General Relativistic versus Newtonian: a universality in radiation hydrodynamics

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We compare Newtonian and general relativistic descriptions of the stationary accretion of self-gravitating fluids onto compact bodies. Spherical symmetry and thin gas approximation are assumed. Luminosity depends, amongst other factors, on the temperature and the contribution of gas to the total mass, in both – general relativistic (L_{GR}) and Newtonian (L_N) – models. We discover a remarkable universal behaviour for transonic flows: the ratio of respective luminosities L_{GR}/L_N is independent of the fractional mass of the gas and depends on asymptotic temperature. It is close to 1 in the regime of low asymptotic temperatures and can grow by one order of magnitude for high temperatures. These conclusions are valid for a wide range of polytropic equations of state.

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I. INTRODUCTION

The Newtonian description of accretion has many advantages over the general relativistic modelling; it is much simpler conceptually, analytically and numerically. Therefore it is of interest to specify limits of the validity of Newtonian models. It is already known, that they can fail in the accretion of transonic test fluids [1]-[2]. It is not suprising that the Newtonian description is not suitable for a class of transonic flows in radiation hydrodynamics. But it is suprising that there emerges a universality unknown in the existing literature. We prove that the ratio L_{GR}/L_N of general relativistic L_{GR} and Newtonian L_N luminosities is sensitive only to the asymptotic temperature, assuming the thin gas approximation and some additional conditions.

Models considered here constitute extensions – Newtonian and general relativistic – of the classical Bondi model [3] to radiation hydrodynamics. They have been formulated by several authors (incomplete list includes [4] – [10]; see also references therein). We make a number of assumptions — spherical symmetry, a polytropic equation of state and the thin gas approximation in the transport equation [11]. It is assumed that the accretion is quasi-stationary.

There are two kinds of possible general relativistic effects. The first is related to the backreaction – and that includes self-gravity and dependence on the fractional mass content (defined later as $1 - y$) of the accreting gas – and the other is related to the asymptotic speed of sound a_∞ (we shall occasionally use the term "asymptotic temperature"; both quantities are proportional). The same boundary conditions are assumed in both models. Each of the two quantities, L_{GR} and L_N, separately depends on y and a_∞, but their ratio L_{GR}/L_N depends only on the asymptotic temperature. This universality means that the enhancement of L_{GR}/L_N can be found by solving the accretion of test fluids in hydrodynamics without radiation. This is a much simpler – algebraic – problem than the original one. These facts are valid for polytropic equations of state $p = K\rho_0^\Gamma$ with $1 < \Gamma < 5/3$.

The order of the rest of this paper is as follows. Section II discusses notation and equations. Section III shows how the validity of the thin gas approximation constrains the choice of boundary data. We describe the form of boundary data for the accretion problem. Section IV discusses the notion of sonic points and transonic flows in radiation hydrodynamics. In the first part of Section V we formulate a low-radiation condition and prove the universality of L_{GR}/L_N. The luminosity of hot accreting gases in a general relativistic model can be much higher than that given by the related Newtonian counterpart. In the second part of Section V we find a sector of luminous transonic flows which again gives a universal ratio, but now the asymptotic temperature is low and $L_{GR}/L_N \approx 1$. Section VI describes the numerical scheme that is used in this work, gives the equation of state and details numerical values for most of boundary data. Section VII reviews numerical results. The last Section summarizes the main conclusions.

II. FORMALISM AND EQUATIONS

A. General relativistic accretion.

The metric

$$ds^2 = -N^2 dt^2 + \hat{a} dr^2 + R^2 \left(d\theta^2 + \sin^2(\theta) d\phi^2\right) \quad (1)$$

uses comoving coordinates $t, r, 0 \leq \theta \leq \pi, 0 \leq \phi < 2\pi$: time, coordinate radius and two angle variables, respectively. R denotes the areal radius and N is the lapse. The radial velocity of gas is given by $U = \frac{1}{N}\frac{dR}{dt}$. We assume relativistic units with $G = c = 1$.

The energy-momentum tensor reads $T_{\mu\nu} = T^B_{\mu\nu} + T^E_{\mu\nu}$, where the baryonic part is given by $T^B_{\mu\nu} = (\rho + p) U_\mu U_\nu + p g_{\mu\nu}$ with the time-like normalized four-velocity U_μ, $U_\mu U^\mu = -1$. The radiation part $T^E_{\mu\nu}$ possesses only four non-zero components, $T^{0E}_0 \equiv -\rho^E = -T^{rE}_r$ and

$T^E_{r0} = T^E_{0r}$. A comoving observer would measure local mass densities, the material density $\rho = T^{B\mu\nu} U_\mu U_\nu$ and the radiation density ρ^E, respectively. The baryonic current reads $j^\mu \equiv \rho_0 U^\mu$, where ρ_0 is the baryonic mass density. Its conservation is expressed by the equation

$$\nabla_\mu j^\mu = 0. \tag{2}$$

Let n_μ be the unit normal to a coordinate sphere lying in the hypersurface $t = const$ and let k be the related mean curvature scalar, $k = \frac{R}{2}\nabla_i n^i = \frac{1}{\sqrt{\hat{a}}}\partial_r R$. The quantity $j = U_\mu n^\nu N T^{\mu E}_\nu / \sqrt{\hat{a}} = N T^{0E}_r / \sqrt{\hat{a}}$ is interpreted as the comoving radiation flux density. We assume the polytropic equation of state $p = K\rho_0^\Gamma$, with constants K and Γ. The internal energy density h and the rest and baryonic mass densities are related by $\rho = \rho_0 + h$, where $h = p/(\Gamma - 1)$.

