Nonlinear r-modes in Rapidly Rotating Relativistic Stars

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We present the first study of nonlinear r-modes in isentropic, rapidly rotating relativistic stars, via 3-D general-relativistic hydrodynamical evolutions (in the Cowling approximation). Our study aims at addressing several questions regarding the properties of r-modes, which are significant for the prospect of detecting gravitational waves from the r-mode instability. We find that (1) on dynamical timescales, there is no strong coupling of r-modes to other modes at amplitudes of order one – unless nonlinear saturation occurs on longer timescales, the maximum r-mode amplitude is of order $\alpha \sim 1$. An absolute upper limit on the amplitude (relevant for the most rapidly rotating stars) is set by causality, $\alpha_{\rm causal} \sim 3c/\Omega R$. (2) The spectrum of r-modes and inertial modes in isentropic stars is discrete. (3) In addition, the kinematical drift associated with r-modes, recently found by Rezzolla, Lamb and Shapiro (2000), appears to be present in our simulations, but an unambiguous confirmation requires more precise initial data. We discuss the implications of our findings for the detectability of gravitational waves from the r-mode instability.

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Considerable effort has been spent in the last two years, in determining the properties of r-modes [1] in rotating compact stars, since the discovery [2,3] that these modes are unstable to the emission of gravitational radiation. This is motivated by the current understanding that the r-mode instability may have several important astrophysical consequences: it provides an explanation for the spin-down of rapidly rotating proto-neutron stars to Crab-like spin-periods and for the spin-distribution of millisecond pulsars and accreting neutron stars, while being a strong source of detectable continuous gravitational radiation (see [4–6] for reviews). In addition, if r-modes induce differential rotation, then the interaction between them and the magnetic field in neutron stars has been proposed as a gamma-ray burst model [7] or as a mechanism for enhancing the star's toroidal magnetic field, which could, in turn, limit the r-mode amplitude [8].

Still, several important uncertainties remain, the resolution of which could significantly modify the above conclusions. In this *Letter* we present nonlinear hydrodynamical simulations of r-modes in rapidly rotating relativistic stars (in the relativistic Cowling approximation [9]) and address the questions of the maximum amplitude that r-modes can reach, the nature of their frequency spectrum, and the existence of a kinematical differential drift, associated with r-mode oscillations [8]. To our knowledge, these are the first simulations of nonaxisymmetric oscillation modes in rotating (Newtonian or relativistic) stars.

The maximum amplitude of unstable r-modes in a fluid star (neglecting the magnetic field) and the precise mechanism by which it is set, are currently unknown, but saturation is thought to occur due to some form of nonlinear hydrodynamical coupling. An example of such a mechanism is the nonlinear coupling of the unstable r-mode to other, stable, modes of pulsation. Presumably, nonlinear saturation is set on a hydrodynamical timescale, although it cannot be excluded that weak hydrodynamical couplings saturate the r-mode amplitude on longer timescales (but shorter than the growth timescale due to gravitational radiation reaction).

For our study we have used a numerical code based on the 3-D `CACTUS` code developed by the AEI-Potsdam/NCSA/Washington University collaboration [10], into which we implemented the 3rd order Piecewise Parabolic Method (PPM) [13] for the hydrodynamics (see [11] for a recent review) and initial data for equilibrium and perturbed rapidly rotating relativistic stars. In [12] it was shown that the PPM method is suitable for long-term evolutions of rotating relativistic stars. The equilibrium initial data are constructed using the `RNS` code [14]. We focus on a particular, representative, rapidly (and uniformly) rotating model with gravitational mass $M = 1.63 M_\odot$, equatorial circumferential radius $R = 17.25$km and spin period $P = 1.26$ms (the gravitational mass and equatorial radius are larger than for a corresponding nonrotating model, because of rotational effects - the ratio of polar to equatorial coordinate radii is 0.7). The star is rotating at 92% of the mass-shedding (Kepler) limit (at same central density) and we use the $N = 1.0$ relativistic polytropic equation of state (see [4]). Unless otherwise noted, we use 88^3 Cartesian grid-points, with an equidistant grid separation of $\Delta x = 0.41$km.

