Minimal modifications of the primordial power spectrum from an adiabatic short distance cutoff

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Abstract

As a simple model for unknown Planck scale physics, we assume that the quantum modes responsible for producing primordial curvature perturbations during inflation are placed in their instantaneous adiabatic vacuum when their proper momentum reaches a fixed high energy scale M. The resulting power spectrum is derived and presented in a form that exhibits the amplitude and frequency of the superimposed oscillations in terms of H/M and the slow roll parameter ϵ. The amplitude of the oscillations is proportional to the third power of H/M. We argue that these small oscillations give the lower bound of the modifications of the power spectrum if the notion of free mode propagation ceases to exist above the critical energy scale M.

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Inflation, in addition to explaining the homogeneity and flatness of the universe, provides a platform for computing the properties of the small inhomogeneities which give rise to the large scale structures [1, 2]. When initially in the vacuum, the quantized fluctuations of the coupled system of metric and scalar inflaton field produce a nearly scale invariant spectrum. Small deviations arise from the time dependence of the Hubble expansion rate during inflation [3]. The predictions of the simplest inflationary scenarios are in good agreement with recent analysis of high-resolution CMBR maps.

However, these predictions are based on the incomplete framework of semiclassical quantum field theory. One particular aspect which has been questioned lately is the choice of the ground state of the quantum modes that generate the fluctuations. In the standard derivation, the vacuum is imposed at the beginning of inflation for all quantum modes. During inflation, the proper wavelengths of these modes are redshifted by many orders of magnitude. Given that inflation lasts more than 70 e-folds, the initial state of the modes responsible for the cosmological structures observed today is assigned at proper wavelengths shorter than the Planck length. Moreover, in the semiclassical framework, one implicitly assumes that their redshift to lower momenta proceeds without interference from Planck scale physics. The validity of these assumptions is unclear; this is known as the trans-Planckian question of inflationary cosmology [4].

The sensitivity of the predictions of inflation, in particular the scale invariance of the power spectrum, with respect to trans-Planckian physics has been probed in a number of different ways, including nonlinear dispersion relations [5, 6, 7, 8, 9, 10], string-inspired short distance cutoffs [11, 12, 13, 14, 15], and modifications derived from non-commutative geometry [16, 17, 18]. Two aspects of the discussion have become particularly important: the question of the leading order correction to the power spectrum in terms of the ratio (called σ in this work) of the Hubble parameter H and the scale of new physics M, and the existence of distinctive features of short distance effects that would allow their unambiguous detection in a general inflationary setting. As a candidate for the latter, the potential violation of the scalar-tensor consistency relation by Planck scale physics has been mentioned [19]. Unfortunately it requires the measurement of the tensor perturbation power spectrum with great accuracy [20]. Alternatively, if models for short distance physics predict a unique pattern imprinted on the scalar power spectrum, this alone might be sufficient for detection.

With regard to the amplitude of the correction, Ref.[20] has pointed out that local effective field theory generically allows only σ^2 and smaller effects, whereas [21] have argued that $O(\sigma)$ effects are possible if adiabaticity of the mode evolution is violated. Danielsson [22] has indeed found a linear correction by imposing that each quantum mode obeys a positive frequency solution at a given energy scale M (see Eq.(33) in [23]). This method has been re-analyzed in Ref.[23] in a general inflationary background, confirming the $O(\sigma)$-amplitude and demonstrating the oscillatory character of the correction to the power spectrum. It has also been noted that some classes of cutoff models predict corrections that depend on M in more complicated ways [24].

In this work, we take the same approach but we replace Danielsson's vacuum condition by the instantaneous adiabatic vacuum at the scale M. The basic underlying physical assumption is that whatever physics determines the dynamics at higher energies, it places the quantum modes into their adiabatic vacuum at some scale $M \lesssim m_{\rm Planck}$, from where on they evolve freely and in the usual way, i.e. without dispersion. Owing to space-time expansion, this state is not exactly identical to the asymptotic vacuum which is imposed in the infinite past. This difference gives rise to a modification of the power spectrum whose shape and amplitude we calculate both exactly and perturbatively. In contrast with [22], we find a leading correction that is cubic in σ, thus predicting a much smaller modification of the spectrum.

