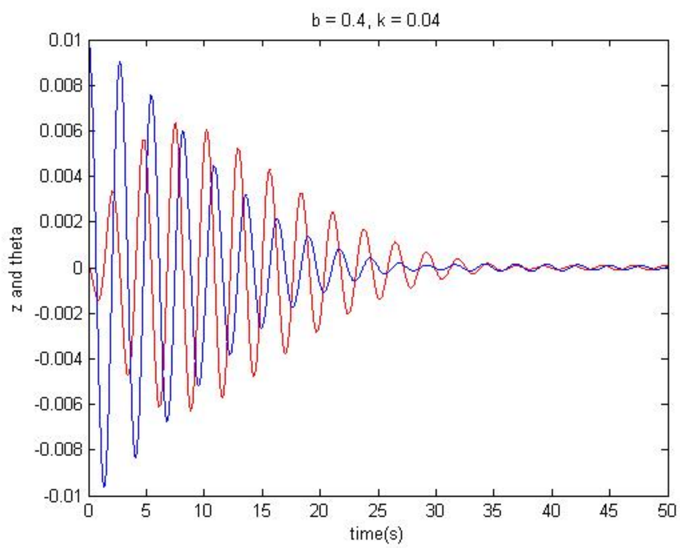

Figure 2: Solution for the coordinates z and θ (overdamped situation)

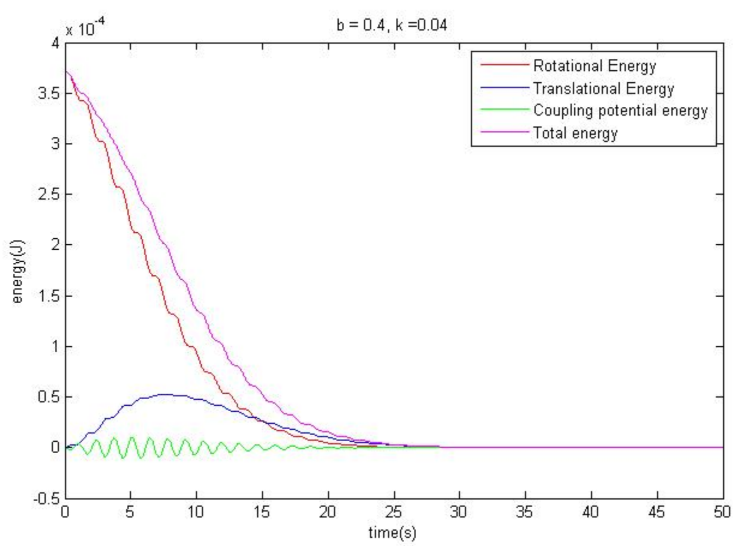

Figure 3: Energies as functions of time (damped situation)

1.1 Stability Analysis

Following the same linearization technique as in the case of undamped one-

We write the equation of the motions in terms of functions with several variables as follows[2]-

$$\frac{d\dot{z}}{dt} = -\omega_o^2 z - b\dot{z} - \frac{\epsilon}{2m}\theta = f_1(z, \theta, x, y) \tag{9}$$

$$\frac{d\dot{\theta}}{dt} = -\omega_o^2 \theta - k\dot{\theta} - \frac{\epsilon}{2I} z = f_2(z, \theta, x, y) \tag{10}$$

$$\frac{dz}{dt} = \dot{z} = x = f_3(z, \theta, x, y) \tag{11}$$

$$\frac{d\theta}{dt} = \dot{\theta} = y = f_4(z, \theta, x, y) \tag{12}$$

To find the fixed points,

$$f_1(z^*, \theta^*, x^*, y^*) = f_2(z^*, \theta^*, x^*, y^*) = f_3(z^*, \theta^*, x^*, y^*) = f_4(z^*, \theta^*, x^*, y^*) = 0$$

where, $(z^*, \theta^*, x^*, y^*)$ is 4-tuple fixed point in four-dimensional hyperspace.