During the time-evolution, we only evolve the hydrodynamical variables, keeping all spacetime variables fixed at their initial, unperturbed, values (for small amplitude pulsations this is equivalent to the Cowling approx-

imation in linear perturbation theory; see [12]). Since r-modes are basically fluid modes, we expect that the adopted approximation is suitable for studying, qualitatively, their basic properties. The computational requirements for a coupled spacetime and hydrodynamical evolution (for the same long-term accuracy as reported here using the Cowling approximation), by far exceed presently available supercomputing resources.

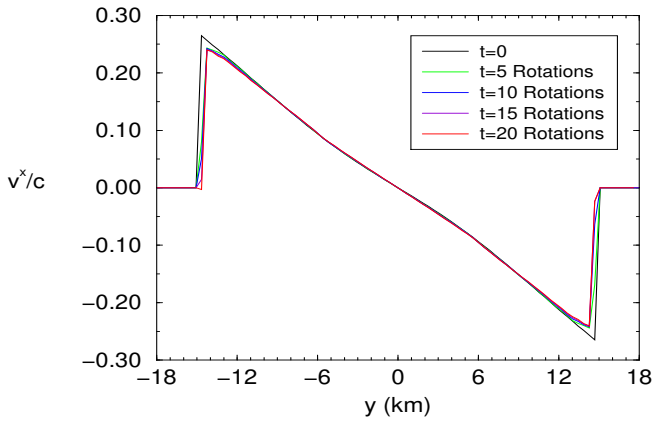

FIG. 1. Time-evolution of the rotational velocity profile for a stationary equilibrium model, using the 3rd order PPM scheme. The star remains stationary, even after 20 rotations. Larger deviations (due to finite-differencing) occur only at the last grid point inside the surface.

Figure 1 clearly demonstrates the ability of the 3-D code to maintain an unperturbed stationary equilibrium configuration (apart from the influence of local truncation errors) in long-term evolutions. The rotational velocity profile (v^x along y-axis) remains nearly unchanged (with the exception of the last grid-point near the surface) over more than 20 rotational periods (all simulations presented here are stopped due to computing time restrictions only and could be continued for longer times).

In order to excite r-modes during the time-evolution, we perturb the initial stationary model, by adding a specific perturbation δv^i to the contravariant components of the equilibrium 3-velocity, v^i, as measured by a local zero-angular-momentum observer. As there is no exact linear eigenfunction available in the literature that would correspond to an $l = m = 2$ r-mode eigenfunction for rapidly rotating relativistic stars in the Cowling approximation, we are using the following approximate eigenfunction, valid in the slow-rotation $O(\Omega)$ limit to the first Post-Newtonian (1PN) order [15,16]

$$\delta v^r = \alpha \frac{6.33\times 10^{-2}}{\tilde{\alpha}} \left(\frac{2M_0}{r_e}\right) \Omega r \left(\frac{r}{r_e}\right) \left(1-\frac{r^2}{r_e^2}\right) \sin^2\theta \cos\theta \sin 2\phi, \quad (1)$$

$$\delta v^\theta = \alpha \frac{\Omega}{\tilde{\alpha}} \left(\frac{r}{r_e}\right) \left\{ 2.11\times 10^{-2} \left[2-3\left(\frac{r}{r_e}\right)^2\right] \left(\frac{2M_0}{r_e}\right) \left(1-\frac{3}{2}\sin^2\theta\right) + 0.315 \left[1-0.319\left(1-\frac{r^2}{r_e^2}\right)\left(\frac{2M_0}{r_e}\right)\right] - 9.64\times 10^{-3}\left(\frac{r}{r_e}\right)^2 \left(\frac{2M_0}{r_e}\right) \left(7\cos^2\theta - 1\right) \right\} \sin\theta \sin 2\phi, \quad (2)$$