There are four conservation equations that originate from the contracted Bianchi identities, $\nabla_\mu T^{\mu B}_\nu = -\nabla_\mu T^{\mu E}_\nu = F_\nu$ (here $\nu = 0, r$). The radiation force density F_ν describes the interaction between baryons and radiation. This formulation of general relativistic radiation hydrodynamics agrees with that of Park [7], Miller and Rezzola [8] and (on the fixed, Schwarzschildean, background) Thorne *et. al* [6].

One can find the mean curvature k from the Einstein constraint equations $G_{\mu 0} = 8\pi T_{\mu 0}$ ([1], [12])

$$k = \sqrt{1 - \frac{2m(R)}{R} + U^2}, \tag{3}$$

where $m(R)$ is the quasilocal mass,

$$m(R) = M - 4\pi \int_R^{R_\infty} dr r^2 \left(\rho + \rho^E + \frac{Uj}{k}\right). \tag{4}$$

The integration in (4) extends from R to the outer boundary R_∞ of the ball of gas. Its external boundary is connected to the Schwarzschild vacuum spacetime by a transient zone of a negligible mass. Thus the asymptotic mass M is approximately equal to $m(R_\infty)$.

In the polar gauge foliation one has a new time $t_S(t, r)$ with $\partial_{t_S} = \partial_t - NU\partial_R$. The quantity $4\pi N k R^2 \left(j\left(1 + \left(\frac{U}{k}\right)^2\right) + 2U\rho^E/k\right)$ is the radiation flux measured by an observer located at R in coordinates (t_S, R). One can show that

$$\partial_{t_S} m(R) = -4\pi N k R^2 \left(j\left(1 + \left(\frac{U}{k}\right)^2\right) + 2\rho^E \frac{U}{k}\right) - 4\pi N U R^2 (\rho + p). \tag{5}$$

The mass contained in the annulus (R, R_∞) changes if the fluxes on the right hand side, one directed outward and the other inward, do not cancel. The local baryonic flux reads $\dot{M} = -4\pi U R^2 \rho_0$, it is not constant ($\partial_R \dot{M} \neq 0$) and its boundary value reads $\dot{M}_\infty$.

The accretion process is said to be stationary (or quasi-stationary) if all relevant physically observables, that are measured at a fixed areal radius R, remain approximately constant during time intervals much smaller than the runaway instability time scale $T = M/\dot{M}_\infty$. That means that $\partial_{t_S} X \equiv (\partial_t - NU\partial_R)X = 0$ for $X = \rho_0, \rho, j, U \ldots$.

The above assumptions imply that in the thin gas approximation $F_0 = 0$ and the radiation force density has only one non-zero component $F_r = \kappa k \sqrt{\hat{a}} \rho_0 j$ [10]. Baryons and radiation interact through the elastic Thomson scattering. κ is a material constant, in standard units $\kappa = \sigma / (m_p c)$ and c, σ and m_p are respectively the speed of light, the Thomson cross section and the proton mass.

The full system of equations in a form suitable for numerics has been obtained in [10]. It consists of:

i) the total energy conservation

$$\dot{M} N \frac{\Gamma - 1}{\Gamma - 1 - a^2} + 2\dot{M} N \frac{\rho^E}{\rho_0} = 4\pi R^2 j N k \left(1 + \frac{U^2}{k^2}\right) + C; \tag{6}$$

the constant C is the asymptotic energy flux inflowing through the sphere of a radius R_∞.

ii) The local radiation energy conservation

$$\left(1 - \frac{2m(R)}{R}\right) \frac{N}{R^2} \frac{d}{dR}\left(R^2 \rho^E\right) = -\kappa k^2 N j \rho_0 + 2N\left(U\rho^E - kj\right)\frac{dU}{dR} + 2k\left(jU - k\rho^E\right)\frac{dN}{dR} + 8\pi N R \left(j^2 - j\rho^E \frac{U}{k}\right). \tag{7}$$

iii) The relativistic Euler equation (below $a = \sqrt{\frac{dp}{d\rho}}$ is the speed of sound)

$$\frac{d}{dR}\ln a^2 = -\frac{\Gamma - 1 - a^2}{a^2 - \frac{U^2}{k^2}} \times \left[\frac{1}{k^2 R}\left(\frac{m(R)}{R} - 2U^2 + 4\pi R^2 \left(\rho_E + p + j\frac{U}{k}\right)\right) - \kappa j \left(1 - \frac{a^2}{\Gamma - 1}\right)\right]. \tag{8}$$

iv) The baryonic mass conservation

$$\frac{dU}{dR} = -\frac{U}{\Gamma - 1 - a^2}\frac{d}{dR}\ln a^2 - \frac{2U}{R} + \frac{4\pi R j}{k}. \tag{9}$$

v) The equation for the lapse

$$\frac{dN}{dR} = N\left(\kappa j \frac{\Gamma - 1 - a^2}{\Gamma - 1} + \frac{d}{dR}\ln\left(\Gamma - 1 - a^2\right)\right). \tag{10}$$

Equations (3), (4) and (6 – 10) give the complete model used in numerical calculations.