$$f_1 = 0 \Rightarrow \omega_o^2 z^* + b\dot{z}^* + \frac{\epsilon}{2m}\theta^* = 0 \tag{13}$$

$$f_2 = 0 \Rightarrow \omega_o^2 \theta^* + k\dot{\theta}^* + \frac{\epsilon}{2I} z^* = 0 \tag{14}$$

$$f_3 = 0 \Rightarrow x^* = \dot{z}^* = 0 \tag{15}$$

$$f_4 = 0 \Rightarrow y^* = \dot{\theta}^* = 0 \tag{16}$$

Solving equations (13) & (14) we get-

$$z^* = \theta^* = 0 \text{ or } \omega_o^4 = \tfrac{\epsilon^2}{4mI}$$

But from our definition of $\omega_o^2 = \frac{k_s}{m} = \frac{\delta}{I}$ we know that, $\omega_o^4 = \frac{k_s \delta}{mI} \neq \frac{\epsilon^2}{4mI}$ and therefore the only values which satisfy the equations (5) & (6) simultaneously are $z^* = \theta^* = 0$. Hence the only fixed point for this system is -

$$(z^*, \theta^*, x^*, y^*) = (0, 0, 0, 0)$$

Now the Jacobian matrix for this fixed point can be written as-

$$J(0,0,0,0) = \begin{pmatrix} \frac{\partial f_1}{\partial z} & \frac{\partial f_1}{\partial \theta} & \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial z} & \frac{\partial f_2}{\partial \theta} & \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \\ \frac{\partial f_3}{\partial z} & \frac{\partial f_3}{\partial \theta} & \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} \\ \frac{\partial f_4}{\partial z} & \frac{\partial f_4}{\partial \theta} & \frac{\partial f_4}{\partial x} & \frac{\partial f_4}{\partial y} \end{pmatrix}_{(0,0,0,0)} = \begin{pmatrix} -\omega_o^2 & -\frac{\epsilon}{2m} & b & 0 \\ -\frac{\epsilon}{2I} & -\omega_o^2 & 0 & k \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

[2] The coordinates have usual SI units such as z in (m), θ in (rad), x in $(\frac{m}{s})$ and y in $(\frac{rad}{s})$

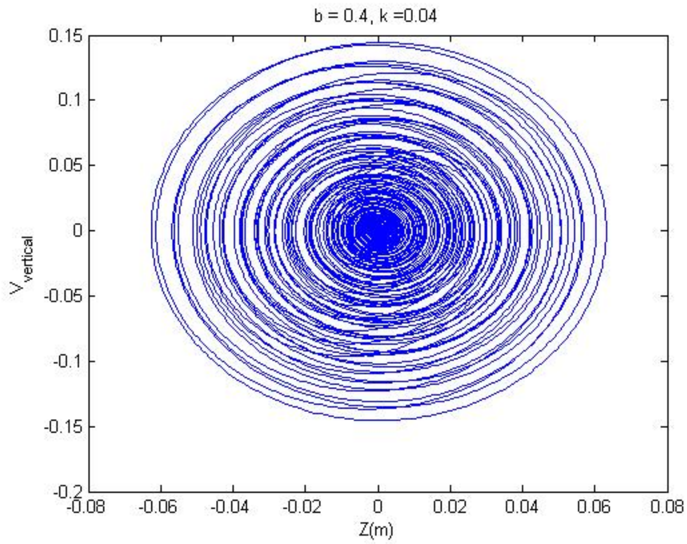

Figure 4: Phase portrait for the Translational motion (damped)

To find the stability of this fixed point we need to calculate the eigenvalues of this matrix. The following expression will give the eigenvalues of this matrix-

$$det(J(0,0,0,0) - \lambda I) = 0$$

where, λ denotes the eigenvalues for this matrix and I is 4 x 4 identity matrix and $det(\cdot)$ denotes the determinant of the matrix. This gives-

$$det \begin{pmatrix} -\omega_o^2 - \lambda & -\frac{\epsilon}{2m} & b & 0 \\ -\frac{\epsilon}{2I} & -\omega_o^2 - \lambda & 0 & k \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 0$$