$$\delta v^\phi = \alpha \frac{\Omega}{\tilde{\alpha}} \left(\frac{r}{r_e}\right) \left\{ 2.11\times 10^{-2} \left[2-3\left(\frac{r}{r_e}\right)^2\right] \left(\frac{2M_0}{r_e}\right) + 0.315 \left[1-0.315\left(1-\frac{r^2}{r_e^2}\right)\left(\frac{2M_0}{r_e}\right)\right] - 9.64\times 10^{-3}\left(\frac{r}{r_e}\right)^2 \left(\frac{2M_0}{r_e}\right) \left[7\left(1-2\sin^2\theta\right) - 1\right] \right\} \cos\theta \cos 2\phi, \quad (3)$$

where Ω is the angular velocity of the equilibrium star, r, θ, ϕ are the usual spherical polar coordinates, M_0 is the baryonic rest mass, r_e is the coordinate radius at the equator, α is the amplitude of the perturbation and $\tilde{\alpha}$ is the lapse function (as defined in the 3+1 formalism). Such expressions are transformed to Cartesian coordinates as this is the coordinate system employed by the 3-D code. With this eigenfunction (multiplied by a chosen initial amplitude), we are able to excite mainly the $l = m = 2$ r-mode. Due to the approximate nature of the eigenfunction, additional modes are also excited, primarily $m = 2$ inertial modes.

Figure 2 displays the evolution of the axial velocity in the equatorial plane (v^z along the y-axis) at a radius of $r \sim 0.75R$. It is clear that the evolution is a superposition of several modes, and that one mode, the $l = m = 2$ r-mode, is the dominant component. The amplitude in this evolution is $\alpha = 1.0$ (in the Newtonian limit, our definition of the amplitude coincides with the definition in [17]). The perturbed star is evolved for more than 24ms (25 $l = m = 2$ r-mode periods), during which the amplitude of the oscillation decreases due to numerical viscosity. Even at amplitudes larger than 1.0, the evolution is still similar to that in Figure 2, with no sign of nonlinear saturation of the r-mode amplitude on a dynamical timescale. This implies, that, unless nonlinear saturation is set at timescales much longer than the dynamical one, unstable r-modes could be driven to large amplitudes (of order one) by gravitational radiation reaction, before saturation sets in. An absolute upper limit on the r-mode amplitude (relevant, perhaps, for the most rapidly rotating stars) is set by causality, requiring $\sqrt{v_i v^i} < c$ (where c is the speed of light), or approximately, $\alpha_{\rm causal} \lesssim 3c/\Omega R$ (for slowly rotating stars, we expect other upper limits to be more relevant).

A Fourier transform of the time-evolution shown in Figure 2, as a function of the frequency in the inertial

frame, shown in Figure 3, reveals that the initial data we are using, excite mainly the $l = m = 2$ r-mode (r_2), with a frequency of 1.03 kHz and, with smaller amplitudes, several inertial modes (i_3, i_4, i_5) and higher harmonics. The frequencies of the various modes are the same at any given point inside the star (the specific data shown in the figure correspond to radii $r \sim 0.5R$ and $r \sim 0.75R$), i.e. the spectrum is a sum of discrete modes only. This is in agreement with the conclusions in [16], regarding the r-mode spectrum in isentropic stars. The possible existence of a continuous part in the frequency spectrum of nonisentropic stars (see e.g. [18]), will be examined in future work.