The asymptotic data for the accretion must satisfy several physical conditions. We assume the inequalities $a^2_\infty \gg M/R_\infty \gg U^2_\infty$ ensuring, as demonstrated by Karkowski et al. [9] and Mach et al. [13], that the assumption of stationary accretion is reasonably well satisfied. They are probably required by the demand of stability (see a discussion in [9] and studies of stability of accreting flows in Newtonian hydrodynamics [13]). In the asymptotic region $j_\infty \approx \rho^E_\infty$ and the total luminosity is well approximated by $L_0 = 4\pi R^2_\infty j_\infty$. The total luminosity is related to the asymptotic accretion rate $\dot{M}_\infty$ by [10]

$$L_0 = \alpha \dot{M}_\infty \equiv \left(1 - \frac{N\left(R_0\right)}{k\left(R_0\right)} \sqrt{1 - \frac{2m\left(R_0\right)}{R_0}} \right) \dot{M}_\infty. \quad (11)$$

Here R_0 is the size of the compact core and the quantity α can be interpreted as a binding energy per unit mass.

B. Newtonian approximation

The notation is as in the preceding part of this section. The mass accretion flux is now R-independent, in contrast to the general relativistic case, $\partial_R \dot{M} = 0$ and the baryonic mass density ρ_0 coincides with ρ. $\phi(R)$ is the Newtonian gravitational potential,

$$\phi(R) = -\frac{M(R)}{R} - 4\pi \int_R^{R_\infty} r\rho(r)dr; \quad (12)$$

$M(R) \equiv M - 4\pi \int_R^{R_\infty} r^2\rho(r)dr$ is the mass contained within the sphere R.

The Newtonian model can be described by two basic equations [9]:

i) the energy conservation equation

$$L_0 - L(R) = \dot{M} \left(\frac{a^2_\infty}{\Gamma - 1} + \frac{U^2_\infty}{2} + \phi(\infty) - \frac{a^2}{\Gamma - 1} - \frac{U^2}{2} - \phi(R) \right). \quad (13)$$

ii) The luminosity equation

$$L = L_0 \exp\left(\frac{-\kappa \dot{M}}{4\pi R} \right) = L_0 \exp\left(\frac{-L_0 \tilde{R}_0}{L_E R} \right). \quad (14)$$

Notice that the luminosity has the same form as in the case of test fluids [4]. Here we introduced the Eddington luminosity $L_E = 4\pi M/\kappa$ while $\tilde{R}_0 \equiv GM/|\phi(R_0)|$ is a kind of modified size measure of the compact body. In the case of test fluids $\tilde{R}_0 = R_0$. We assume $L_0 = |\phi\left(R_0\right)|\dot{M}$; notice, however, that for small α this relation (with $\alpha = |\phi(R_0)|$) appears as the Newtonian limit of Eq. (11).

III. THIN GAS APPROXIMATION AND BOUNDARY DATA

The thin gas approximation demands that the optical thicknes [11] of the cloud is smaller than one, i.e.

$$\tau = \int_{R_0}^{R_\infty} n(r)\sigma dr < 1, \quad (15)$$

where n is the baryonic number density. Notice that $n = \frac{\rho_0}{m_p}$ if we assume the monoatomic hydrogenic gas. Assuming that n decreases to the asymptotic value n_∞, we arrive at $1 > R_\infty n_\infty \sigma$. Thus the rough condition for the validity of the thin gas approximation is that the radiation free path $l \equiv 1/(n_\infty \sigma)$ is not shorter than the size of the cloud, $l > R_\infty$. This implies $\rho_0 < \frac{m_p}{R_\infty \sigma}$ and (taking into account that $\rho_0 \approx \rho$) estimates the mass of gas, $M_g < \frac{4\pi}{3\kappa c} R^2_\infty$. Denote the solar mass by $M_\odot$ and define $10^s \equiv \frac{R_\infty}{M}$. One obtains an estimate consistent with the thin gas approximation

$$\frac{M_g}{M} < 10^{-21} \times 10^{2s} \times \frac{M}{M_\odot}. \quad (16)$$

We choose s and M that give the right hand side of (16) of the order of unity. In such a case a significant part of the total mass M would be contributed by the gas itself. That could allow for the strong impact of backreaction and selfgravitation onto accretion. It is clear that there is a scaling freedom – one can trade the size (represented by the exponent s) for the total mass without changing the bound in (16).

The boundary data set is the same for the Newtonian and general relativistic models. Thus we specify in both cases the same values of asymptotic masses M, masses of the core, the binding energy per unit mass $\alpha = |\phi\left(R_0\right)|$, the asymptotic speed of sound a_∞ and the size R_∞. The total luminosity L_0 is not a free data, but it results from equations. We assume identical equations of state in the two models.

IV. SONIC POINTS AND LUMINOSITY

We shall study transonic flows. For these flows there exists a radius R_* such that $a_* = |\vec{U}_*|$; the speed of sound is equal to the length of the spatial part of the velocity vector. Henceforth all quantities denoted by asterisk will refer to a sonic point.