The procedure of assigning the vacuum when the proper momentum of each mode crosses M can be considered a minimal model for short distance effects since the adiabatic vacuum is the closest analogue to the usual Minkowski vacuum in an expanding space-time. Indeed, any other choice of the initial state will produce larger effects unless the amplitudes to find pairs of adiabatic quanta are fine tuned so as to "undo" the Bogoliubov transformation with respect to the asymptotic vacuum, see the conclusions for more details. Hence the modification of the power spectrum we found can be interpreted as the minimal one if modes are "created" at the scale M.

Regarding the pattern that short distance physics imprints on the microwave background, Easther et al. [14] have shown numerically for one specific model that nearly sinusoidal oscillations in $\log(k)$ may be superimposed on the power spectrum. This has been confirmed in [23] using the cutoff criterion of [22]. We also find a distinctive oscillatory pattern and present a simple expression for its oscillations in wavenumber space. These oscillations arise through the wavenumber dependence of σ, which is governed by the parameter ϵ measuring the deviation of the inflationary background from de Sitter space. Since in the slow-roll framework, ϵ can be determined from the spectral index and the tensor/scalar ratio [25], our model makes an explicit and simple prediction for the amplitude and shape of the correction. We hope that this result will be useful in future studies of the detectability of short distance effects in cosmological measurements.

The power spectrum of curvature perturbations produced during inflation can be calculated from the dynamics of a minimally coupled, massless scalar field in an FRW space time [3]. The k-mode frequency component of this field obeys

$$(\partial_\eta^2 + \omega_k^2(\eta))\phi_k = 0 \tag{1}$$

where η is the conformal time. The time dependent frequency is given by

$$\omega_k^2(\eta) = k^2 - \frac{\partial_\eta^2 a(\eta)}{a(\eta)} . \tag{2}$$

For simplicity, we restrict our discussion to power-law inflation with

$$a(\eta) = \left(\frac{H_0}{q}\eta\right)^q \tag{3}$$

where H_0 is the value of the Hubble rate, $H = \partial_\eta a/a^2$, evaluated at $\eta_0 = q/H_0 < 0$. The parameter q is related to the first slow roll parameter ϵ by $q = -1/(1-\epsilon)$, where $\epsilon > 0$ is defined as

$$\epsilon = -\frac{\partial_\eta H}{aH^2} = -\frac{\partial_t H}{H^2} . \tag{4}$$

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Our basic assumption is that the unknown physics governed by quantum gravity delivers each mode k in the instantaneous adiabatic vacuum at the time η_k when the proper momentum $k/a(\eta)$ equals some high energy scale M. Below this scale, the propagation obeys Eq. (1). This line of thought has been originally developed in the context of Hawking radiation in [26]. The time η_k is given by $k = a(\eta_k)M$ and can be written as

$$\eta_k = \frac{q}{k\sigma(\eta_k)} \tag{5}$$

where we defined

$$\sigma(\eta) = \frac{H(\eta)}{M} . \tag{6}$$

If we take M to be of the order of the Planck mass, the amplitude of CMB temperature fluctuations implies that $\sigma \lesssim 10^{-5}$. On the other hand, certain strongly coupled M-theory scenarios allow somewhat larger values for σ, reaching up to order unity [13, 20]. We will nevertheless assume that σ is a small parameter for the purpose of determining the lowest order correction. The time dependence of σ can be re-expressed in terms of k using $\eta_k = q/H_0\ (k/M)^{1/q}$ as

$$\sigma_k \equiv \sigma(\eta_k) = \frac{H_0}{M}\left(\frac{k}{M}\right)^{-\epsilon} = \sigma_0 \left(\frac{k}{M}\right)^{-\epsilon} \ . \tag{7}$$

This result is in agreement with Ref. [23] where a different time coordinate was used. It should also be noted that M itself can slowly vary with k. The formalism used in this work can readily handle this extension as it is expressed entirely in terms of σ_k.