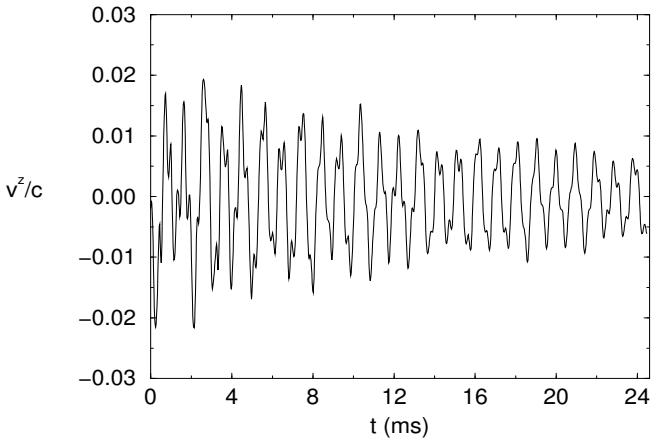

FIG. 2. Evolution of the axial velocity in the equatorial plane for an amplitude of $\alpha = 1.0$, at $r = 0.75R$. The evolution is a superposition of (mainly) the $l = m = 2$ r-mode and several inertial modes. The amplitude of the oscillation decreases due to numerical (finite-differencing) viscosity of the code. A beating between the $l = m = 2$ r-mode and the $l_0 = 4, m = 2$ inertial mode can also be seen.

In order to identify some of the peaks in the Fourier transform with specific inertial modes, we compare the ratio of their frequencies over the frequency of the $l = m = 2$ r-mode to the corresponding ratios derived from the normal mode linear eigenfrequencies, for a Newtonian $N = 1.0$ polytrope [19]. This comparison shows good agreement and allows us to identify, in Figure 3, the $m = 2$ inertial modes with $l_o = 3, 4, 5$ [19], having corresponding frequencies $f = 0.68, 1.16$ and 0.38 kHz.

We investigate the influence of the numerical viscosity of the code (due to finite-differencing) on the evolution, by comparing evolutions at different resolutions. Figure 4 shows the late-time evolution of the axial velocity in the equatorial plane, at $r \sim 0.75R$ for grid-spacings of $\Delta x = 0.59$km and $\Delta x = 0.41$km. The decrease in the amplitude of the oscillation (when compared to the initial amplitude) scales as roughly first order with resolution. This indicates that the numerical viscosity damping the oscillations is dominated by the truncation error at the surface of the star, which is only first-order (compared to the second-order accuracy in the interior; see related discussion in [12]).

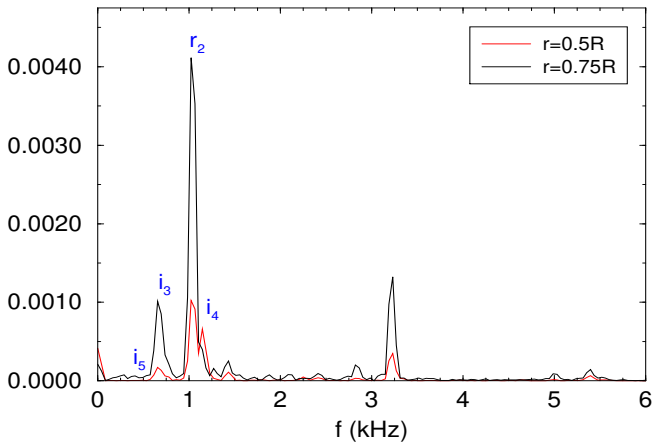

FIG. 3. Fourier transform of the evolution shown in Figure 2, showing the frequencies of the modes in the inertial frame. The frequencies of the modes are the same at different radii (discrete spectrum) and the ratio of the r-mode frequency to the frequencies of the identified inertial modes agrees with published results.

In a recent paper [8] it is indicated that the hydrodynamical oscillatory motion of an r-mode in a rotating star may be accompanied by a differential drift (of kinematical origin). In the equatorial plane, the drift velocity has opposite sign, compared to the rotational velocity of the unperturbed star. In a rotating star with a poloidal magnetic field, this kinematical drift may wind up the magnetic field lines and limit the effect of the r-mode instability [7,8]. The differential drift reported in [8] is second order in the r-mode amplitude, but it is derived from the first-order in amplitude definition of the r-mode velocity. Still, in the restricted case of a spherical shell [20], the full nonlinear hydrodynamical equations yield the same drift as in [8], independent of the amplitude of the oscillations.