It is clear from the inspection of equations that the regularity of solutions demands a particular relation for the fraction m_*/R_*; here m_* is the mass within the sonic sphere. In the Newtonian model the three characteristics, a_{*N}, U_{*N}, and m_{*N}/R_{*N} are related as below [9]

$$a^2_{*N} = U^2_{*N} = \frac{m_{*N}}{2R_{*N}} \left(1 - \frac{L_{*N}\kappa}{4\pi m_{*N}} \right) = \frac{m_{*N}}{2R_{*N}} \left(1 - \frac{L_{*N} M}{L_E m_{*N}} \right). \quad (17)$$

In the last equation appears the Eddington luminosity L_E. It is clear that the necessary condition for the critical Newtonian flow – that is, possessing a sonic point – reads

$$\frac{L_{*N}M}{L_E m_{*N}} < 1. \tag{18}$$

Define $x \equiv L_0/L_E$ and $y \equiv m_*/M$. Since $L_* \leq L_0$, the inequality $x < y$ becomes the necessary condition for a sonic point. In the general relativistic model, at the sonic point $a^2 = \frac{U^2}{k^2}$; the denominator of the right hand side of Eq. (8) vanishes and that implies the vanishing of the numerator. One obtains

$$\frac{1}{k^2R}\left(\frac{m}{R} - 2U^2 + 4\pi R^2\left(\rho_E + p + j\frac{U}{k}\right)\right) = \kappa j\left(1 - \frac{a^2}{\Gamma - 1}\right). \tag{19}$$

Let us remark, that in the Newtonian limit $a^2 \ll 1$ and $4\pi R^2_{*GR}\left(\rho_{*E} + p_* + j\frac{U_*}{k_*}\right) \ll \frac{m_{*GR}}{R_{*GR}}$. Therefore, in this limit Eq. (19) coincides with Eq. (17). It is obvious that radiation pushes the sonic point inward; if the size of a compact object is bigger than the value of R_* predicted by (18) and (19), then the flow becomes subsonic.

V. UNIVERSALITY IN L_{GR}/L_N

We assume in this Section the polytropic equation of state $p = Kn^\Gamma$ with $\Gamma < 5/3 - \epsilon$, for some small $\epsilon > 0$. This restriction is due to the peculiar character of the equation of state corresponding to $\Gamma = 5/3$ The constancy of L_{GR}/L_N is valid for all Γ's, although the specific value of this ratio depends on the equation of state.

A. Low luminosities

The natural reference quantity for radiating systems is the Eddington luminosity L_E. It can be roughly described as the luminosity at which the infall of gas is prevented. Thus one might define weakly radiating systems as radiating with a luminosity L_0 (herein $L_0 = L_{GR}$ or $L_0 = L_N$) that is much smaller than the Eddington luminosity, $L_0 \ll L_E$. We will adopt a different definition, for reasons that will become clear.

The (XY) condition. We will say that an accretion system satisfies the (XY) condition if $x \ll y$.

Notice the trivial fact that $y < 1$. If (XY) holds, that is $x \ll y$, then obviously $L_0 \ll L_E$. Thus the (XY) assumption is stronger than just the statement $L_0 \ll L_E$. Another interesting fact is that (XY) guarantees that the characteristics of the sonic point are essentially unchanged by the radiation – see Eqs. (17) and (19). The luminosity is the product of α by the (asymptotic) mass accretion rate, and since the mass accretion rate can be formulated completely in terms of the sonic point parameters, it becomes luminosity independent if $x \ll y$. The general relativistic mass accretion rate $\dot{M}_{GR}$ within the steadily accreting fluid can be expressed as below (see Eq. (6.1) in [1])

$$\dot{M}_{GR} = \pi m^2_{*GR}\rho_{\infty GR}\frac{R^2_{*GR}}{m^2_{*GR}}\left(\frac{a^2_{*GR}}{a^2_\infty}\right)^{\frac{(5-3\Gamma)}{2(\Gamma-1)}}\left(1 + \frac{a^2_{*GR}}{\Gamma}\right) \times\frac{1 + 3a^2_{*GR}}{a^3_\infty}. \tag{20}$$

The corresponding Newtonian expression reads

$$\dot{M}_N = \pi m^2_{*N}\rho_{\infty N}\frac{R^2_{*N}}{m^2_{*N}}\left(\frac{a^2_{*N}}{a^2_\infty}\right)^{\frac{(5-3\Gamma)}{2(\Gamma-1)}}\frac{1}{a^3_\infty}. \tag{21}$$