Solving this we get following eigenvalues and eigenvectors-

Eigenvalues (λ)	Eigenvectors (ν)
$1 > 0$	$\nu_1 = (\frac{\epsilon k}{2m(-1+\frac{\epsilon}{2m}k-2\omega_o^2-\omega_o^4)}, \frac{-k(1+\omega_o^2)}{(-1+\frac{\epsilon}{2m}k-2\omega_o^2-\omega_o^4)}, 0, 1)$
$1 > 0$	$\nu_2 = (\frac{b(1+\omega_o^2)}{2m(-1+\frac{\epsilon}{2m}k+2\omega_o^2+\omega_o^4)}, \frac{bk}{2I(-1+\frac{\epsilon}{2m}k-2\omega_o^2-\omega_o^4)}, 1, 0)$
$-\sqrt{\frac{\epsilon^2}{4mI}} - \omega_o^2 < 0$	$\nu_3 = (\sqrt{\frac{I}{m}}, 0, 0, 0)$
$\sqrt{\frac{\epsilon^2}{4mI}} - \omega_o^2 < 0$	$\nu_4 = (-\sqrt{\frac{I}{m}}, 0, 0, 0)$

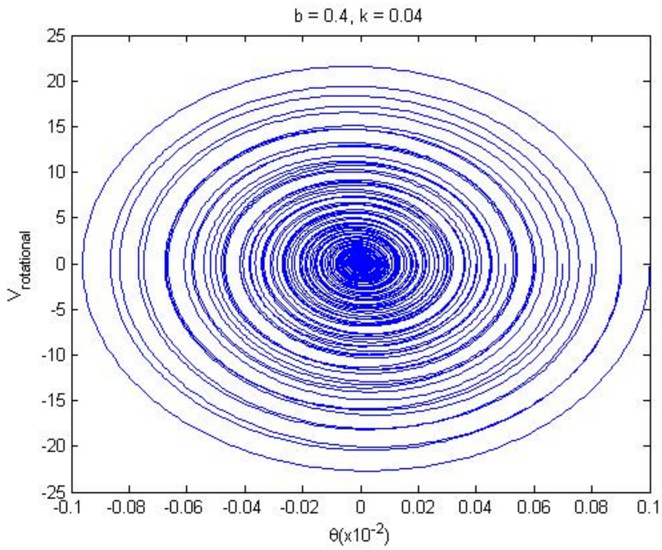

Figure 5: Phase portrait for the Rotational motion (damped)

Form the above table we can easily compare the eigenvalues and eigenvectors of damped system with undamped system. The only difference comes in the first two eigenvectors. The flowlines here also are repelling from the fixed point in the directions ν_1 and ν_2 and attracting in the directions ν_3 and ν_4 in 4-dimension phase portrait. In the damped case also the origin is a universal sink and we can clealry observe the spirals (as z & θ decay in time in this case). The separate phase portrait for translational and rotational are plotted numerically in MATLAB (See Figure 4 & Figure 5).

2 Table for Parameters[3]

Parameters	Values (units)
Coupling constant: ϵ	$9.27 \times 10^{-3} N$
Natural frequency: ω_o	$2.31 rad/s$
Mass: m	$0.49 kg$
Moment of Intertia: I	$1.29 \times 10^{-4} kg - m^2$

3 Acknowledgement

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[3]Values taken from Ref-[1].

4 References

[1] Robert Berg and Todd Marshall ,"*Wilberforce pendulum oscillations and normal modes*" (1990).
[2] L.A. Wilberforce, "*On the vibrations of a loaded spiral spring*" (1894).
[3] Ulrich Köpf, "*Wolberforce pendulum revisited*" (1989).
[4] Matthew Mewes, "*The Slinky Wilberforce pendulum: A simple coupled oscillator*" (2013).
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