Using our code we attempted to investigate the presence of such a kinematical drift in the numerical evolutions. Our numerical results do indicate that the perturbed star is rotating slower near the surface (compared to the rotational velocity of the unperturbed star) and that this drift scales roughly as α^2 for amplitudes of order $\alpha \sim 1.0$, although its magnitude is significantly smaller than estimated in [8]. We cannot test the presence of the drift for amplitudes much less than $\alpha \sim 1.0$, since it becomes smaller than numerical truncation errors. Even at $\alpha \sim 1.0$, it is not completely unambiguous that the drift we find is the same as that predicted by [8], as it could be due to the fact that our initial data do not correspond exactly to a single $l = m = 2$ r-mode and thus also contain other modes and violate (to some extent, which is still acceptable for the main purpose of this work) the relativistic Hamiltonian and momentum constraints. However, the fact that our drift scales as α^2, points to an association with the results reported in [8].

Our preliminary findings on the differential drift need to be confirmed by simulations with more precise eigenfunctions and higher resolution, before a definite conclusion can be drawn.

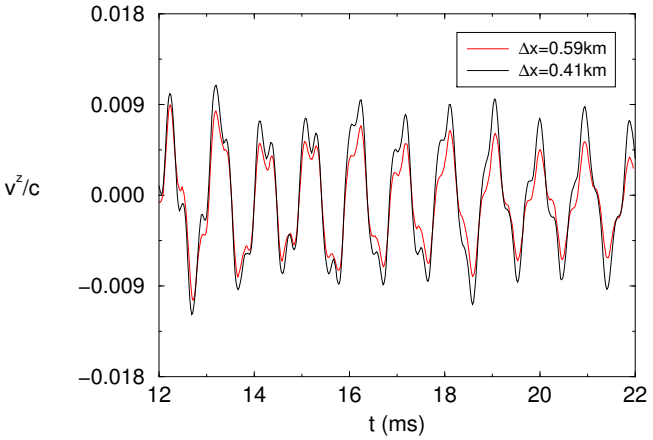

FIG. 4. Late-time evolution of the axial velocity in the equatorial plane for $\alpha = 1.0$, at $r = 0.75R$, for two different resolutions. The decrease in the oscillation amplitude scales as roughly first order with resolution, i.e. it is dominated by the first-order truncation error at the surface of the star.

In summary, our hydrodynamical evolutions of nonlinear r-modes show that (1) r-modes in rapidly rotating relativistic stars could attain a large amplitude, of order one, when driven unstable by gravitational radiation reaction - it remains to be investigated whether nonlinear saturation could set in at timescales longer than the dynamical one, (2) r-modes and inertial modes, in isentropic stars, are discrete and (3) a kinematical drift associated with r-mode oscillations, appears to be present in our simulations, although an unambiguous confirmation will require more precise initial data.

Our finding that gravitational radiation could drive unstable r-modes to a large amplitude, implies that r-modes can easily melt the crust in newly-born neutron stars [21], leaving the initial conclusions about the r-mode instability being a strong source of gravitational waves (see [5,6]) essentially unchanged. Our results also imply that r-modes could be excited to large amplitudes during the violent formation of a proto-neutron star, after a supernova core-collapse [22]. Our confirmation of the discrete nature of the frequency spectrum of r-modes and inertial modes (at least in isentropic stars) significantly simplifies the expected gravitational wave signal compared to a signal emitted by a source with a continuous part in its frequency spectrum. The indications for the existence of the kinematical drift, call for a more detailed consideration of the interaction between r-modes and magnetic fields.

We will present details and tests of our numerical method, as well as extensive higher-resolution results for various equations of state and rotation rates, in a forthcoming paper [23]. In future work, we plan to study r-modes in nonisentropic stars and implement an accelerated gravitational-radiation reaction force (see [24]).

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