Eq. (21) has been derived by Kinasiewicz in [14], but it follows also from (20) in the limit of small sound speeds, $a_{*GR} << 1$. The way of writing these two expressions is not accidental. It has been shown in [15] that characteristics of the sonic point – a^2_{*GR} and $\frac{R^2_{*GR}}{m^2_{*GR}}$ – do not depend on the fraction of mass carried by the gas. These quantities are dictated just by the asymptotic speed of sound a_∞ in a test fluid model. An analogous result holds in the Newtonian model, as shown in [14]. Therefore $L_{GR}/L_N = \dot{M}_{GR}/\dot{M}_N$ is equal to the ratio $F \times \frac{m^2_{*GR}\rho_{\infty GR}}{m^2_{*N}\rho_{\infty N}}$, where the coefficient F depends on Γ (and thus on the equation of state) and on the sonic point parameters a_{*N}, a_{*GR}, $\frac{R^2_{*GR}}{m^2_{*GR}}$ and $\frac{R^2_{*N}}{m^2_{*N}}$. Therefore the coefficient F is independent of the mass fraction y. Now the masses are approximately equal, $m_{*GR} \approx m_{*N}$; this is because the masses within the sonic point are well approximated by the masses of the cores, and the latter are equal by definition. The equality of masses of the cores in both models is one of our boundary conditions. The asymptotic gas densities $\rho_{\infty GR}$ and $\rho_{\infty N}$ are approximately equal to $(M - m_{*GR})/V$ ([14], [15]); in order to show that one should invoke assumptions concerning boundary conditions $U^2_\infty \ll \frac{M}{R_\infty} \ll a^2_\infty$. The calculation is long but straightforward. Thus, we finally obtain $L_{GR}/L_N = F$; the ratio of luminosities is independent of the fraction of mass carried by the gas, in the regime of low luminosities. This means that the appropriate information on the ratio of the relativistic and Newtonian luminosities, L_{GR}/L_N, can be obtained just by the analysis of accreting systems with test gas (and for these see, for instance, results in [1] and [2]). This is despite the fact that actual values of both luminosities taken separately depend on the contribution of the gas to total mass. We already know that in accretion without radiation the mass accretion rates are maximal when $m_* = 2M/3$ and they tend to zero at both ends: i) $m_* \to M$ (when the density ρ_∞ tends to zero) and ii) $m_*/M \to 0$ (when the mass of the core is negligible in comparison to the mass of the fluid) [15]. That implies, for weakly radiating systems, that luminosities behave in a similar way. But still their ratio is constant and independent of the parameter y.

B. High luminosities.

The argument of the former subsection does not apply to accreting luminous systems, when the total luminosity L_0 is close to the Eddington limit or more generally – when the (XY) condition is broken. That this can happen, is easily illustrated by the Newtonian model. In this model one obtains an equation that relates luminosity x and the mass content y (see Eq. (21) in [9], in units adapted to the convention of this paper):

$$x = \alpha \frac{\chi_\infty M^2}{4a_\infty^3}(1-y)(y-x)^2\left(\frac{2}{5-3\Gamma}\right)^{(5-3\Gamma)/(2(\Gamma-1))}. \tag{22}$$

Here χ_∞ is a constant. The argument that was used above relied on the fact that the right hand side of (22) does not depend on x if $x \ll y$. But if x is relatively large, then $\dot{M}$ becomes x-dependent, and (22) yields a relation $x = x(y)$. A similar reasoning can also be applied to the general relativistic model; again $\dot{M}$ depends on x if x is large. In conclusion, for luminous systems the ratio of L_{GR}/L_N can become x and y-dependent.

Luminous systems are characterized by small values of the asymptotic speed of sound, $a_\infty \ll 1$. We restrict our attention to systems that satisfy the following

X(1-Y) condition. We will say that an accretion system satisfies the $X(1-Y)$ condition if $x \gg 1-y$ and $x < y/2$.

Since $y > x$, the above implies $y > 2/3$. Thus X(1-Y) selects a subclass of luminous accretion systems with moderate contribution of the gas to total mass. Luminous test fluids belong to this category.

Define an auxiliary quantity

$$\hat{L} \equiv -2\dot{M}N\frac{\rho^E}{\rho_0} + 4\pi R^2 jNk\left(1+\frac{U^2}{k^2}\right). \tag{23}$$

$\hat{L}$ represents local luminosity as measured by an observer stationary at R [10]. It follows from Eq. (6) that at the boundary of the accretion cloud $\hat{L}(R_\infty) = L_{GR}$; $\hat{L}(R_\infty)$ is the total luminosity. By differentiating Eq. (6) and employing Eq. (10) one can easily derive the following equation

$$\frac{d}{dR}\hat{L} = \dot{M}N\kappa j + N\frac{\Gamma-1}{\Gamma-1-a^2}\frac{d}{dR}\dot{M}. \tag{24}$$

We have $\frac{d}{dR}\dot{M} = -16\pi^2 R^3\frac{\rho_0}{k}j$ [10]. It is convenient to replace $\dot{M}$ by $\dot{M}_\infty \equiv \dot{M} + \kappa\int_R^{R_\infty} drjr^3\frac{\rho_0}{k}$; the new object is constant and coincides with $\dot{M}$ at the boundary R_∞. Then,

$$\begin{aligned}\frac{d}{dR}\hat{L} &= \dot{M}_\infty N\kappa j + \\ &16\pi^2 jN\left(-R^3\frac{\rho_0}{k}\frac{\Gamma-1}{\Gamma-1-a^2}+\kappa\int_R^{R_\infty} drjr^3\frac{\rho_0}{k}\right).\end{aligned} \tag{25}$$

Now, the assumptions $a_\infty \ll 1$ and $x < y/2$ imply that in the region extending from the sonic point to R_∞ the infall velocity U is small and the position of the sonic point R_* is large ($R_* \gg M$). Therefore the metric function k and the lapse N are close to unity. The argument bases on the approximate validity of Newtonian relations (17) at general relativistic sonic points. The second term on the right hand side of (25),

$$\delta L \equiv 16\pi^2 jN\left(-R^3\frac{\rho_0}{k}\frac{\Gamma-1}{\Gamma-1-a^2}+\kappa\int_R^{R_\infty} drjr^3\frac{\rho_0}{k}\right) \tag{26}$$

is bounded from above by the first term on the right hand of (25), $\delta L \le |\dot{M}_\infty N\kappa j| \times \frac{(1-y)}{x}$. Indeed, the first term on the right hand side of (25) can be written as $j\frac{4\pi ML_0}{\alpha L_E}$; but that is equal to $j\frac{4\pi Mx}{\alpha}$, which in turn is larger than $j4\pi Mx$. The first term of δL is bounded by $12\pi|j|M_g = 12\pi M(1-y)$. The second term of δL is much smaller than the first one if sonic radii are much larger than M. Therefore the conclusion follows.