To properly define the instantaneous adiabatic vacuum [27] at time η_k we follow Section 2.4 of [28] where more details can be found. We first require that the exact solution $\varphi_k^{\eta_k}(\eta)$ of Eq. (1) is a positive frequency mode at $\eta = \eta_k$:

$$i\partial_\eta \varphi_k^{\eta_k}(\eta)|_{\eta=\eta_k} = \omega_k(\eta_k)\,\varphi_k^{\eta_k}(\eta_k) \tag{8}$$

We then express this solution as

$$\varphi_k^{\eta_k}(\eta) = c_k^{\eta_k}(\eta)\varphi_k^{\rm WKB}(\eta) + d_k^{\eta_k}(\eta)\left[\varphi_k^{\rm WKB}(\eta)\right]^* \tag{9}$$

where $\varphi_k^{\rm WKB}(\eta)$ is the usual WKB solution of unit Wronskian:

$$\varphi_k^{\rm WKB}(\eta) = \frac{1}{\sqrt{2\omega_k(\eta)}}\exp\left(-i\int_{\eta_k}^{\eta}\omega_k(\tilde\eta)d\tilde\eta\right) \ . \tag{10}$$

We also require that $i\partial_\eta\varphi_k^{\eta_k} = \omega_k\left[c_k^{\eta_k}\varphi_k^{\rm WKB} - d_k^{\eta_k}(\varphi_k^{\rm WKB})^*\right]$ for all η. Then, the coefficients $c_k^{\eta_k}$ and $d_k^{\eta_k}$ obey the coupled first order equations

$$\begin{aligned}
\partial_\eta c_k^{\eta_k}(\eta) &= \frac{1}{2}\frac{\partial_\eta\omega_k(\eta)}{\omega_k(\eta)}\, d_k^{\eta_k}(\eta)\,\exp\left(2i\int_{\eta_k}^{\eta}\omega_k(\tilde\eta)d\tilde\eta\right) \\
\partial_\eta d_k^{\eta_k}(\eta) &= \frac{1}{2}\frac{\partial_\eta\omega_k(\eta)}{\omega_k(\eta)}\, c_k^{\eta_k}(\eta)\,\exp\left(-2i\int_{\eta_k}^{\eta}\omega_k(\tilde\eta)d\tilde\eta\right) \ .
\end{aligned} \tag{11}$$

These three first order differential equations are identical to Eq. (1) and they guarantee that $|c_k^{\eta_k}(\eta)|^2 - |d_k^{\eta_k}(\eta)|^2 = const$, i.e. the conservation of the Wronskian. The conditions that $\varphi_k^{\eta_k}(\eta)$ have unit Wronskian and be positive frequency at time η_k translate into

$$c_k^{\eta_k}(\eta)|_{\eta=\eta_k} = 1\ ,\ \ d_k^{\eta_k}(\eta)|_{\eta=\eta_k} = 0\ . \tag{12}$$

The instantaneous vacuum at η_k is then defined as the state annihilated by the (η-independent) destruction operator associated with $\varphi_k^{\eta_k}(\eta)$.

The solution $\varphi_k^{\eta_k}(\eta)$ can be also expressed in terms of the asymptotic positive and negative energy solutions. The positive frequency solution is

$$\varphi_k^{-\infty} = \frac{1}{2}\sqrt{-\eta\pi}\,\mathcal{H}_{1/2-q}(-k\eta)\ , \tag{13}$$

where $\mathcal{H}_{1/2-q}$ is the Hankel function of the first kind. In the conventional derivation [3], this solution is used to define the vacuum state at the onset of inflation. The decomposition of $\varphi_k^{\eta_k}$ reads

$$\varphi_k^{\eta_k}(\eta) = \alpha_k^{\eta_k}\varphi_k^{-\infty} + \beta_k^{\eta_k}(\varphi_k^{-\infty})^*\ , \tag{14}$$

where the Bogoliubov coefficients $\alpha_k^{\eta_k}$ and $\beta_k^{\eta_k}$ are independent of η. They are related to the coefficients of Eq. (9) by $\alpha_k^{\eta_k} = c_k^{\eta_k}(-\infty)$ and $\beta_k^{\eta_k} = d_k^{\eta_k}(-\infty)$. $|\beta_k^{\eta_k}|^2$ gives the mean occupation number of adiabatic instantaneous quanta present in the asymptotic vacuum, or, equivalently, the number of asymptotic quanta present in the instantaneous vacuum. We will use this latter interpretation to determine the modifications of the power spectrum associated with the replacement of the asymptotic vacuum by the instantaneous one.