Thence in the annular region (R_*, R_∞) the function $\hat{L}$ satisfies with good accuracy the differential equation

$$\frac{d}{dR}\hat{L} \approx \dot{M}_\infty\kappa\frac{\hat{L}}{4\pi R^2}, \tag{27}$$

which is solved by $\hat{L} = L_0\exp\left(-\kappa\frac{\dot{M}_\infty}{4\pi R}\right)$. Thus we obtain the same form of a solution as in the Newtonian case. Furthermore, one can approximate Eq. (6) by a suitable Newtonian model in the region (R_*, R_∞). Indeed, it is easy to show that $\dot{M} \approx \dot{M}_\infty$. Expanding the lapse N – keeping only the first order terms in a^2, $M(R)/R$ and U^2 – we arrive at the Newtonian equation (13). That means that under adopted boundary conditions the total general relativistic model is well approximated by the Newtonian model in the annulus (R_*, R_∞), and thus by Eq. (22). But Eq. (22) has a unique solution, assuming $x < y < 1$. Therefore the Newtonian limit of the general relativistic model and the Newtonian solution do coincide and the ratio L_{GR}/L_N is not only constant, but it is equal to 1.

VI. NUMERICS.

We compare two accreting systems, a Newtonian one and its general relativistic counterpart, that have indentical sizes, the same asymptotic masses and identical masses of compact cores, equal asymptotic temperature and the same binding energy. Thus, it is legitimate to say that the boundary data are ultimately: the asymptotic mass, the mass of the core, the binding energy per unit mass $\alpha = |\phi(R_0)|$, the asymptotic speed of sound a_∞ and the size of the system R_∞. The total luminosity L_0 is not part of these data but is the sought result of the two models.

It appears convenient in numerical calculations to specify temporarily $\rho_{0\infty}$ and L_0 instead of the mass of the core. Conceptually the computational technique is the same in the two models. For a given $\rho_{0\infty}$ one randomly chooses L_0 (equivalently one could choose an accretion rate, due to relation $\dot{M} = L_0/\alpha$). This choice completely specifies $L(r)$ in the Newtonian model – see formula (14). Asymptotic radiation data for the general relativistic system in turn are given by $j_\infty = \rho^E_\infty = L_0/\left(4\pi R^2_\infty\right)$, and the mass accretion rate $\dot{M}_\infty = L_0/\alpha$. During the numerical integration one gets a subsonic solution (if the chosen L_0 is smaller than a critical luminosity) or finds no solution at all (if L_0 is greater than a critical value). Using the bisection method one finds this critical luminosity for which the gas flow becomes transonic. The mass of the core results from computations. Notice, that for a given $\rho_{0\infty}$ masses of the core usually differ in the Newtonian and the general relativistic models. The difference is particularly noticeable for high asymptotic sound speeds a_∞. One should change the value of $\rho_{0\infty}$ and repeat the procedure until finally the masses of both cores are the same for both critical flows.

In this way one obtains a boundary of the solution set (in the plane $L_0 - M_{core}$) that consists exclusively of transonic solutions, if the mean free path of photons is larger than the size of the system R_∞.

From a mathematical point of view we have a system of ordinary first order differential equations. The general relativistic problem also includes the integro-algebraic constraint Eq. (6). Numerical calculations start from the values adopted at the outer boundary R_∞ and continue inward until the equality $\alpha = 1 - \frac{N(R)}{k(R)}\sqrt{1 - \frac{2m(R)}{R}}$ (in the GR case) is met at some R; this value of the areal radius is denoted as R_0 and interpreted as the radius of the compact core of the accreting system. For the Newtonian model the calculation continues until the gravitational potential ϕ becomes equal to $-\alpha$. The numerical integration employs the 8th order Runge-Kutta method [16]. The main numerical difficulty is encountered in the vicinity of the sonic point. In the general relativistic case the denominator and the numerator of Eq. (19) vanish for $a^2 = \frac{U^2}{k^2}$. In numerical computation, the division by very small numbers may cause errors and lead to unphysical solutions, therefore a special regularization technique had to be implemented. We omit further discussion of related technicalities, but let us mention that because of this difficulty with the sonic point there appear small numerical errors for $M_{core} \approx 1$ and $M_{core} \approx 0.1$ (see Fig. 3). In the Newtonian model one has to deal with the same problem.

We choose specific numerical data, but since the accreting system possesses a simple scaling property – as discussed in one of preceding sections – one can extend the validity of all conclusions to a large family of systems with appropriately scaled masses M and sizes R_∞.

We assume standard gravitational units $G = c = 1$, the polytropic index $\Gamma = 3/2$, the size $R_\infty = 10^8 \times M$ and the mass $M = 10^6 M_\odot$, where $M_\odot$ is the Solar mass. In the scaling $M = 1$ one gets $\kappa = 3.6258 \times 10^{22}\,(M_\odot/M)$, that is $\kappa = 3.6258 \times 10^{16}$. The Eddington luminosity reads $L_E = 3.4658 \times 10^{-22} M/M_\odot = 3.4658 \times 10^{-16}$. These data are arranged to ensure the validity of the thin gas approximation. The optical thickness (15) is always smaller than 1.

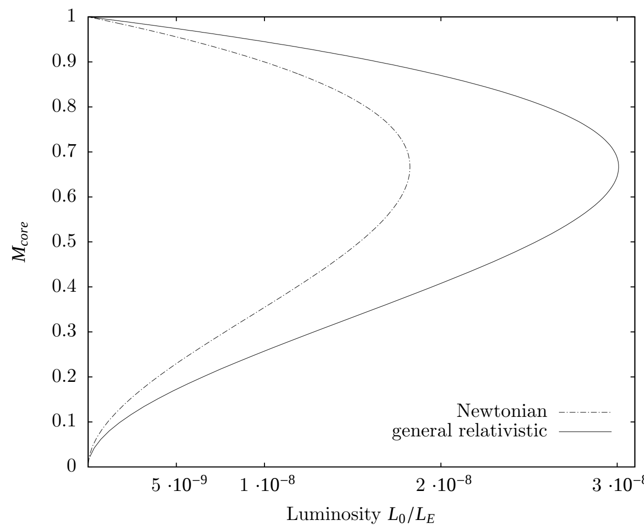

FIG. 1: Luminosity of general relativistic and Newtonian models. $\alpha = 0.9$ and $a^2_\infty = 10^{-1}$. The abscissa shows the luminosity in terms of the Eddington luminosity L_E and the ordinate shows the mass of the compact core.

VII. RESULTS

Figures 1–2 show accreting solutions on the luminosity-(mass of the central core) diagram for $\alpha = 0.9$. The squared speed of sound is $a^2_\infty = 10^{-1}$ and $a^2_\infty = 10^{-6}$, respectively. The two figures show transonic solution sets for the Newtonian and general relativistic models; they are depicted by dashed and solid lines, respectively. Comparing these figures, we notice that the brightness of a system increases sharply with the decrease of a^2_∞. In the case illustrated in the second figure, maximal luminosities go up to one quarter of the Eddington luminosity. In the test gas limit, the interaction between gas and radiation is negligible and the gas accretion can be approximated by the purely hydrodynamic description. Such a case was already analyzed in [1], with the same conclusion as suggested by the comparison of Figs 1 and 2: the larger the asymptotic speed of sound, the larger the gap between the general relativistic and the Newtonian predictions. The general relativistic model gives significantly larger accretion rates for high asymptotic temperatures. These figures clearly demonstrate that luminosities depend on the fraction of mass deposited in the gas and become maximal when this fraction is not bigger than 1/3. Again, this aspect of the description of the regime of weakly radiating sources agrees with the purely hydrodynamic study of [15]. The position of this maximum

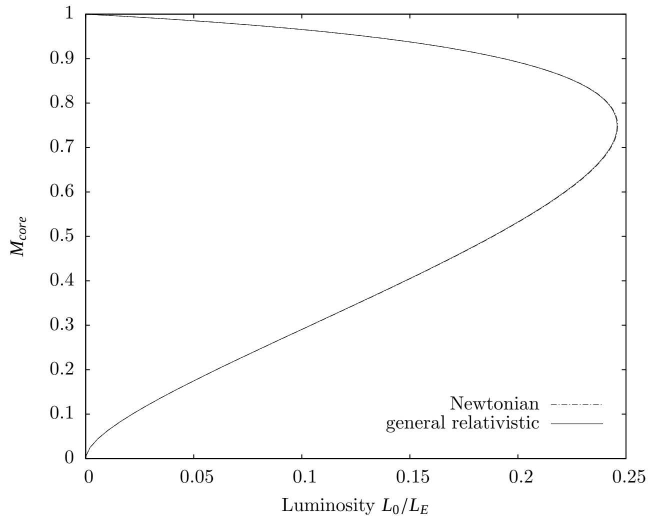

FIG. 2: Luminosity of general relativistic and Newtonian models. $\alpha = 0.9$ and $a_\infty^2 = 10^{-6}$. The abscissa and ordinate are as in Fig. 1.

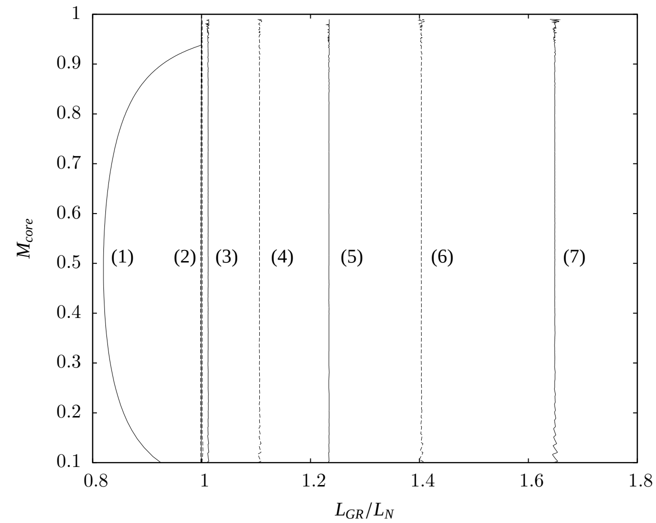

FIG. 3: Binding energy $\alpha = 0.9$. The values of L_{GR}/L_N are shown on the abscissa. The mass fraction y is put on the ordinate. Asymptotic squared speeds of sound are: 10^{-7} (line no 1); $10^{-6}, 10^{-5}, 10^{-4}$ (the three close lines are grouped as line 2); 10^{-3} (line 3); 10^{-2} (line 4), 2.5×10^{-2} (line 5), 5×10^{-2} (line 6), 10^{-1} (line 7).