As explained in [3], the power spectrum of scalar perturbations is governed by the norm of the solution evaluated at horizon crossing, i.e. when $k/a(\eta) = H(\eta)$. Thus, when working with the instantaneous vacuum, it is given by

$$\mathcal{P}_{\mathcal{R}}^{\eta_k} = \frac{k^3}{2\pi^2}\left|\frac{\varphi_k^{\eta_k}}{z}\right|^2_{k=aH}\ ,\ z = -m_{\text{Planck}}\,a\sqrt{\epsilon}\ , \tag{15}$$

where

$$|\varphi_k^{\eta_k}(\eta)|^2 = \frac{\pi}{4}(-\eta)\left[(1+2|\beta_k^{\eta_k}|^2)|\mathcal{H}_{1/2-q}|^2 + 2\Re\left\{\alpha_k^{\eta_k}(\beta_k^{\eta_k})^*\left(\mathcal{H}_{1/2-q}\right)^2\right\}\right]\ . \tag{16}$$

In the following, terms quadratic and higher order in $|\beta_k^{\eta_k}|$ will be neglected assuming that $|\beta_k^{\eta_k}| \ll 1$. The first order deviation from the usual spectrum $\mathcal{P}_{\mathcal{R}}^{-\infty}$ obtained with the asymptotic mode of Eq. (13) is

$$\delta_k \equiv \frac{\mathcal{P}_{\mathcal{R}}^{\eta_k} - \mathcal{P}_{\mathcal{R}}^{-\infty}}{\mathcal{P}_{\mathcal{R}}^{-\infty}} \simeq 2\Re\left[\frac{\beta_k^{\eta_k}}{\alpha_k^{\eta_k}}\,\frac{(\mathcal{H}_{1/2-q})^*}{\mathcal{H}_{1/2-q}}\right] \tag{17}$$

evaluated at $k/a = H$. Therefore, $|\delta_k| \sim 2|\beta_k^{\eta_k}|$ provides the magnitude of the deviation from the standard result when the adiabatic vacuum is imposed at η_k instead of $\eta \to -\infty$.

What remains is to derive the coefficients $\alpha_k^{\eta_k}$ and $\beta_k^{\eta_k}$. They are given by

$$\begin{aligned}(\alpha_k^{\eta_k})^* &= (\varphi_k^{\eta_k})^*\, i\overleftrightarrow{\partial}_\eta\, \varphi_k^{-\infty}\ ,\\ \beta_k^{\eta_k} &= \varphi_k^{\eta_k}\, i\overleftrightarrow{\partial}_\eta\, \varphi_k^{-\infty}\ .\end{aligned} \tag{18}$$

Using Eqs.(8), (9), (12) and evaluating (for simplicity) the r.h.s. at η_k, we get

$$\beta_k^{\eta_k} = i\left(\varphi_k^{\rm WKB}\,\partial_\eta\varphi_k^{-\infty} + i\omega_k\varphi_k^{\rm WKB}\,\varphi_k^{-\infty}\right)|_{\eta=\eta_k}\ , \tag{19}$$

and a similar expression for $\alpha_k^{\eta_k}$. This yields

$$\beta_k^{\eta_k} = \frac{-i}{4}\sqrt{\frac{\pi}{2|\phi_k|}}\left[x_k\left(\mathcal{H}_{-1/2-q}(x_k) - \mathcal{H}_{3/2-q}(x_k)\right) + (1+i2\phi_k)\,\mathcal{H}_{1/2-q}(x_k)\right]\ , \tag{20}$$

and $\alpha_k^{\eta_k} = [\beta_k^{\eta_k}(-\phi_k)]^*$, where $x_k = -q/\sigma_k$, and, using Eq.(7),

$$\phi_k \equiv \omega(\eta_k)\,\eta_k \ = \ -\sqrt{x_k^2 - q(q-1)} \tag{21}$$

Regarding the norm of the correction $|\delta_k| \sim 2|\beta_k^{\eta_k}|$, we first give the exact expression for $\epsilon = 0$ (i.e., in de Sitter space)

$$\begin{aligned} |\beta_k^{\eta_k}| \ &= \ \frac{1}{2}\left(\frac{2 - 2\sigma_0^2 - \sigma_0^4}{\sqrt{1-2\sigma_0^2}} - 2\right)^{1/2} \\ &= \ \frac{1}{2}\sigma_0^3\left[1 + \frac{9}{8}\sigma_0^2 + O(\sigma_0^4)\right]\ . \end{aligned} \tag{22}$$