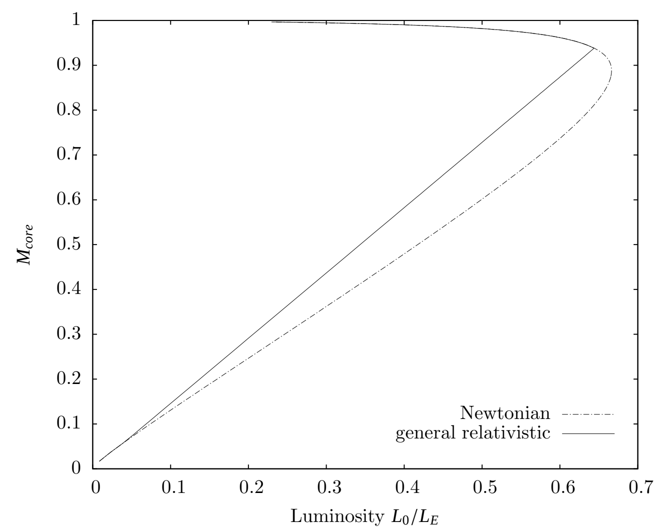

FIG. 4: Binding energy $\alpha = 0.9$. The luminosities are shown on the abscissa while the mass fraction y (the core mass) is put on the ordinate. Here $a_\infty^2 = 10^{-7}$.

depends weakly on the relative luminosity L/L_E and it shifts from $y = 2/3$ in Fig. 1 towards $y = 0.75$ in Fig. 2. It is clear that this effect is due to the influence of the radiation; the higher the luminosity, the larger the mass of the core at the maximum.

Fig. 3 reveals a feature of spherical accretion that confirms the analytic proof (made in one of preceding sections), that L_{GR}/L_N should be constant, at least for small luminosities. While each individual quantity L_{GR}, L_N depends on the contribution of the gas to the total mass, their ratio is roughly constant at a given asymptotic temperature. In the six sets of transonic flows (lines 2-7 on Fig. 4) the ratio L_{GR}/L_N is independent of the mass of accreting gas. We would like to call the reader's attention to line 2, where the maximal value of $x = 1/4$ is achieved at $y = 0.75$ (see Fig. 2). Thus the maximal value of $x/y \approx 1/3$, $x \approx 1 - y$ and still the fraction L_{GR}/L_N is constant. This agrees with the analytic result shown in the second part of Section V, although the proof of this requires $x \gg 1 - y$. That suggests that analytic results can be proven under less stringent conditions than stated in Sec. V.

Line 1 in Fig. 3 and Fig 4. display data where the backreaction effect causes L_{GR}/L_N to vary (and in particular, L_{GR}/L_N can be made significantly smaller than 1). Notice, however, that in the general relativistic model the flows cease to be transonic for $y < 0.94$. In contrast, they are always transonic in the Newtonian model. There is a small segment just below $y = 1$, where the **X(1-Y)** condition is met and the ratio of luminosities equals 1.

VIII. CONCLUSIONS

There are interesting universal properties hidden in generalizations of the classical Bondi accretion model. It is already known that when radiation is absent, transonic flows (that maximize mass accretion rates) correspond to the case when $y \equiv m_*/M = 2/3$, irrespective of the equation of state and the asymptotic speed of sound [15], [14]. The mass of the core is about 2/3 of the total mass of an accreting system. This paper deals with radiating accretion flows. We compare luminosities corresponding to transonic solutions of the general relativistic and Newtonian accretion models, assuming the same polytropic equation of state and identical boundary data – asymptotic speed of sound a_∞, size R_∞, total (asymptotic) mass and fraction $1-y$ of the total mass contributed by gas. We focus our attention on the investigation of the relation between their relative luminosity (L_{GR}/L_N) and y. When accreting systems are characterized by low luminosity and the condition (XY) of Section V holds true (that is $L_{GR} \ll L_E \times y$ and $L_N \ll L_E \times y$) then the ratio L_{GR}/L_N is independent of y and can be significantly larger than 1. We have found an example with the largest value of L_{GR}/L_N exceeding 1.6, but in earlier investigation of test fluids with the polytropic index close to 5/3 the ratio of mass accretion rates $\dot{M}_{GR}/\dot{M}_N$ exceeded 10 [2], which suggests that L_{GR}/L_N can grow by one order of magnitude. On the other hand, when the condition X(1-Y) of Section V is valid (that is, a transonic flow is highly luminous, $x \equiv L_0/L_E \gg 1-y$ and $x < y/2$, but the contribution of gas to the mass is small), then $L_{GR}/L_N \approx 1$. These properties of the ratio L_{GR}/L_N have been derived analytically and confirmed (under less stringent conditions) numerically.

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