In order to generalize this expression to $\epsilon \neq 0$, we use the asymptotic expansion of the Hankel function [29]:

$$\begin{aligned} \mathcal{H}_\nu(z) \ &= \ [w_1(z) + iw_2(z)]\exp(iz - i\nu\pi/2 - i\pi/4) \\ w_1(z) \ &\sim \ \sum_{n=0}^{m}(-1)^n c_{2n} z^{-2n} \\ w_2(z) \ &\sim \ \sum_{n=0}^{m}(-1)^n c_{2n+1} z^{-2n-1} \\ c_n \ &= \ \frac{1}{8^n\, n!}\prod_{j=1}^{n}\left(4\nu^2 - (2j-1)^2\right)\ ,\ \ c_0 = 1\ , \end{aligned} \tag{23}$$

where m controls the order of the expansion. Expanding Eq.(20) around $\epsilon = \sigma = 0$ yields

$$|\beta_k^{\eta_k}| = \left(\frac{1}{2} - \frac{3}{2}\epsilon\left\{\frac{1}{2} + \ln(k/M)\right\} + O(\epsilon^2)\right)\sigma_0^3 + O(\epsilon^3, \sigma_0^5)\ . \tag{24}$$

Notice that this result is two powers of σ smaller than the modification found by Danielsson [22]. The reason is that the state he used is not the adiabatic vacuum. In fact, the Bogoliubov coefficients relating his modes to Eq.(9) are $\beta_k = O(\sigma_k), \alpha_k \simeq 1$, explaining the linear modification in σ_k.

Eq.(20) allows us to describe the oscillatory pattern superimposed on the nearly scale invariant power spectrum. In Eq.(17), the dependence on k follows from $\beta_k^{\eta_k}/\alpha_k^{\eta_k}$. Then Eq. (20) shows that δ is modulated by $\cos(2x_k + \chi_k)$ where χ_k is a slowly varying function of k. Rapid oscillations are generated by the k-dependence of $\sigma_k = \sigma(\eta_k)$. Hence, any deviation from de Sitter space will be reflected in an oscillatory pattern with frequency $(\epsilon/\sigma_0)\ln(k/M)$ if $|\ln(k/M)| \ll \epsilon^{-1}$. A similar pattern has also been found by Easther et al. [14, 23]. As ϵ can independently be measured from the slope of the power spectrum, our result provides a simple prediction for the signal of an adiabatic cutoff at the scale M that does not rely on the measurement of the tensor perturbation spectrum.

In conclusion, in spite of the smallness of δ in our model, this result can be seen in an optimistic light. σ may be larger than the normally assumed value of 10^{-5}, allowing a potential detection of short distance effects in CMBR measurements [14, 20], especially in combination with a measurement of the tensor spectrum. Moreover, it is by no means obvious that nature chooses to place newly born modes in the instantaneous adiabatic vacuum at η_k. Indeed, when using a nonlinear dispersion relation which modifies $\omega_k(\eta)$ for $k/a > M$, one might obtain a state characterized by Bogoliubov coefficients α_k, β_k which signal the departure from the adiabatic vacuum. In this case, the leading correction will be given by $|\delta_k| \simeq |\beta_k/\alpha_k + \beta_k^{\eta_k}/\alpha_k^{\eta_k}|$ in place of Eq. (17). At this point, two remarks should be made. First, since $|\delta_k|$ is given by a sum of amplitudes and not a sum of squares, one cannot exclude the possibility of a partial cancellation of the two Bogoliubov transformations. However, an exact cancellation is unlikely since β_k and $\beta_k^{\eta_k}$ have different origins. Secondly, one should keep in mind that the coefficients β_k must obey $|\beta_k| \ll \sigma_k$ in order not to violate the hypothesis that the dominant contribution to the Friedman equation be the inflaton potential energy [9, 30, 31].

Finally, the simple prediction for the frequency of the superimposed oscillation in the power spectrum given by Eq.(20) may allow a systematic search for such features in cosmological data. If an oscillation is detected whose frequency, for some σ, is consistent with the value of ϵ inferred from the scalar spectral index and the tensor/scalar ratio [25], it would represent a strong indication of modified short distance physics. It is certainly worthwhile exploring the parameter space that would make this detection feasible.

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